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SOME PROPERTIES OF STARLIKE AND CLOSE TO CONVEX FUNCTIONS WITH RESPECT TO SYMMETRIC CONJUGATE POINTS

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Abstract: In the present paper, we introduce and investigate some new subclasses of α- starlike and α - close to convex functions with respect to symmetric conjugate points. Inclusion relationships, integral representations and some interesting convolution properties for these functions are obtained.

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1. Introduction, Definitions and Preliminaries

Let $\mathcal{A}_p$ be the class of functions $f(z)$, of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. And let $\mathcal{A}$=$\mathcal{A}_1$.

We denote by $\mathcal{S}^{\star}$, $\mathcal{C}$, $\mathcal{K}$ and $\mathcal{C}^{\star}$ the familiar subclasses of $\mathcal{A}$ consisting of

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functions which are respectively starlike, convex, close-to-convex and quasi-convex in $\mathbb{U}$. The references of the field are [5, 6], it covers most of the topics.

Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of all functions which are univalent in $\mathbb{U}$. Also, let $\mathcal{P}$ denote the class of functions of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$$

which are analytic and convex in $\mathbb{U}$ and satisfy the condition

$$\Re(p(z)) > 0, \ (z \in \mathbb{U}).$$

Let $f(z)$ and $g(z)$ be analytic in $\mathbb{U}$. Then we say that the function $f(z)$ is subordinate to $g(z)$ in $\mathbb{U}$, if there exists an analytic function $w(z)$ in $\mathbb{U}$ such that $|w(z)| < |z|$ and $f(z) = g(w(z))$, denoted by $f(z) \prec g(z)$. If $g(z)$ is univalent in $\mathbb{U}$, then the subordination is equivalent to $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.

Let k be a positive integer and $j = 0, 1, 2, \ldots (k-1)$. A domain D is said to be (j, k)-fold symmetric if a rotation of D about the origin through an angle $2\pi j/k$ carries D onto itself. A function $f \in \mathcal{A}$ is said to be (j, k)-symmetrical if for each $z \in \mathbb{U}$

$$f(\varepsilon z) = \varepsilon^j f(z), \tag{1.2}$$

where $\varepsilon = \exp(2\pi i/k)$. The family of (j, k)-symmetrical functions will be denoted by $\mathcal{F}_k^j$. For every function f defined on a symmetrical subset $\mathbb{U}$ of $\mathbb{C}$, there exits a unique sequence of (j, k)-symmetrical functions $f_{j,k}(z), j = 0, 1, \ldots, k-1$ such that

$$f = \sum_{j=0}^{k-1} f_{j,k}.$$

Moreover,

$$f_{j,k}(z) = \frac{1}{k} \sum_{\nu=0}^{k-1} \frac{f(\varepsilon^\nu z)}{\varepsilon^{\nu p j}}, \quad (f \in \mathcal{A}_p;\ k = 1, 2, \ldots;\ j = 0, 1, 2, \ldots (k-1)). \tag{1.3}$$

This decomposition is a generalization of the well known fact that each function defined on a symmetrical subset $\mathbb{U}$ of $\mathbb{C}$ can be uniquely represented as the sum of an even function and an odd functions (see Theorem 1 of [7]). It is obvious that $f_{j,k}(z)$ is a linear operator from $\mathbb{U}$ into $\mathbb{U}$. The notion of (j, k)-symmetrical functions was first introduced and studied by P. Liczberski and J. Polubiński in [7]. The class of (j, k)-symmetrical functions was extended to the class $(j,\, k)$-symmetrical conjugate functions in [12]. For fixed positive integers j and k, let $f_{2j,\,k}(z)$ be defined by the following equality

$$f_{2j,\,k}(z)=\frac{1}{2k}\sum_{\nu=0}^{k-1}\left[\varepsilon^{-\nu pj}f(\varepsilon^{\nu}z)+\varepsilon^{\nu pj}\overline{f(\varepsilon^{\nu}\overline{z})}\right],\quad (f\in\mathcal{A}_p). \tag{1.4}$$

If ν is an integer, then the following identities follow directly from (1.4):

$$\begin{aligned} f_{2j,\,k}'(z)&=\frac{1}{2k}\sum_{\nu=0}^{k-1}\left[\varepsilon^{-\nu pj+\nu}f'(\varepsilon^{\nu}z)+\varepsilon^{\nu pj-\nu}\overline{f'(\varepsilon^{\nu}\overline{z})}\right],\\ f_{2j,\,k}''(z)&=\frac{1}{2k}\sum_{\nu=0}^{k-1}\left[\varepsilon^{-\nu pj+2\nu}f''(\varepsilon^{\nu}z)+\varepsilon^{\nu pj-2\nu}\overline{f''(\varepsilon^{\nu}\overline{z})}\right],\end{aligned} \tag{1.5}$$

and

$$\begin{aligned} f_{2j,\,k}(\varepsilon^{\nu}z)&=\varepsilon^{\nu pj}f_{2j,\,k}(z),\quad f_{2j,\,k}(z)=\overline{f_{2j,\,k}(\overline{z})},\\ f_{2j,\,k}'(\varepsilon^{\nu}z)&=\varepsilon^{\nu pj-\nu}f_{2j,\,k}'(z),\quad f_{2j,\,k}'(\overline{z})=\overline{f_{2j,\,k}'(z)}.\end{aligned} \tag{1.6}$$

Motivated by the concept introduced by Sakaguchi in [11], recently several subclasses of analytic functions with respect to k-symmetric points were introduced and studied by various authors (see [1, 2, 14, 15, 16]). Parvatham in ([10]) introduced and investigated $K_n(\alpha, h)$ - so called class of α starlike functions with respect to n symmetric points.

Definition 1.1. The function $f\in\mathcal{A}_p$ and $\dfrac{f(z)\,f'(z)}{z}\neq 0$ in $\mathbb{U}$ is said to be in the class $\mathcal{S}_p^{j,\,k}(\alpha,\phi)$ if and only if it satisfies the condition

$$\frac{1}{p}\left(\frac{\alpha z\,(zf'(z))'+(1-\alpha)\,zf'(z)}{\alpha zf_{2j,k}'(z)+(1-\alpha)\,f_{2j,k}(z)}\right)\prec\phi(z),\qquad (z\in\mathbb{U}), \tag{1.7}$$

where, $\phi\in\mathcal{P}$, $0\leq\alpha\leq 1$ and $f_{2j,\,k}(z)\neq 0$ is defined by the equality (1.4). Similarly, we say that a function $f\in\mathcal{A}_p$ is in the class $\mathcal{C}_p^{j,\,k}(\alpha,\phi)$ if and only if

$$zf'\in\mathcal{S}_p^{j,\,k}(\alpha,\phi).$$

Definition 1.2. The function $f\in\mathcal{A}_p$ and $\dfrac{f(z)\,f'(z)}{z}\neq 0$ in $\mathbb{U}$ is said to be in the class $\mathcal{M}_p^{j,\,k}(\alpha,\phi)$ if and only if it satisfies the condition

$$\frac{1}{p}\left((1-\alpha)\,\frac{zf'(z)}{f_{2j,k}(z)}+\alpha\frac{(zf'(z))'}{f_{2j,k}'(z)}\right)\prec\phi(z),\qquad (z\in\mathbb{U}), \tag{1.8}$$

where, $\phi\in\mathcal{P}$ and $\alpha\geq 0$.

Definition 1.3. The function $f \in \mathcal{A}_p$ and $\frac{f(z) f'(z)}{z} \neq 0$ in $\mathbb{U}$ is said to be in the class $\mathcal{K}_p^{j,k}(\alpha, \phi)$ if and only if it satisfies the condition

$$\frac{1}{p}\left(\frac{\alpha z (z f'(z))' + (1-\alpha) z f'(z)}{\alpha z g'_{2j,k}(z) + (1-\alpha) g_{2j,k}(z)}\right) \prec \phi(z), \qquad (z \in \mathbb{U}), \tag{1.9}$$

where, $\phi \in \mathcal{P}$, $0 \leq \alpha \leq 1$ and $g_{2j,k}(z) \neq 0$ is defined by (1.4) with $g_{2j,k}(z) \in \mathcal{S}_p^{j,k}(\alpha, \phi)$.

Remark 1.1. In view of definitions 1.1 and 1.2, we know that the classes $\mathcal{S}_p^{j,k}(\alpha, \phi)$ and $\mathcal{C}_p^{j,k}(\alpha, \phi)$ unify the classes of α starlike and α convex functions with respect to $(2j, k)$ symmetric conjugate points. Furhter we note that several other new and well known subclasses of analytic functions can be obtained as special cases of $\mathcal{S}_p^{j,k}(\alpha, \phi)$ and $\mathcal{C}_p^{j,k}(\alpha, \phi)$.

If we put $\alpha = 0$, then the classes $\mathcal{S}_p^{j,k}(\alpha, \phi)$ and $\mathcal{C}_p^{j,k}(\alpha, \phi)$ reduces to the respective class $\mathcal{S}_p^{j,k}(\phi)$ and $\mathcal{C}_p^{j,k}(\phi)$ introduced by Karthikeyan [12]. Also, if we let $p = j = k = 1$, $\alpha = 0$ and $h(z) = \frac{1+z}{1-z}$ then the class $\mathcal{S}_p^{j,k}(\alpha, \phi)$ reduces to the class $\mathcal{S}_c^*$ investigated by EL Ashwa and Thomas in [4].

Lemma 1.1. *[3, 8] Let $\beta, \gamma \in \mathbb{C}$. Suppose that ϕ is convex and univalent in $\mathbb{U}$ with $\phi(0) = 1$ and $\Re(\beta\phi(z) + \gamma) > 0$, $(z \in \mathbb{U})$ and let p be analytic in $\mathbb{U}$ with $p(0) = 1$, then the subordination*

$$p(z) + \frac{z p'(z)}{\beta p(z) + \gamma} \prec \phi(z) \Rightarrow p(z) \prec \phi(z).$$

Lemma 1.2. *[9] Let $\beta, \gamma \in \mathbb{C}$. Suppose that ϕ is convex and univalent in $\mathbb{U}$ with $\phi(0) = 1$ and $\Re(\beta\phi(z) + \gamma) > 0$, $(z \in \mathbb{U})$. Also, let $q(z) \prec \phi(z)$. If $u \in \mathcal{P}$ and satisfies the subordination*

$$u(z) + \frac{z u'(z)}{\beta q(z) + \gamma} \prec \phi(z),$$

then,

$$u(z) \prec \phi(z).$$

Lemma 1.3. *[13] Let $\phi \in \mathcal{P}$. Then,*

$$\mathcal{S}_p^{j,k}(\phi) \subset \mathcal{C} \subset \mathcal{S}.$$

Lemma 1.4. *[13] Let $\phi \in \mathcal{P}$. Then,*

$$\mathcal{C}_p^{j,k}(\phi) \subset \mathcal{C}^* \subset \mathcal{C}.$$

2. Inclusion Relationship

Theorem 2.1. *Let $\phi \in \mathcal{P}$, $0 \le \alpha \le 1$ and let $f \in \mathcal{S}_p^{j,k}(\alpha, \phi)$. Then $f_{2j,k} \in \mathcal{S}_p^{j,k}(\alpha, \phi)$. Furthermore for any $z \in \mathbb{U}$,*

$$\mathcal{S}_p^{j,k}(\alpha, \phi) \subset \mathcal{S}_p^{j,k}(\phi) \subset \mathcal{C} \subset \mathcal{S}.$$

Proof. Let $f \in \mathcal{S}_p^{j,k}(\alpha, \phi)$. From the definition of $\mathcal{S}_p^{j,k}(\alpha, \phi)$, we have

$$\frac{1}{p}\left(\frac{\alpha z\left(zf'(z)\right)' + (1-\alpha) zf'(z)}{\alpha z f'_{2j,k}(z) + (1-\alpha) f_{2j,k}(z)}\right) \prec \phi(z), \qquad (z \in \mathbb{U}). \tag{2.1}$$

On a simple computation, we have

$$\frac{1}{p}\left(\frac{zf'(z) + \alpha z^2 f''(z)}{(1-\alpha) f_{2j,k}(z) + \alpha z f'_{2j,k}(z)}\right) \prec \phi(z). \tag{2.2}$$

If we replace z by $\varepsilon^\nu z$ in (2.2), then (2.2) will be of the form

$$\frac{1}{p}\left(\frac{\varepsilon^\nu z f'(\varepsilon^\nu z) + \alpha(\varepsilon^\nu z)^2 f''(\varepsilon^\nu z)}{(1-\alpha) f_{2j,k}(\varepsilon^\nu z) + \alpha z \varepsilon^\nu f'_{2j,k}(\varepsilon^\nu z)}\right) \prec \phi(z)$$
$$(z \in \mathcal{U};\ \nu = 0, 1, 2, \ldots, k-1). \tag{2.3}$$

Also we have,

$$\frac{1}{p}\left(\frac{\overline{\varepsilon^\nu \overline{z}} \overline{f'(\varepsilon^\nu \overline{z})} + \alpha \overline{(\varepsilon^\nu \overline{z})}^2 \overline{f''(\varepsilon^\nu \overline{z})}}{(1-\alpha) \overline{f_{2j,k}(\varepsilon^\nu \overline{z})} + \alpha \overline{z \varepsilon^\nu} \overline{f'_{2j,k}(\varepsilon^\nu \overline{z})}}\right) \prec \phi(z)$$
$$(z \in \mathcal{U};\ \nu = 0, 1, 2, \ldots, k-1). \tag{2.4}$$

Using the equality (1.6), (2.3) and (2.4) can be rewritten as

$$\frac{1}{p}\left(\frac{\varepsilon^{\nu - \nu p j}\left(zf'(\varepsilon^\nu z) + \alpha z^2 \varepsilon^\nu f''(\varepsilon^\nu z)\right)}{(1-\alpha) f_{2j,k}(z) + \alpha z f'_{2j,k}(z)}\right) \prec \phi(z) \tag{2.5}$$

and

$$\frac{1}{p}\left(\frac{\varepsilon^{-\nu + \nu p j}\left(z\overline{f'(\varepsilon^\nu \overline{z})} + \alpha z^2 \varepsilon^{-\nu} \overline{f''(\varepsilon^\nu \overline{z})}\right)}{(1-\alpha) f_{2j,k}(z) + \alpha z f'_{2j,k}(z)}\right) \prec \phi(z)$$

$$(z \in \mathcal{U};\ \nu = 0,\ 1,\ 2,\ \ldots,\ k-1). \quad (2.6)$$

Adding (2.5) and (2.6), we get

$$\frac{1}{p}\left(\frac{\frac{z}{2}\left(\varepsilon^{\nu-\nu pj} f'(\varepsilon^{\nu} z) + \varepsilon^{-\nu+\nu pj}\overline{f'(\varepsilon^{\nu}\overline{z})}\right) + \frac{\alpha z^2}{2}\left(\varepsilon^{2\nu-\nu pj} f''(\varepsilon^{\nu} z) + \varepsilon^{-2\nu+\nu pj}\overline{f''(\varepsilon^{\nu}\overline{z})}\right)}{(1-\alpha) f_{2j,k}(z) + \alpha z f'_{2j,k}(z)}\right) \prec \phi(z). \quad (2.7)$$

Let $\nu = 0,\ 1,\ 2,\ \ldots,\ k-1$ in (2.7) respectively and summing them, we get

$$\frac{1}{p}\frac{\frac{z}{2k}\sum_{\nu=0}^{k-1}\left[\varepsilon^{\nu-\nu pj} f'(\varepsilon^{\nu} z) + \varepsilon^{-\nu+\nu pj}\overline{f'(\varepsilon^{\nu}\overline{z})}\right] + \frac{\alpha z^2}{2k}\sum_{\nu=0}^{k-1}\left[\varepsilon^{2\nu-\nu pj} f''(\varepsilon^{\nu} z) + \varepsilon^{-2\nu+\nu pj}\overline{f''(\varepsilon^{\nu}\overline{z})}\right]}{(1-\alpha) f_{2j,k}(z) + \alpha z f'_{2j,k}(z)} \prec \phi(z), \quad (2.8)$$

or equivalently,

$$\frac{1}{p}\frac{z f'_{2j,k}(z) + \alpha z^2 f''_{2j,k}(z)}{(1-\alpha) f_{2j,k}(z) + \alpha z f'_{2j,k}(z)} \prec \phi(z), \quad (2.9)$$

which implies that $f_{2j,k} \in \mathcal{S}_p^{j,\,k}(\alpha, \phi)$.

If we set

$$p(z) = \frac{z f'_{2j,k}(z)}{p f_{2j,k}(z)}, \qquad (z \in \mathbb{U}),$$

then (2.9) can be written as

$$\frac{1}{p}\frac{z f'_{2j,k}(z) + \alpha z^2 f''_{2j,k}(z)}{(1-\alpha) f_{2j,k}(z) + \alpha z f'_{2j,k}(z)} = p(z) + \frac{\alpha z p'(z)}{(1-\alpha) + p\alpha p(z)} \prec \phi(z). \quad (2.10)$$

It is very clear that $p(z) \prec \phi(z)$, since α is a real number with the condition imposed $0 \leq \alpha \leq 1$.

Setting

$$h(z) = \frac{z f'(z)}{p f_{2j,k}(z)}, \qquad (z \in \mathbb{U}).$$

Now

$$\begin{aligned}
&\frac{1}{p}\left(\frac{(1-\alpha)\,zf'(z)+\alpha z\,(zf'(z))'}{(1-\alpha)\,f_{2j,k}(z)+\alpha zf'_{2j,k}(z)}\right)\\
=&\frac{1}{p}\frac{(1-\alpha)\,ph(z)\,f_{2j,k}(z)+\alpha zp\left[h'(z)\,f_{2j,k}(z)+h(z)\,f'_{2j,k}(z)\right]}{(1-\alpha)\,f_{2j,k}(z)+\alpha zf'_{2j,k}(z)}\\
=&\frac{(1-\alpha)\,h(z)+\alpha zh'(z)+\frac{\alpha zf'_{2j,k}(z)}{f_{2j,k}(z)}.h(z)}{(1-\alpha)+\frac{\alpha zf'_{2j,k}(z)}{f_{2j,k}(z)}}\\
=&h(z)+\frac{\alpha zh'(z)}{(1-\alpha)+p\alpha p(z)}.
\end{aligned} \tag{2.11}$$

Now, by applying the above proof for $p(z)\prec\phi(z)$ and using Lemma 1.2 in (2.11), we know that

$$h(z)=\frac{zf'(z)}{pf_{2j,k}(z)}\prec\phi(z),\qquad (z\in\mathbb{U}),$$

which implies that

$$\mathcal{S}_p^{j,k}(\alpha,\phi)\subset\mathcal{S}_p^{j,k}(\phi).$$

And also

$$\mathcal{S}_p^{j,k}(\alpha,\phi)\subset\mathcal{S}_p^{j,k}(\phi)\subset\mathcal{C}\subset\mathcal{S}.$$

□

By means of Lemma 1.4 and making use of similar arguments given in the proof for Theorem 2.1, we easily get the following inclusion relationship for the class $\mathcal{C}_p^{j,k}(\alpha,\phi)$.

Corollary 2.2. *Let $\phi\in\mathcal{P}$ and $0\le\alpha\le1$, then*

$$\mathcal{C}_p^{j,k}(\alpha,\phi)\subset\mathcal{C}_p^{j,k}(\phi)\subset\mathcal{C}'\subset\mathcal{C}.$$

Theorem 2.3. *Let $\phi\in\mathcal{P}$ and $0\le\alpha\le1$, then*

$$\mathcal{K}_p^{j,k}(\alpha,\phi;g)\subset\mathcal{K}_p^{j,k}(\phi;g).$$

Proof. Let $f \in \mathcal{K}_p^{j,k}(\alpha, \phi)$. Setting $p(z) = \dfrac{zf'(z)}{pg_{2j,k}(z)}$ and $q(z) = \dfrac{zg'_{2j,k}(z)}{pg_{2j,k}(z)}$, we have,

$$\frac{1}{p}\frac{\alpha z (zf'(z))' + (1-\alpha) zf'(z)}{\alpha z g'_{2j,k}(z) + (1-\alpha) g_{2j,k}(z)} = \frac{\alpha z p'(z) + p(z)\left[(1-\alpha) + \alpha q(z)\right]}{(1-\alpha) + p\alpha q(z)}$$
$$= p(z) + \frac{\alpha z p'(z)}{(1-\alpha) + p\alpha q(z)} \prec \phi(z)$$

since $f \in \mathcal{K}_p^{j,k}(\alpha, \phi)$. Here $q(z) \prec \phi(z)$(by lemma). Again an application of Lemma (1.2) yields $p(z) = \dfrac{zf'(z)}{pg_{2j,k}(z)}$ which estabilish the theorem. □

Theorem 2.4. *Let $\phi \in \mathcal{P}$ and $0 \le \alpha \le 1$, then*

$$\mathcal{M}_p^{j,k}(\alpha, \phi) \subset \mathcal{S}_p^{j,k}(\phi) \subset \mathcal{C} \subset \mathcal{S}.$$

Proof. Suppose that $f \in \mathcal{M}_p^{j,k}(\alpha, \phi)$. It follows form (1.8) that

$$\frac{1}{p}\left((1-\alpha)\frac{zf'(z)}{f_{2j,k}(z)} + \alpha\frac{(zf'(z))'}{f'_{2j,k}(z)}\right)$$
$$= \frac{(1-\alpha)}{p}\frac{zf'(z)}{f_{2j,k}(z)} + \frac{\alpha}{p}\frac{f'(z) + zf''(z)}{f'_{2j,k}(z)} \prec \phi(z) \quad (2.12)$$

If we replace z by $\varepsilon^\nu z$ ($\nu = 0, 1, 2, \dots k-1$)in (2.12), then (2.12) will be of the form

$$\frac{(1-\alpha)}{p}\frac{\varepsilon^\nu z f'(\varepsilon^\nu z)}{f_{2j,k}(\varepsilon^\nu z)} + \frac{\alpha}{p}\frac{f'(\varepsilon^\nu z) + \varepsilon^\nu z f''(\varepsilon^\nu z)}{f'_{2j,k}(\varepsilon^\nu z)} \prec \phi(z). \quad (2.13)$$

From (2.13), we have,

$$\frac{(1-\alpha)}{p}\frac{\overline{\varepsilon^\nu \overline{z}}\,\overline{f'(\varepsilon^\nu \overline{z})}}{\overline{f_{2j,k}(\varepsilon^\nu \overline{z})}} + \frac{\alpha}{p}\frac{\overline{f'(\varepsilon^\nu \overline{z})} + \overline{\varepsilon^\nu \overline{z}}\,\overline{f''(\varepsilon^\nu \overline{z})}}{\overline{f'_{2j,k}(\varepsilon^\nu \overline{z})}} \prec \phi(z). \quad (2.14)$$

Proceeding as in Theorem (2.1), we have

$$\frac{1}{p}\left((1-\alpha)\frac{zf'_{2j,k}(z)}{f_{2j,k}(z)} + \alpha\frac{\left(zf'_{2j,k}(z)\right)'}{f'_{2j,k}(z)}\right) \prec \phi(z). \quad (2.15)$$

Letting

$$p(z) = \frac{zf'_{2j,k}(z)}{pf_{2j,k}(z)}, \qquad (z \in \mathbb{U}).$$

Then, (2.15) can be written as

$$\frac{1}{p}\left((1-\alpha)\frac{zf'_{2j,k}(z)}{f_{2j,k}(z)} + \alpha\frac{\left(zf'_{2j,k}(z)\right)'}{f'_{2j,k}(z)}\right) = p(z) + \alpha\frac{zp'(z)}{pp(z)} \prec \phi(z).$$

By Lemma 1.1, we have

$$p(z) = \frac{zf'_{2j,k}(z)}{pf_{2j,k}(z)} \prec \phi(z).$$

If we let

$$h(z) = \frac{zf'(z)}{pf_{2j,k}(z)},$$

then (1.8), can be written as follows:

$$\frac{1}{p}\left((1-\alpha)\frac{zf'(z)}{f_{2j,k}(z)} + \alpha\frac{(zf'(z))'}{f'_{2j,k}(z)}\right) = h(z) + \alpha\frac{zh'(z)}{pp(z)} \prec \phi(z).$$

Since

$$p(z) \prec \phi(z),$$

By Lemma 1.2, yields

$$\mathcal{M}_p^{j,k}(\alpha,\phi) \subset \mathcal{S}_p^{j,k}(\phi).$$

Then Lemma 1.3, we have

$$\mathcal{M}_p^{j,k}(\alpha,\phi) \subset \mathcal{S}_p^{j,k}(\phi) \subset \mathcal{C} \subset \mathcal{S}. \qquad \square$$

3. Integral Representation

Theorem 3.1. *Let* $f \in \mathcal{S}_p^{j,k}(\alpha, \phi)$ *with* $0 < \alpha \leq 1$. *Then*

$$f_{2j,k}(z) = \frac{1}{\alpha} z^{1-\frac{1}{\alpha}} \int_0^z exp\left(\frac{p}{2k}\sum_{\nu=0}^{k-1}\int_0^u \frac{\phi(w(\varepsilon^\nu\zeta)) + \overline{\phi(w(\varepsilon^\nu\overline{\zeta}))} - 2}{\zeta} d\zeta\right) u^{\frac{1}{\alpha}+p-2} du, \quad (3.1)$$

where, $f_{2j,k}(z)$ *is defined by (1.4),* w *is analytic in* $\mathbb{U}$ *with*

$$w(0) = 0 \text{ and } |w(z)| < 1, \qquad (z \in \mathbb{U}).$$

Proof. Suppose that $f \in \mathcal{S}_p^{j,k}(\alpha, \phi)$. We know that the condition (1.7) can be written as

$$\frac{1}{p}\left(\frac{\alpha z (z f'(z))' + (1-\alpha) z f'(z)}{\alpha z f'_{2j,k}(z) + (1-\alpha) f_{2j,k}(z)}\right) = \phi(w(z)), \qquad (z \in \mathbb{U}) \qquad (3.2)$$

where, w is analytic in $\mathbb{U}$ with

$$w(0) = 0 \text{ and } |w(z)| < 1, \qquad (z \in \mathbb{U}).$$

By similar application of the arguments given in the proof for Theorem 2.1 to (3.2), we get,

$$\frac{1}{p}\frac{\alpha z \left(z f'_{2j,k}(z)\right)' + (1-\alpha) z f'_{2j,k}(z)}{\alpha z f'_{2j,k}(z) + (1-\alpha) f_{2j,k}(z)} = \frac{1}{2k}\sum_{\nu=0}^{k-1}\left(\phi(w(\varepsilon^\nu z)) + \overline{\phi(w(\varepsilon^\nu \overline{z}))}\right). \quad (3.3)$$

From (3.3),

$$\frac{\alpha \left(z f'_{2j,k}(z)\right)' + (1-\alpha) f'_{2j,k}(z)}{\alpha z f'_{2j,k}(z) + (1-\alpha) f_{2j,k}(z)} - \frac{p}{z} = \frac{p}{2k}\sum_{\nu=0}^{k-1}\frac{\left(\phi(w(\varepsilon^\nu z)) + \overline{\phi(w(\varepsilon^\nu \overline{z}))} - 2\right)}{z}. \quad (3.4)$$

Integrating this equality, we get

$$\log\left(\frac{\alpha z f'_{2j,k}(z)+(1-\alpha) f_{2j,k}(z)}{z^p}\right)$$
$$=\frac{p}{2k}\sum_{\nu=0}^{k-1}\int_0^z \frac{\left(\phi\left(w(\varepsilon^\nu\zeta)\right)+\overline{\phi\left(w(\varepsilon^\nu\overline{\zeta})\right)}-2\right)}{\zeta}d\zeta,$$

or equivalently,

$$\alpha z f'_{2j,k}(z)+(1-\alpha) f_{2j,k}(z)$$
$$=z^p\exp\left\{\frac{p}{2k}\sum_{\nu=0}^{k-1}\int_0^z \frac{\left(\phi\left(w(\varepsilon^\nu\zeta)\right)+\overline{\phi\left(w(\varepsilon^\nu\overline{\zeta})\right)}-2\right)}{\zeta}d\zeta\right\}. \quad (3.5)$$

Now we can derive (3.1) from (3.5). □

Theorem 3.2. *Let $f\in\mathcal{S}_p^{j,k}(\alpha,\phi)$ with $0<\alpha\leq 1$. Then*

$$f(z)=$$
$$\frac{1}{\alpha}z^{1-\frac{1}{\alpha}}\int_0^z\int_0^u exp\left(\frac{p}{2k}\sum_{\nu=0}^{k-1}\int_0^\eta \frac{\phi\left(w(\varepsilon^\nu\zeta)\right)+\overline{\phi\left(w(\varepsilon^\nu\overline{\zeta})+\right)}-2}{\zeta}d\zeta\right)\phi\left(w(\eta)\right)d\eta$$
$$u^{\frac{1}{\alpha}+p-2}du \quad (3.6)$$

where, w is analytic in $\mathbb{U}$ with $w(0)=0$ and $|w(z)|<1$.

Proof. Suppose that $f\in\mathcal{S}_p^{j,k}(\alpha,\phi)$. Then by (3.2) and (3.5),

$$\alpha\left(zf'(z)\right)'+(1-\alpha)f'(z)=\frac{\alpha z f'_{2j,k}(z)+(1-\alpha) f_{2j,k}(z)}{z}.\phi\left(w(z)\right)$$
$$=\exp\left\{\frac{p}{2k}\sum_{\nu=0}^{k-1}\int_0^z \frac{\left(\phi\left(w(\varepsilon^\nu\zeta)\right)+\overline{\phi\left(w(\varepsilon^\nu\overline{\zeta})\right)}-2\right)}{\zeta}d\zeta\right\}\phi\left(w(z)\right). \quad (3.7)$$

Integrating the above equality two times, will give the assertions of the theorem. □

4. Convolution Conditions

Let $f, g \in \mathcal{A}_p$, where $f(z)$ is given by (1.1) and $g(z)$ is defined by

$$g(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n,$$

then the Hadamard product (or convolution) $f * g$ is defined by

$$(f * g)(z) = z^p + \sum_{n=p+1}^{\infty} a_n b_n z^n = (g * f)(z).$$

We now derive some convolution properties for the function classes $\mathcal{S}_p^{j,k}(\alpha, \phi)$ and $\mathcal{C}_p^{j,k}(\alpha, \phi)$.

Theorem 4.1. *Let $f \in \mathcal{S}_p^{j,k}(\alpha, \phi)$ if and only if*

$$\frac{1}{zp}\left\{f*\left[(1-\alpha)\left(\frac{z}{(1-z)^2}-\frac{\phi(e^{i\theta})}{2}h\right)+\alpha z\left(\frac{z}{(1-z)^2}-\frac{\phi(e^{i\theta})}{2}h\right)'\right](z)\right\} - \frac{1}{zp}\left\{\phi(e^{i\theta}).\overline{f*\left(\frac{1-\alpha}{2}h+\frac{\alpha z}{2}h'\right)(\overline{z})}\right\} \neq 0. \quad (4.1)$$

for all $z \in \mathbb{U}$ and $0 \leq \theta < 2\pi$ where h is given by

$$h(z) = \frac{1}{k}\sum_{\nu=0}^{k-1} \frac{z}{1-\varepsilon^{\nu} z}. \quad (4.2)$$

Proof. Let $f \in \mathcal{S}_p^{j,k}(\alpha, \phi)$, then

$$\frac{1}{p}\left(\frac{\alpha z\left(zf'(z)\right)' + (1-\alpha) zf'(z)}{\alpha z f'_{2j,k}(z) + (1-\alpha) f_{2j,k}(z)}\right) \prec \phi(z),$$

it is equivalent to

$$\frac{1}{p}\frac{\alpha z\left(zf'(z)\right)' + (1-\alpha) zf'(z)}{\alpha z f'_{2j,k}(z) + (1-\alpha) f_{2j,k}(z)} \neq \phi(e^{i\theta}) \quad (4.3)$$

for all $z \in \mathbb{U}$ and $0 \leq \theta < 2\pi$. Now (4.3) can be written as in the form of

$$\frac{1}{p}\left\{(1-\alpha)zf'(z)+\alpha z\left(zf'(z)\right)'-\left[\alpha z f'_{2j,k}(z)+(1-\alpha)f_{2j,k}(z)\right]\phi\left(e^{i\theta}\right)\right\} \neq 0. \quad (4.4)$$

It is well known that,

$$zf'(z)=f(z)*\frac{z}{(1-z)^2}. \quad (4.5)$$

From the definition of $f_{2j,k}(z)$,

$$f_{2j,k}(z)=\frac{1}{2}\left((f*h)(z)+\overline{(f*h)(\overline{z})}\right) \quad (4.6)$$

where h is defined by (4.2). Substitute (4.5)and (4.6)in (4.4), we get (4.7). □

Theorem 4.2. *Let $f \in \mathcal{A}$ $\phi \in \mathcal{P}$.Then $f \in \mathcal{C}_p^{j,k}(\alpha,\phi)$ if and only if*

$$\frac{1}{zp}\left\{f*z\left[(1-\alpha)\left(\frac{z}{(1-z)^2}-\frac{\phi(e^{i\theta})}{2}h\right)+\alpha z\left(\frac{z}{(1-z)^2}-\frac{\phi(e^{i\theta})}{2}h\right)'\right]'(z)\right\}$$
$$-\frac{1}{zp}\left\{\phi(e^{i\theta}).\overline{f*\left[z\left(\frac{1-\alpha}{2}h+\frac{\alpha z}{2}h'\right)\right]'(\overline{z})}\right\}\neq 0. \quad (4.7)$$

for all $z \in \mathbb{U}$ and $0 \le \theta < 2\pi$, where h is given by (4.2).

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