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EVEN SEQUENTIAL HARMONIOUS LABELING
OF SOME CYCLE RELATED GRAPHS

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Abstract: In this paper, we generalize this result on cycles by showing that the kC_n snake with string $1, 1, \cdots, 1$ when $n \equiv 0 \pmod 4$ are even sequential harmonious graph. Also we show that the kC_4 snake with m-pendant edges for each $k, m \geq 1$, (for linear case and for general case). Moreover, we show that, all subdivision of $2m_k$ - snake are even sequential harmonious for each $k, m \geq 1$. Finally we present some examples to illustrate the proposed theories.

1. Introduction

Graph labeling is an active area of research in graph theory which has mainly

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evolved through its many applications in coding theory, communication networks, mobile telecommunication system. Optimal circuits layouts or graph decompositions problems, no name just a few of them.

Most graph labeling methods trace their origin to one introduced by Rosa [1] called such a labeling a β-valuation and Golomb subsequently called graceful labeling, and one introduced by Graham and Sloane [2] called harmonious labeling. Several infinite families of graceful and harmonious graphs have been reported. Many illustrious works on graceful graphs brought a tide to different ways of labeling the elements of graph such as odd graceful.

All graphs in this paper are finite, simple graphs and undirected graph. The symbols $V(G)$ and $E(G)$ denote the vertex set and edge set of a graph G. The cardinality of the vertex set is called the order of G. The cardinality of the edge set is called the size of G. A graph with p vertices and q edges is called a $G(p, q)$ graph [4].

A graph G of size q is odd-graceful, if there is an injection $f : V(G) \rightarrow \{0, 1, 2, , 2q - 1\}$ such that, when each edge xy is assigned the label or weight $f(x) - f(y)$, the resulting edge labels are $\{1, 3, 5, \cdots, 2q - 1\}$. This definition was introduced by Gnanajothi. Many researchers have studied odd graceful labeling. Seoud and Abdel-Aal determine all connected odd graceful graphs of order less than or equal to 6. For a dynamic survey of various graph labeling problems we refer to Gallian [3]. A graph G is said to be odd harmonious if there exists an injection $f : V(G) \rightarrow \{0, 1, 2, \cdots, 2q - 1\}$ such that the induced function $f : E(G) \rightarrow \{1, 3, \cdots, 2q - 1\}$ defined by $f(uv) = f(u) + f(v)$ is a bijection. Then f is said to be an odd harmonious labeling of G. [8]. A graph G with p vertices and q edges is said to have an even sequential harmonious labeling if there exists a function $f : V(G) \rightarrow \{0, 1, 2, \cdots, 2q\}$ such that the induced map $f : E(G) \rightarrow \{2, 4, 6, \cdots, 2q\}$ defined by

$$f(uv) = f(u) + f(v), \quad \text{if } f(u) + f(v) \text{ is even}$$
$$f(uv) = f(u) + f(v) + 1, \quad \text{if } f(u) + f(v) \text{ is odd,}$$

then the resulting edge labels are distinct. A graph G is said to be an even sequential harmonious graph if it admits an even sequential harmonious labeling.

Consider a Graph kC_n- snake with $k \geq 2$. Let $u_1, u_2, \cdots, u_{k-1}$ be the consecutive cut vertices of G. Let d_i be the distance between u_i and u_{i+1} in G, and the string $(d_1, d_2, \cdots, d_{k-2})$ of integers. Hence, any graph $G = kC_n$ -snake, can be represented by a string. For instance the string (from left to right) of the $5C_4$- snake of figure 4 is 2,2,2,2,2. Gracefulness of the kind of kC_4-snake studied by Gnanajothi have string 2, 2,.. 2. We obtain in theorem

2.4 an even sequential harmonious labeling of the snake $1, 1 \cdots, 1$ if the string o given kC_n-snake is $, \lfloor \frac{n}{2} \rfloor, \cdots, \lfloor \frac{n}{2} \rfloor$, we say that snake kC_n is linear.

This paper can be divided into two sections. Section 1, we show that the graphs kC_4-snake $\odot mk_1$ (the graph obtained by joining m pendant edges to each vertex of kC_4-snake $k, m \geq 1$) for linear and general cases of kC_4-snake for each $k \geq 1$ are even sequential harmonious. We also obtain an even harmonious labeling of kC_n with the sequences string is $1, 1, \cdots, 1$ and when $n \equiv 0 (mod 4)$ in section 2, we show that an even sequential harmonious labeling of the all subdivision of of double triangular snakes finally, we prove that the all Subdivision of $2m \triangle_k$ snake are even sequential harmonious for each $k, m > 1$.

2. Main Results

Theorem 2.1. *The linear graphs kC_4- snake $\odot mk_1$ are even sequential harmonious for $k, m \geq 1$.*

Proof. Consider the linear graph $G = kC_4$-snake, $k \geq 1$ which has the vertices w_i, u_j and v_j where $i = 0, 1, 2, \cdots, k, j = 1, 2, 3, \cdots, k$. In order to get the linear kC_4 - snake $\odot mk_1, k, m \geq 1$, we add m pendant edges w_i^l, u_j^l andv_j^l to each vertex of w_i, u_j, v_j respectively such that $l = 0, 1, 2, \cdots, m$. Now, let G be the linear kC_4 - snake $\odot mk_1, k, m \geq 1$ be described as indicated in figure 1.
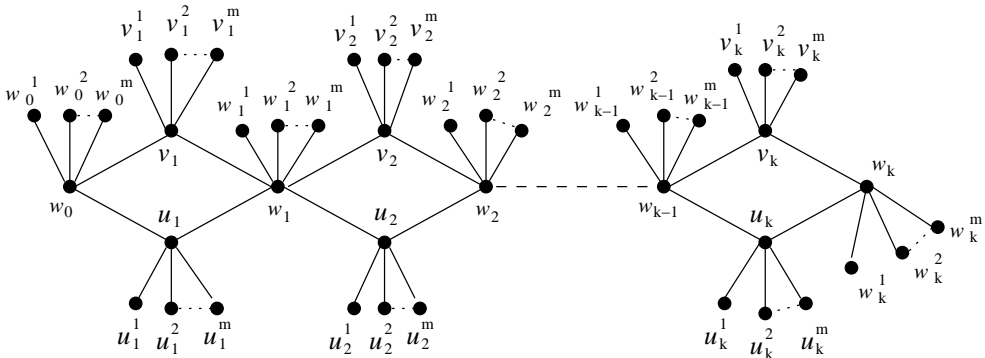

Figure 1

It is clear that the number of edges of the graph G is $q = 4k + m(3k + 1)$. We define the labeling function $f : V(G) \rightarrow \{0, 1, 2, \cdots, 8k + 2m(3k + 1)\}$ as

follows:

$$f(w_i) = 4i; \quad 0 \le i \le k,$$
$$f(v_i) = 4i - 3; \quad 1 \le i \le k,$$
$$f(u_i) = 4i - 1; \quad 1 \le i \le k,$$
$$f(w_i^j) = 2[q - j - (m+2)i] + 1, \quad 0 \le i \le k, \quad 1 \le j \le m,$$
$$f(u_i^j) = 4k(m+2) - 2j - 4(m+1)i + 2m + 2, 1 \le i \le k, \quad 1 \le j \le m,$$
$$f(v_i^j) = 4k(m+2) - 2j - 4(m+1)(i-1), 1 \le i \le k, \quad 1 \le j \le m.$$

Let A,B,C,D,E denote the edge set.

$$A = \{e_i = w_{i-1}v_i, f(e_i) = 8i - 6 : 1 \le i \le k\},$$
$$B = \{e_i = w_{i-1}u_i/e_i = 8i - 4 : 1 \le i \le k\},$$
$$C = \{e_i = v_iw_i/e_i = 8i - 2 : 1 \le i \le k\},$$
$$D = \{e_i =_i^u w_i/e_i = 8i : 1 \le i \le k\}.$$

The remaining odd edge labels from $8k + l$ to $2k + 2m(3k + l)$ are obtained from the following:

$$E = \{e_j = u_j u_j^j / f(e_j) = 4k(m+2) - 2j - 4mi + 2m + 2, \ 1 \le i \le k, 1 \le j \le m\},$$
$$F = \{e_j = v_j v_j^j / f(e_j) = 4k(m+2) - 2j - 4mi + 4m + 2, \ 1 \le i \le k, 1 \le j \le m\},$$
$$G = \{e_j = w_i w_i^j / f(ej) = 2[q - j - mi] + 2, 0 \le i \le k, \ 1 \le j \le m\}.$$

Hence the graph G is an even sequential harmonious.   □

Example 2.2. An even sequential harmonious labeling of the graph $3C_4$-snake $\odot 2k_1$ is shown in figure 2.

Theorem 2.3. *The following graphs are an even sequential harmonious.*
(i) kC_4 - snake for each $k \ge 1$.
(ii) kC_4 - snake $\odot mk_1$ for each $k, m \ge 1$.

Proof. The graph kC_4 - snake can be considered as a bipartite graph (one partite set has black vertices and the other has white vertices) it is possible to embed it, on a square grid as is showed in the figure 3.

Let G be the graph kC_4 - snake $\odot mk_1$ which obtained by joining m- pendant edges to each vertex of kC_4 - snake. Then $| V(G) |= (m+1)(3k+1)$ and $| E(G) = 4k + m(3k+1) |$. Now, we are running the following steps sequentially
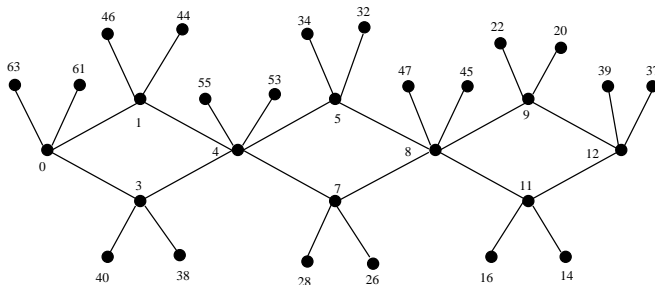
Figure 2: The graph linear $3C_4$ - snake $\odot 2k_1$ with its even sequential harmonious labeling
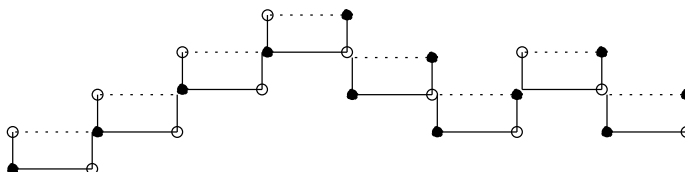

Figure 3

in order to prove the Theorem ;

Step 1. Since kC_4-snake is a bipartite graph so it has one partite set has black vertices and the other has white vertices as is showed in figure 3. Put black vertices in string ordered by diagonals from left to right and inside each diagonal from bottom to top, assign to them from an arithmetic progression of difference 4 which first term is zero, when we move to another diagonal we use an arithmetic progression of difference 2, counting until the last black vertex has been numbered. Similarly, put the White vertices on a string ordered for diagonals from left to right and inside each diagonal from top to bottom, starting with the first diagonal assign numbers from an arithmetic progression of difference 2 which first term is 1, when we move to another diagonal we use an arithmetic progression of differences 2 counting until the last white vertex has been numbered.

Step 2. In this step, We are labeling the vertices of m-pendant edges which contact with the white diagonals, from the right to left and inside each white diagonal from bottom to top, assign to them from an arithmetic progression of

difference 2, which first term is z such that $z = y + 2$ where y is the last vertex labeling of black diagonal. when we move to a new vertex of the white diagonal, the first vertex of m-pendant edges is labeled by an arithmetic progression of difference 4, but the arithmetic progression of difference 2 has been used with the remain $(m - l)$ vertices of m-pendant edges. We move from a vertex to another of the white diagonals until the last white vertex.

Step 3. Finally we are labeling the vertices of m-pendant edges which contact with the black diagonals, from left to right and inside each black diagonal from bottom to top, assign to them from an arithmetic progression of difference (-2), which first term is $(2q - 1)$ where q is the size of G, When to move to a new vertex of the black diagonal, the first vertex of m-pendant edges is labeled by an arithmetic progression of difference (-6) but the arithmetics progression of difference (-2) has been used with the remain $(m - l)$ vertices of m-pendant edges. We move from a vertex to another of the black diagonal until the last black vertex.

Now, we have complete the proof by running the above steps, we mention only the vertices labels and the reader can fulfill the proof as we did in the previous theorem where step 1 give us an even sequential harmonious labeling of the graph kC_4 - snake for each $k \geq 1$, and step 1-step 3 give us an even sequential harmonious labeling of the graph kC_4-snake $\odot mk_1$ for each $k, m \geq 1$ (in general case). □

The following example illustrates the last result.

Example 2.4. An even sequential harmonious labeling of the graph $5C_4$-snake $\odot 2k_1$ (for general case) is shown in Figure 4.

The graphs kC_4-snake when the sequence string is $(1, 1, 1, \cdots, 1)$ then $n \equiv 0 (mod4)$ are studied in the following Theorem:

Theorem 2.5. The graphs kC_{4m}-snake for each $k, m \geq 1$, with string $(1, 1, \cdots, 1)$ are even sequential harmonious.

Proof. Let $G = kC_{4m}$-snake $= (n - 1)C_{4m}$-snake can be described as indicated in Figure 5.

It is clear that $|E(G)| = 4m(n - 1)$. We define the labeling function $f : V(G) \rightarrow \{0, 1, 2, \cdots, 8m(n - 1)\}$ as follows:
$$f(v_i^1) = \begin{cases} 4m(i - 1), \ i = 1, 3, 5, \cdots, n \text{ or } n - 1 \\ 4m(i - 1) + 1, \ i = 2, 4, 6, \cdots, n - 1 \text{ or } n. \end{cases}$$
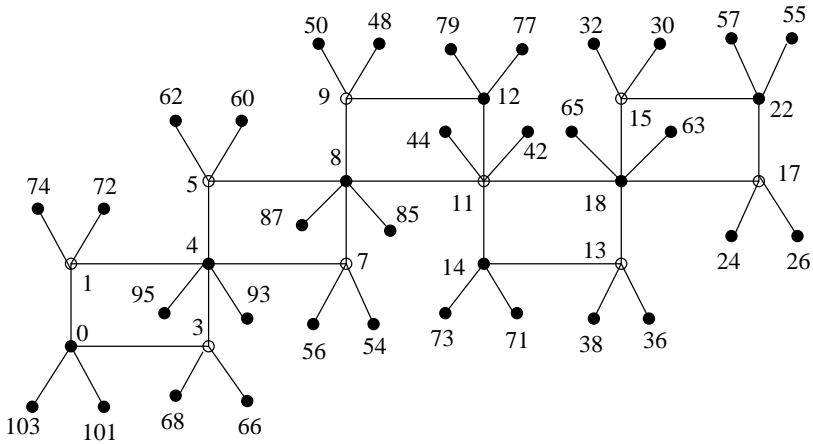
Figure 4: The graph $5C_4$ - snake $\odot 2k_1$ (for general case) with its even sequential harmonious labeling

$f(v_1^j) = j - 1, 1 \leq j \leq 2m.$

For labeling the vertices $v_{i1}^j, 2 \leq i \leq n, 2 \leq j \leq 2m$.
We consider the following two cases:

Case i: If i is odd, $3 \leq i \leq n$, we have the following labeling,for each $2 \leq j \leq 2m$
$f(v_{i1}^j) = f(v_i^1) - j + 1.$

Case ii: If i is even, $2 \leq i \leq n$, we have the following labeling,for each $2 \leq j \leq 2m$:
$$f(v_{i1}^j) = \begin{cases} f(v_i^1) - j + 1, i = 1, 3, 5, \cdots, 2m - 1 \\ f(v_i^1) - j - 1, i = 2, 4, 6, \cdots, 2m \end{cases}.$$

Now we label the remaining vertices $v_{i2}^j, 2 \leq i \leq n - 1, 1 \leq j \leq 2m$ as follows:
$f(v_{i2}^j) = f(v_{i1}^1 + 2(j - 1)), 1 \leq j \leq 2m, 2 \leq i \leq n - 1.$
It follows that f admits an even sequential harmonious labeling for $(n-1)C_4m$ - snake. Hence $(n - 1)C_4m$ - snake for each $n \geq 2, m \geq 1$ with the string $(1, 1, \cdots, 1)$ are even sequential harmonious graphs. $\square$

Example 2.6. Even sequential harmonious labeling of graph $4C_8$ - snake with the string $(1, 1, \cdots, 1)$ is shown in Figure 6.
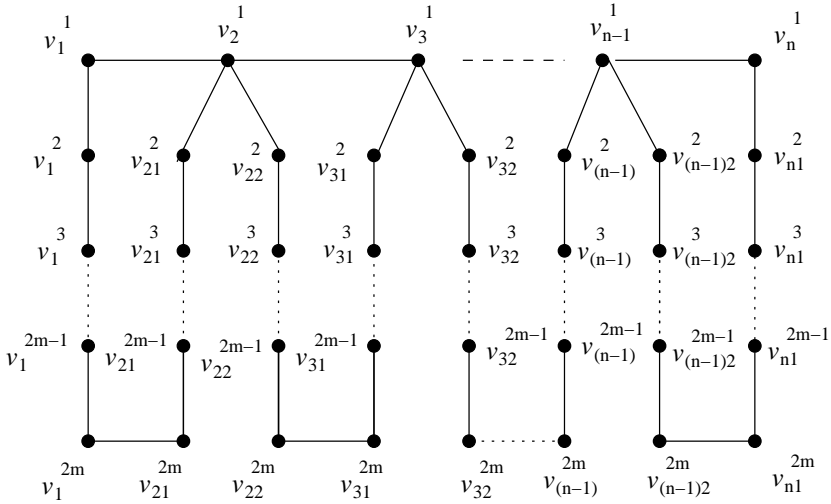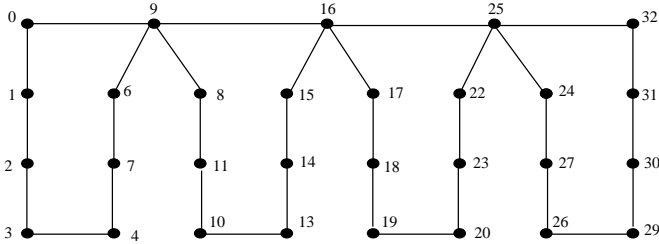
Figure 5

Figure 6: The graph $4C_8$ - snake with its even sequential harmonious labeling

3. Subdivision of Double Triangles Snake

Rosa [8] defined a triangular snake (or $\triangle$ - snake) as a connected graph in which all blocks are triangles and the block cut point graph is a path. Let $\triangle_k$ - snake be a $\triangle$-snake with k blocks while $n\triangle_k$ - snake be a $\triangle$-snake with k blocks and every block has n number of triangles with one common edge.

David moulton [9] proved that triangular snakes with p triangles are graceful if p is congruent to 0 or 1 modulo 4. Xu [10] proved that they are harmonious if and only if p is congruent to 0 ,1 or 3 modulo 4. A double triangular snake is a graph that formed by two triangular snakes have a common path. The

harmonious labeling of double triangle snake introduced by Xi Yue et al [6]. It is known that, the graphs which contain odd cycles are not even sequential harmonious so we used the subdivision notation for odd cycle in order to prove that all subdivision of double triangular snakes are even sequential harmonious.

Theorem 3.1. *All subdivision of double triangular snakes* $(2\triangle_k$- *snake,* $k \geq 1)$ *are even sequential harmonious.*

Proof. Let $G = 2\triangle_k$-snake has q edges and p vertices. The graph G consists of the vertices $(u_1, u_2, \cdots, u_{k+1}), (v_1, v_2, \cdots, v_k),$ $(w_1, w_2, \cdots, w_k)$. Therefore we get the subdivision of double triangular snakes $S(G)$ by subdividing every edge of double triangular $2\triangle_k$-snake exactly once. Let y_i be the newly added vertex between u_i and u_{i+1} while w_{i1} and w_{i2} are newly added vertices between $w_i u_i$ and $w_i u_{i+1}$ respectively where $1 \leq i \leq k$. Finally v_{i1} and v_{i2} are newly added vertices between $v_i u_i$ and $v_i u_{i+1}$ respectively, such that $1 \leq i \leq k$. Let the graph $S(G)$ as described as indicated in Figure 7.
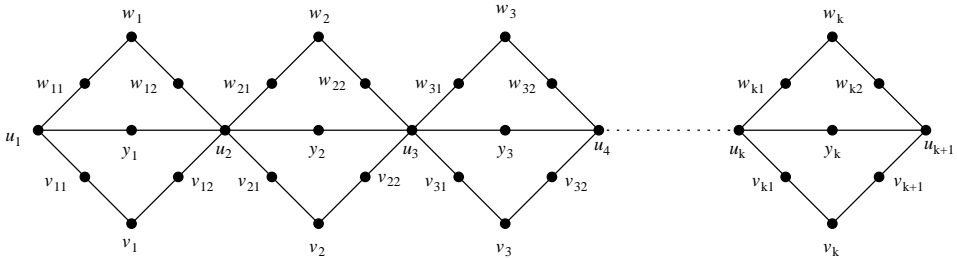

Figure 7: The subdivision of double triangular snakes $(2\triangle_k$-snake)

It is clear that the number of edges of the graph $S(G)$ is $10k$. We define the labeling function : $f : V(S(G)) \rightarrow \{0, 1, 2, \cdots, 20k\}$ as follows:
$f(u_i) = 6(i-1), 1 \leq i \leq k+1 = n,$
$f(y_i) = 14i - 11, 1 \leq i \leq k = n - 1,$
$f(w_i) = 6i + 4, 1 \leq i \leq k = n - 1,$
$f(v_i) = 6i - 4, 1 \leq i \leq k = n - 1,$
$f(w_{ij}) = 14i + 8j - 21, 1 \leq i \leq k = n - 1, j = 1, 2,$
$f(v_{ij}) = 14i + 6j - 15, 1 \leq i \leq k = n - 1, j = 1, 2.$
The edge labels will be as follows:
• The vertices u_i and $v_{i1}, 1 \leq i \leq k$, induce the edge labels
$\{20i - 18, 1 \leq i \leq k\} = \{2, 22, \cdots, 20k - 18\}.$
• The vertices u_i and $y_i, 1 \leq i \leq k$, induce the edge labels

$\{20i - 16, 1 \leq i \leq k\} = \{4, 24, \cdots, 20k - 16\}$.
• The vertices u_i and $v_{i1}, 1 \leq i \leq k$, induce the edge labels
$\{20i - 14, 1 \leq i \leq k\} = \{6, 26, \cdots, 20k - 14\}$.
• The vertices v_{i1} and $v_i, 1 \leq i \leq k$, induce the edge labels
$\{20i - 12, 1 \leq i \leq k\} = \{8, 28, \cdots, 20k - 12\}$.
• The vertices y_i and $u_{i+1}, 1 \leq i \leq k$, induce the edge labels
$\{20i - 10, 1 \leq i \leq k\} = \{10, 30, \cdots, 20k - 10\}$.
• The vertices w_{i1} and $w_i, 1 \leq i \leq k$, induce the edge labels
$\{20i - 8, 1 \leq i \leq k\} = \{12, 32, \cdots, 20k - 8\}$.
• The vertices v_i and $v_{i2}), 1 \leq i \leq k$, induce the edge labels
$\{20i - 6, 1 \leq i \leq k\} = \{14, 34, \cdots, 20k - 6\}$.
• The vertices w_{i2} and $u_{i+1}, 1 \leq i \leq k$, induce the edge labels
$\{20i - 4, 1 \leq i \leq k\} = \{16, 36, \cdots, 20k - 4\}$.
• The vertices v_{i2} and $u_{i+1}, 1 \leq i \leq k$, induce the edge labels
$\{20i - 2, 1 \leq i \leq k\} = \{18, 38, \cdots, 20k - 2\}$.
• The vertices w_i and $w_{i2}, 1 \leq i \leq k$, induce the edge labels
$\{20i, 1 \leq i \leq k\} = \{20, 40, \cdots, 20k\}$.

So we obtain all the edge labels $\{2, 4, 6, \cdots, 20k\}$.
Hence the subdivision of double triangular snakes ($2\triangle_k$-snake, $k \geq 1$) are even sequential harmonious. □

Theorem 3.2. *All subdivision of $2m\triangle_k$-snake, $m, k \geq 1$ are even sequential harmonious.*

Proof. Let $G = 2m\triangle_k$-snake has q edges and p vertices. The graph G consists of the vertices $(u_1, u_2, \cdots, u_{k+1}), (v_i^1, v_i^2, \cdots, v_i^m)$, $(w_i^1, w_i^2, \cdots, w_i^m), 1 \leq i \leq k$. Therefore we get the generalised subdivision of double triangular snakes $S(G)$ by subdividing every edge of double triangular $2m\triangle_k$-snake exactly once. Let y_i be the newly added vertex between u_i and u_{i+1} while w_{i1}^j and w_{i2}^j are newly added vertices between $u_i w_i^j$ and $w_i^j u_{i+1}$ respectively where $1 \leq i \leq k$. Finally v_{i1}^j and v_{i2}^j are newly added vertices between $u_i v_i^j$ and $v_i^j u_{i+1}$ respectively, such that $1 \leq i \leq k, 1 \leq j \leq m$ .Let the graph $S(G)$ as described as indicated in Figure 8.

We define the labeling function $f : V(S(G)) \rightarrow \{0, 1, 2, \cdots, 2k(8m + 2)\}$ as follows: $f(u_i) = (4m + 2)(i - 1), 1 \leq i \leq k + 1 = n$,
$f(y_i) = (12m + 2)i - 10m - 1, 1 \leq i \leq k$,
$f(v_i^1) = (4m + 2)(i - 1) + 4l - 2, 1 \leq i \leq k, 1 \leq l \leq m$,
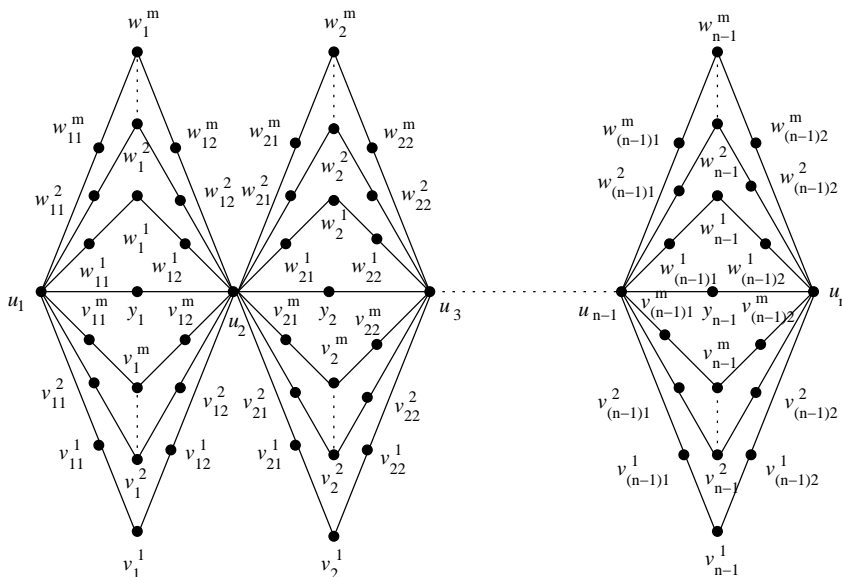$f(w_i^j) = (4m + 6) + (4m + 2)(i - 1) + 4(j - 1), 1 \leq i \leq k, 1 \leq j \leq m$

Figure 8: The subdivision of $2m\triangle_k$- snake

$f(w_{ij}^j) = (2m-1) + (12m+2)(i-1) + (4m+2)(l-1) - 2(j-1), 1 \leq i \leq k, 1 \leq l \leq 2, 1 \leq j \leq m$
$f(v_{ij}^j) = (4m+1) + (12m+2)(i-1) + (6m+2)(l-1), 1 \leq i \leq k, 1 \leq l \leq 2.$

In a view of the above defined labeling pattern f is even sequential harmonious for the graph $S(G)$. Hence S($2m\triangle_k$-snake) is even sequential harmonious for all $m \geq 1, k \geq 1$. $\square$

Illustration 3.3. *An even sequential harmonious labeling of the graph subdivision of $6\triangle_3$-snake is shown in Figure 9.*

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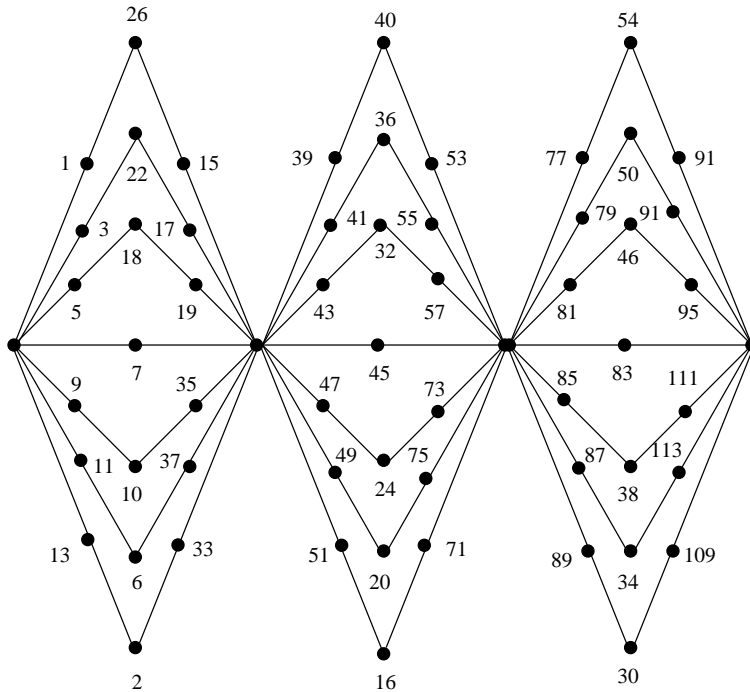
Figure 9: The graph subdivision of $6\triangle_3$-snake with its even sequential harmonious labeling

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