SOME MULTIPLICATIVE PRODUCT OF n -DIMENSIONAL DISTRIBUTIONS. PART II

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Abstract: The method of approximants delta sequences, the convolution and the change of variables is used to obtain multiplicative products of n -dimensional distributions.

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1. Introduction

This the second article in a series of two devoted to product of distributions obtained through the use of delta sequences and thus a change of variables. This way of work allows that products of one-dimensional distributions are generalized.

In order to obtain product of n -dimensional distributions we will use an uni-dimensional delta sequences in $\mathbb{R}$ and then by changing variables will be obtained the desired generalizations. The method of the delta sequences that have been actively used by B. Fisher, essentially involves a sequence of functions starting from a fixed function g belonging to the space D of infinitely differentiable functions with compact support as follows. If S and T are distributions, their multiplicative product is defined by the formula

$$S.T = \lim_{n \rightarrow \infty} \{S * g_n(x)\} \{T * g_n(x)\}$$

if the limit exists for every mollifier $g_n(x)$. The symbol $*$ denotes, as usual, convolution. The functions so defined form a sequence of functions in $\mathcal{D}$ which converge to the Dirac delta. By a mollifier we mean a sequence $g_n(x) = ng(nx)$, where the function has the properties

1. $g(x) \geq 0$
2. $g(x) \in C_0^\infty$
3. $\int_{-\infty}^{\infty} g(x) dx = 1$
4. $g(x) = g(-x)$
5. $\text{sup } g(x) = [-1, 1]$
6. $g(x)$ is increasing for $-1 \leq x \leq 0$ and decreasing for $0 \leq x \leq 1$.

In this definition we follow Gonzalez Dominguez (cf. [1]) who entered a slight variant of the definition due to Mikusinski. This method and other analogous also shown in the review article by Li (cf. [3]). In order to give a sense to all products evaluated in the paragraphs II, or to regularize them, we deal with multidimensional generalizations obtained by means of the following change of variables. Let ϕ_s be a distribution of one variable s and let $u(x) \in C^\infty(\mathbb{R}^n)$ be such that the $n - 1$ -dimensional manifold $u(x_1, x_2, \dots, x_n) = 0$ has no critical point; $\phi_n(x)$ denotes the distribution defined on $\mathbb{R}^n$ by the formula (called the Leray formula)

$$\int_{\mathbb{R}^n} f(x) \phi_u(x) dx_1 \dots dx_n = \int_{-\infty}^{\infty} \phi_s ds \int_{u(x)=s} f(x) \omega_u(x, dx)$$

here ω_u is an $n - 1$ -dimensional form on u defined as follows

$$du \wedge d\omega = dx_1 \wedge dx_2 \wedge \dots \wedge dx_n$$

the manifold $u(x) = s$ has the orientation such that $\omega_u(x, dx) > 0$.

By this way we extend some uni-dimensional distributional multiplicative products to certain kinds of n -dimensional distributions called causal and anti-causal distributions.

Definition 1. Let $x = (x_1, x_2, \dots, x_n)$ be a point of the n -dimensional euclidean space $\mathbb{R}^n$. Let $P(x)$ be a non degenerate quadratic form in n variables of the form

$$P = P(x) = x_1^2 + \dots + x_p^2 - x_{p+1}^2 - \dots - x_{p+q}^2 \quad (1.1)$$

where $n = p + q$.

The distributions $(P \pm i0)^\lambda$ are defined by

$$(P \pm i0)^\lambda = \lim_{\epsilon \rightarrow 0} (P \pm i\epsilon|x|^2)^\lambda \quad (1.2)$$

where $\epsilon > 0$, $|x|^2 = x_1^2 + \dots + x_n^2$; $\lambda \in \mathbb{C}$.

The distributions $(m^2 + P \pm i0)^\lambda$ are defined in an analogue manner as the distributions $(P \pm i0)^\lambda$. Let us put (cf[4], p 289)

$$(m^2 + P \pm i0)^\lambda = \lim_{\epsilon \rightarrow 0} (m^2 + P \pm i\epsilon|x|^2)^\lambda, \quad (1.3)$$

where ϵ is an arbitrary positive number, and m is a positive real number.

It is useful to state an equivalent definition of the distributions $(m^2 + P \pm i0)^\lambda$. In this definition appear the distributions

$$(m^2 + P)_+^\lambda = \begin{cases} (m^2 + P)^\lambda, & \text{if } (m^2 + P) \geq 0; \\ 0, & \text{if } (m^2 + P) < 0. \end{cases} \quad (1.4)$$

$$(m^2 + P)_-^\lambda = \begin{cases} 0, & \text{if } (m^2 + P) > 0; \\ (m^2 + P)^\lambda, & \text{if } (m^2 + P) \leq 0. \end{cases} \quad (1.5)$$

We can prove, without difficulty that the following formula is valid ([5], p. 566)

$$(m^2 + P \pm i0)^\lambda = (m^2 + P)_+^\lambda + e^{\pm i\pi\lambda} (m^2 + P)_-^\lambda \quad (1.6)$$

From this formula we conclude immediately that

$$(m^2 + P \pm i0)^\lambda = (m^2 + P - i0)^\lambda = (m^2 + P)^\lambda, \quad (1.7)$$

when $\lambda = k = \text{positive integer}$.

The distribution $(m^2 + P \pm i0)^\lambda$ are entire distributional function of λ . This is the principal difference between the distributions, formally analogue $(P \pm i0)^\lambda$ which have poles at the point $\lambda = -\frac{n}{2} - k$, $k = 0, 1, \dots$. It can be proved (cf [5] p. 573, formula (2.14) and p. 575 formula (3.5)) that

$$(m^2 + P \pm i0)^{-k} = Pf(m^2 + P)^{-k} \mp i\pi \frac{(-1)^{k-1}}{(k-1)!} \delta^{(k-1)}(m^2 + P),$$

$$k = 1, 2, \dots \quad (1.8)$$

The following formula is true for every $\lambda, \mu \in \mathbb{C}$, and $m^2 \neq 0$,

$$(m^2 + P \pm i0)^\lambda (m^2 + P \pm i0)^\mu = (m^2 + P \pm i0)^{\lambda+\mu} \quad (1.9)$$

(cf. [6], p. 23, formula (1.3.16)), and

$$[\operatorname{sgn}(m^2 + P)]|m^2 + P|^\lambda = (m^2 + P)_+^\lambda - (m^2 + P)_-^\lambda, \quad (1.10)$$

and

$$|m^2 + P|^\lambda = (m^2 + P)_+^\lambda - (m^2 + P)_-^\lambda, \quad (1.11)$$

and

$$\begin{aligned} \{(m^2 + P \pm i0)^\lambda \ln^r(m^2 + P + i0)\} \cdot \{(m^2 + P \pm i0)^\mu \ln^s(m^2 + P + i0)\} \\ = (m^2 + P \pm i0)^{\lambda+\mu} \ln^{r+s}(m^2 + P + i0), \end{aligned} \quad (1.12)$$

for all $\lambda, \mu, \lambda + \mu \in \mathbb{C}$, where

$$(m^2 + P \pm i0)^\lambda \ln^r(m^2 + P + i0) = \frac{\partial^r}{\partial \lambda^r} (m^2 + P + i0)^\lambda, \quad (1.13)$$

where

$$\begin{aligned} \ln(m^2 + P + i0) &= \lim_{\epsilon \rightarrow 0} (m^2 + P + i\epsilon|x|^2) \\ &= \ln|m^2 + P| + i\pi H(-(m^2 + P)) \end{aligned} \quad (1.14)$$

and

$$H(m^2 + P) = \begin{cases} 1, & \text{for } m^2 + P > 0; \\ 0, & \text{for } m^2 + P < 0. \end{cases} \quad (1.15)$$

In particular we have, taking into account (1.7) and (1.14) that

$$\begin{aligned} (m^2 + P + i0)^r \ln(m^2 + P + i0) \\ = (m^2 + P)^r \ln|m^2 + P| + (-1)^r i\pi (m^2 + P)_-^r. \end{aligned} \quad (1.16)$$

and

$$\begin{aligned} (m^2 + P + i0)^r \ln(m^2 + P + i0) &= (m^2 + P)^r \ln|m^2 + P| \\ &+ 2(-1)^r i\pi (m^2 + P)_-^r \ln(m^2 + P)_- - (-1)^r \pi^2 (m^2 + P)_-^r. \end{aligned} \quad (1.17)$$

From (1.17) we have

$$(m^2 + P + i0)^{-r+\frac{1}{2}} = (m^2 + P)_+^{-r+\frac{1}{2}} + (-1)^r i (m^2 + P)_-^{-r+\frac{1}{2}}. \quad (1.18)$$

Then, we obtain

$$\begin{aligned} (m^2 + P + i0)^{-r+\frac{1}{2}} \ln(m^2 + P + i0) \\ = (m^2 + P)_+^{-r+\frac{1}{2}} \ln(m^2 + P)_+ - (-1)^r \pi (m^2 + P)_-^{-r+\frac{1}{2}} \\ + (-1)^r i (m^2 + P)_-^{-r+\frac{1}{2}} \ln(m^2 + P)_- \end{aligned} \quad (1.19)$$

and

$$\begin{aligned} (m^2 + P + i0)^{-r+\frac{1}{2}} \ln(m^2 + P + i0) &= (m^2 + P)_+^{-r+\frac{1}{2}} \ln(m^2 + P)_+ \\ &- (-1)^r \pi (m^2 + P)_-^{-r+\frac{1}{2}} - 2(-1)^r \pi (m^2 + P)_-^{-r+\frac{1}{2}} \ln(m^2 + P)_- \\ &+ (-1)^r i (m^2 + P)_-^{-r+\frac{1}{2}} \ln^2(m^2 + P). \end{aligned} \quad (1.20)$$

By considering the formula ([5], p. 577, form (4.9))

$$(m^2 + P + i0)^{-r} = (m^2 + P)^{-r} - \frac{(-1)^{r-1} i \pi}{(r-1)!} \delta^{(r-1)}(m^2 + P) \quad (1.21)$$

we have

$$\begin{aligned} (m^2 + P + i0)^{-r} \ln(m^2 + P + i0) &= (-1)^r i \pi (m^2 + P)_-^{-r} \\ &- \frac{(-1)^r \pi^2 \delta^{(r-1)}(m^2 + P)}{2(r-1)!} + (m^2 + P)^{-r} \ln |m^2 + P| \end{aligned} \quad (1.22)$$

and

$$\begin{aligned} (m^2 + P + i0)^{-r} \ln(m^2 + P + i0) &= \frac{(-1)^{-r-1} i \pi^3 \delta^{(r-1)}(m^2 + P)}{3(r-1)!} \\ &- (-1)^r \pi^2 (m^2 + P)_-^{-r} + 2i \pi (-1)^r (m^2 + P)_-^{-r} \ln(m^2 + P)_- \\ &+ (m^2 + P)^{-r} \ln^2 |m^2 + P|. \end{aligned} \quad (1.23)$$

2. Main Results

Lemma 1. For r , and $s = 1, 2, 3, \dots$ we have

$$\begin{aligned}
 (m^2 + P)^{-r} [(m^2 + P)^{-s} \ln |m^2 + P|] - \frac{(-1)^s \pi^2}{2(s-1)!} (m^2 + P)^{-r} \delta^{(s-1)}(m^2 + P) - \\
 - \frac{(-1)^{r+s} \pi^2}{(r-1)!} (m^2 + P)^{-s} \delta^{(r-1)}(m^2 + P) = \\
 (m^2 + P)^{-r-s} \ln |m^2 + P| - \frac{(-1)^{r+s} \pi^2 \delta^{(r+s-1)}}{2(r+s-1)!} (m^2 + P) \quad (2.1)
 \end{aligned}$$

and

$$\begin{aligned}
 (m^2 + P)^{-r} [(m^2 + P)^{-s} \ln |m^2 + P|] + (-1)^s \pi^2 (m^2 + P)^{-r} \delta^{(s-1)}(m^2 + P) + \\
 + \frac{(-1)^r \pi^2}{(r-1)!} (m^2 + P)^{-s} \delta^{(r-1)}(m^2 + P) = \\
 (m^2 + P)^{-r-s} \ln |m^2 + P| - \frac{(-1)^{r+s} \pi^2 \delta^{(r+s-1)}}{2(r+s+1)!} (m^2 + P) \quad (2.2)
 \end{aligned}$$

Proof. From (1.12) we have

$$\begin{aligned}
 (m^2 + P + i0)^{-r} \{ (m^2 + P + i0)^{-s} \ln(m^2 + P + i0) \} = \\
 (m^2 + P + i0)^{-r-s} \ln(m^2 + P + i0) \quad (2.3)
 \end{aligned}$$

valid for $r, s = 1, 2, \dots$

Taking into account the following formulas

$$(m^2 + P + i0)^{-r} = (m^2 + P)^{-r} - \frac{(-1)^{r-1} i \pi}{(r-1)!} \delta^{(r-1)}(m^2 + P), \quad (2.4)$$

$r = 1, 2, \dots$ and

$$\begin{aligned}
 (m^2 + P)^{-r} \ln(m^2 + P + i0) = (-1)^r i \pi (m^2 + P)^{-r} - \\
 - \frac{(-1)^r \pi \delta^{(r-1)}(m^2 + P)}{2(r-1)!} + (m^2 + P)^{-r} \ln |m^2 + P|, \quad (2.5)
 \end{aligned}$$

$r = 1, 2, \dots$, we can write (2.3) in the equivalently form

$$\{ (m^2 + P)^{-r} + \frac{(-1)^r}{(r-1)!} i \pi \delta^{(r-1)}(m^2 + P) \}.$$

$$\begin{aligned} & \{(-1)^s i\pi(m^2 + P)_-^{-s} - \frac{(-1)^s \pi^2}{2(s-1)!} \delta^{(s-1)}(m^2 + P) + (m^2 + P)^{-s} \ln |m^2 + P|\} = \\ & (-1)^{r+s} i\pi(m^2 + P)_-^{-r-s} - \frac{(-1)^{r+s} \pi^2}{2(r+s+1)!} \delta^{(r+s+1)}(m^2 + P) + (m^2 + P)^{-r-s} \ln |m^2 + P| \end{aligned} \quad (2.6)$$

valid for $r, s = 1, 2, \dots$

Expanding and equating the real parts of (2.6) we have

$$\begin{aligned} & (m^2 + P)^{-r} [(m^2 + P)^{-s} \ln |m^2 + P|] - \frac{(-1)^s \pi^2}{2(s-1)!} (m^2 + P)^{-r} \delta^{(s-1)}(m^2 + P) - \\ & - \frac{(-1)^{r+s} \pi^2}{(r-1)!} (m^2 + P)_-^{-s} \delta^{(r-1)}(m^2 + P) = \\ & (m^2 + P)^{-r-s} \ln |m^2 + P| - \frac{(-1)^{r+s} \pi^2 \delta^{(r+s+1)}}{2(r+s-1)!} (m^2 + P). \end{aligned}$$

Replacing $(m^2 + P)$ by $-(m^2 + P)$ in the last equation we have

$$\begin{aligned} & (m^2 + P)^{-r} [(m^2 + P)^{-s} \ln |m^2 + P|] + \frac{(-1)^s \pi^2}{2(s-1)!} (m^2 + P)^{-r} \delta^{(s-1)}(m^2 + P) + \\ & + \frac{(-1)^{r+s} \pi^2}{(r-1)!} (m^2 + P)_+^{-s} \delta^{(r-1)}(m^2 + P) = \\ & (m^2 + P)^{-r-s} \ln |m^2 + P| - \frac{(-1)^{r+s} \pi^2 \delta^{(r+s+1)}}{2(r+s-1)!} (m^2 + P). \quad \square \end{aligned}$$

Lemma 2. For $r, s = 1, 2, \dots$, we have

$$\begin{aligned} & 2(m^2 + P)^{-r} [(m^2 + P)^{-s} \ln |m^2 + P|] + \frac{(-1)^r \pi^2}{(r-1)!} \cdot \\ & [sgn(m^2 + P)(m^2 + P)^{-s}] \delta^{(r-1)}(m^2 + P) = \\ & 2(m^2 + P)^{-r-s} \ln |m^2 + P| \end{aligned} \quad (2.7)$$

and

$$\begin{aligned} & (-1)^s (m^2 + P)^{-r} (m^2 + P)_-^{-s} - \frac{(-1)^{r+s} \pi^2}{2(r-1)!(s-1)!} \delta^{(r-1)}(m^2 + P) \cdot \\ & \delta^{(s-1)}(m^2 + P) + \frac{(-1)^r}{(r-1)!} [(m^2 + P)^{-s} \ln |m^2 + P|] \delta^{(r-1)}(m^2 + P) = \\ & (-1)^{r+s} (m^2 + P)_-^{-r+s}. \end{aligned} \quad (2.8)$$

Proof. Adding the equations (2.1) and (2.2) we have

$$2(m^2 + P)^{-r} [(m^2 + P)^{-s} \ln |m^2 + P|] + \frac{(-1)^r \pi^2}{(r-1)!}.$$

$$[sgn(m^2 + P)(m^2 + P)^{-s}] \delta^{(r-1)}(m^2 + P) = \\ 2(m^2 + P)^{-r-s} \ln |m^2 + P|$$

Expanding and equating the imaginary parts of the equation (2.6) we obtain

$$(-1)^s (m^2 + P)^{-r} (m^2 + P)^{-s} - \frac{(-1)^{r+s} \pi^2}{2(r-1)!(s-1)!} \delta^{(r-1)}(m^2 + P).$$

$$\delta^{(s-1)}(m^2 + P) + \frac{(-1)^r}{(r-1)!} [(m^2 + P)^{-s} \ln |m^2 + P|] \delta^{(r-1)}(m^2 + P) = \\ (-1)^{r+s} (m^2 + P)^{-r+s}. \quad \square$$

Lemma 3. For $r, s = 1, 2, \dots$ it result

$$(m^2 + P)^{-r} (m^2 + P)^{-s}_+ - \frac{(-1)^{r+s} \pi^2}{2(r-1)!(s-1)!} \delta^{(r-1)}(m^2 + P).$$

$$\delta^{(r-1)}(m^2 + P) + \frac{(-1)^r}{(r-1)!} [(m^2 + P)^{-s} \ln |m^2 + P|] \delta^{(r-1)}(m^2 + P) = \\ (m^2 + P)^{-r-s}_+ \quad (2.9)$$

and

$$(m^2 + P)^{-r} [sgn(m^2 + P)(m^2 + P)^{-s}] - \frac{2(-1)^r}{(r-1)!} \cdot \\ [(m^2 + P)^{-s} \ln |m^2 + P|] \delta^{(r-1)}(m^2 + P) = \\ sgn(m^2 + P)(m^2 + P)^{-r-s} \quad (2.10)$$

Proof. Replacing $(m^2 + P)$ by $-(m^2 + P)$ in the equation (2.8) we have (2.9); and subtracting (2.9) from (2.8) we have (2.10). $\square$

Lemma 4. For $r, s = 1, 2, \dots$, we have

$$(m^2 + P)^{-r} [(m^2 + P)^s \ln^2 |m^2 + P|] - (-1)^s \pi^2 (m^2 + P)^{-r} (m^2 + P)^s_- - \\ - \frac{2(-1)^{r+s}}{(r-1)!} \pi^2 [(m^2 + P)^s_- \ln(m^2 + P)_-] \delta^{(r-1)}(m^2 + P) =$$

$$(m^2 + P)^{-r+s} \ln^2 |m^2 + P| - (-1)^{r+s} \pi^2 (m^2 + P)_-^{-r+s} \quad (2.11)$$

and

$$\begin{aligned} & (m^2 + P)^{-r} [(m^2 + P)^s \ln^2 |m^2 + P|] - \pi^2 (m^2 + P)^{-r} (m^2 + P)_+^s + \\ & + \frac{2(-1)^r}{(r-1)!} \pi^2 [(m^2 + P)_+^s \ln(m^2 + P)_+] \delta^{(r-1)}(m^2 + P) = \\ & (m^2 + P)^{-r+s} \ln^2 |m^2 + P| - \pi^2 (m^2 + P)_+^{-r+s} \end{aligned} \quad (2.12)$$

Proof. Taking into account the equation (1.12) we have

$$\begin{aligned} & (m^2 + P + i0)^{-r} \{(m^2 + P + i0)^s \ln^2(m^2 + P + i0)\} = \\ & (m^2 + P + i0)^{-r+s} \ln^2(m^2 + P + i0) \end{aligned} \quad (2.13)$$

valid for $r, s = 1, 2, \dots$

Equivalently, from (1.17) and (1.22) we can write

$$\begin{aligned} & \{(m^2 + P)^{-r} + \frac{(-1)^r i \pi}{(r-1)!} \delta^{(r-1)}(m^2 + P)\} \cdot \\ & \{(m^2 + P)^s \ln^2 |m^2 + P| - (-1)^s \pi^2 (m^2 + P)_-^s + 2(-1)^s i \pi (m^2 + P)_-^s \ln(m^2 + P)_- \} = \\ & (m^2 + P)^{-r+s} \ln^2 |m^2 + P| - (-1)^{r+s} \pi^2 (m^2 + P)_-^{r+s} + \\ & 2(-1)^{r+s} i \pi (m^2 + P)_-^{r+s} \ln(m^2 + P)_- \\ & - \frac{(-1)^{r+s} i \pi^3}{3(r-s-1)!} \delta^{(r-s-1)}(m^2 + P) \end{aligned} \quad (2.14)$$

valid for $r, s = 1, 2, \dots$

Expanding the equation (2.14) and equating the real parts, it results (2.11), and replacing $(m^2 + P)$ for $-(m^2 + P)$ we have (2.12). $\square$

Lemma 5. For $r, s = 1, 2, \dots$:

$$\begin{aligned} & (m^2 + P)^{-r} [\operatorname{sgn}(m^2 + P)(m^2 + P)^s] \\ & - \frac{2(-1)^r}{(r-1)!} \left[(m^2 + P)^s \ln |m^2 + P| \delta^{(r-1)}(m^2 + P) \right] \\ & = \operatorname{sgn}(m^2 + P)(m^2 + P)^{-r+s} \end{aligned} \quad (2.15)$$

and

$$(m^2 + P)^{-r} [(m^2 + P)^r \ln^2 |m^2 + P|]$$

$$\begin{aligned}
& + \frac{(-1)^r \pi^2}{(r-1)!} [\operatorname{sgn}(m^2 + P) \ln |m^2 + P|] \delta^{(r-1)}(m^2 + P) \\
& = (m^2 + P)^{-r+s} \ln^2 |m^2 + P|. \quad (2.16)
\end{aligned}$$

Proof. Subtracting (2.12) from (2.11) we obtain (2.15). Adding the equation (2.12) to (2.11) results (2.16). $\square$

Lemma 6. For $r, s = 1, 2, \dots$, it results

$$\begin{aligned}
& 2(-1)^s (m^2 + P)^{-r} [(m^2 + P)_-^s \ln(m^2 + P)_-] + \\
& \frac{(-1)^r}{(r-1)!} [(m^2 + P)^s \ln^2 |m^2 + P|] \cdot \delta^{(r-1)}(m^2 + P) - \\
& - \frac{(-1)^{r+s} \pi^2}{(r-1)!} (m^2 + P)_-^s \delta^{(r-1)}(m^2 + P) = \\
& 2(-1)^{r+s} (m^2 + P)_-^{-r+s} \ln(m^2 + P)_- - \frac{(-1)^{r+s} \pi^2}{3(r-s-1)!} \delta^{(r-s-1)}(m^2 + P) \quad (2.17)
\end{aligned}$$

and

$$\begin{aligned}
& 2(m^2 + P)^{-r} [(m^2 + P)_+^s \ln(m^2 + P)_+] \\
& - \frac{(-1)^r}{(r-1)!} [(m^2 + P)^s \ln^2 |m^2 + P|] \cdot \delta^{(r-1)}(m^2 + P) \\
& + \frac{(-1)^r \pi^2}{(r-1)!} (m^2 + P)_+^s (m^2 + P) \\
& = 2(m^2 + P)_+^{-r+s} \ln(m^2 + P)_+ + \frac{(-1)^{r+s} \pi^2}{3(r-s-1)!} \delta^{(r-s-1)}(m^2 + P). \quad (2.18)
\end{aligned}$$

Proof. Equating the imaginary parts of (2.14) we have (2.17), and replacing $(m^2 + P)$ for $-(m^2 + P)$ in (2.17) we obtain (2.18). $\square$

Lemma 7. For $r, s = 1, 2, \dots$, we have

$$\begin{aligned}
& 2(m^2 + P)^{-r} [(m^2 + P)^s \ln |m^2 + P|] \\
& + \frac{(-1)^r \pi^2}{(r-1)!} [\operatorname{sgn}(m^2 + P)(m^2 + P)^s] \delta^{(r-1)}(m^2 + P) \\
& = 2(m^2 + P)^{-r+s} \ln |m^2 + P| \quad (2.19)
\end{aligned}$$

and

$$\begin{aligned}
& (m^2 + P)^{-r} [sgn(m^2 + P)(m^2 + P)^s \ln |m^2 + P|] \\
& - \frac{(-1)^r}{(r-1)!} [(m^2 + P)^s \ln^2 |m^2 + P|] \delta^{(r-1)}(m^2 + P) \\
& = \frac{(-1)^{r+s} \pi^2}{6(r-s-1)!} \delta^{(r-s-1)}(m^2 + P) \\
& + sgn(m^2 + P)(m^2 + P)^{-r+s} \ln |m^2 + P|. \quad (2.20)
\end{aligned}$$

Proof. Adding the equation (2.17) to the equation (2.18) result (2.19) and subtracting (2.17) to the equation (2.18) we obtain (2.20). $\square$

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