**SOLVING THE DIRICHLET PROBLEM WITH
DISCONTINUOUS COEFFICIENTS IN BOUNDED MULTIPLY
CONNECTED REGIONS USING A BOUNDARY INTEGRAL
EQUATION WITH THE GENERALIZED NEUMANN KERNEL**

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Abstract: This paper presents a numerical method for solving the Dirichlet problem with discontinuous coefficients in bounded multiply connected regions. The method is based on reducing the problem of solving the Dirichlet problem with discontinuous coefficients to a problem of solving Dirichlet problem with continuous coefficients. The Dirichlet problem with continuous coefficients is then solved using a combination of a uniquely solvable boundary integral equation with generalized Neumann kernel and the Fast Multipole Method. Numerical results and comparison are given to illustrate the efficiency of the suggested method.

Received: March 5, 2014

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url: www.acadpubl.eu

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AMS Subject Classification: 30E25, 31B10

Key Words: Dirichlet problem with discontinuous coefficients, boundary integral equations, generalized Neumann kernel, simply connected region, multiply connected region

1. Introduction

The boundary integral equation method is a classical method for solving the Dirichlet and Neumann boundary value problems. The classical boundary integral method for the Dirichlet problem and the Neumann problem with continuous coefficients are in the form of second kind Fredholm integral equations with the Neumann kernel. These integral equations are derived by representing the solutions as the potential of double layer for the Dirichlet problem and single layer for the Neumann problem [1].

Recently, the interplay of the Riemann-Hilbert problem and integral equation with the generalized Neumann kernel has been investigated in [18, 19]. It has been shown that the problem of conformal mapping, Dirichlet problem, Neumann problem, and mixed Dirichlet-Neumann problem can all be treated as Riemann-Hilbert problems [12, 15, 16, 19, 20]. Hence they can be solved efficiently using integral equations with the generalized Neumann kernel.

Dirichlet problem with discontinuous coefficients in rectangular regions have been solved by separation of variable method, Fourier series and Green function [6, 7]. Also, Dirichlet problem with discontinuous coefficients in smooth boundary regions have been solved by conformal mapping and Poisson's integral formula [2, 4, 8, 9].

In this paper, we solve Laplace's equation on bounded simply connected region with Dirichlet condition with discontinuous coefficients via an integral equation with the generalized Neumann kernel. The results of this paper extend the results presented in [15] where the Dirichlet problem involved continuous coefficients only.

2. Dirichlet Problem with Discontinuous Coefficients in Bounded Simply Connected Region

2.1. Preliminaries

Let G be a bounded simply connected region in the extended complex plane $\overline{\mathbb{C}}$ with boundary Γ in counterclockwise direction (see Figure 1). Suppose that α

is a fixed point in G .

Figure 1. A bounded simply connected region in the extended complex plane

The curve Γ is assumed to have a 2π -periodic twice continuously differentiable complex function $\eta(t)$ with non-vanishing first derivative, i.e.,

$$\dot{\eta}(t) = \frac{d(\eta(t))}{dt} \neq 0, \quad t \in J := [0, 2\pi]. \quad (1)$$

Assume that H be the space of all real Hölder continuous functions on the boundary Γ . From now on, for complex-valued or real-valued function $\psi \in H$ defined on the boundary Γ and for $t \in J$, we will not distinguish between $\psi(\eta(t))$ and $\psi(t)$.

We define the function A by

$$A(t) = \eta(t) - \alpha, \quad (2)$$

with a fixed point $\alpha \in G$. The generalized Neumann kernel formed with A is defined by [18]

$$N(s, t) := \frac{1}{\pi} \operatorname{Im} \left(\frac{A(s)}{A(t)} \frac{\dot{\eta}(t)}{\eta(t) - \eta(s)} \right). \quad (3)$$

The kernel N is continuous with

$$N(t, t) := \frac{1}{\pi} \operatorname{Im} \left(\frac{1}{2} \frac{\ddot{\eta}(t)}{\dot{\eta}(t)} - \frac{\dot{A}(t)}{A(t)} \right). \quad (4)$$

Define also a real kernel M by [18]

$$M(s, t) := \frac{1}{\pi} \operatorname{Re} \left(\frac{A(s)}{A(t)} \frac{\dot{\eta}(t)}{\eta(t) - \eta(s)} \right). \quad (5)$$

The kernel M can be represented by

$$M(s, t) := -\frac{1}{2\pi} \cos \frac{s-t}{2} + M_1(s, t), \quad (6)$$

with a continuous kernel M_1 which takes on the diagonal the values

$$M_1(t, t) := \frac{1}{\pi} \operatorname{Re} \left(\frac{1}{2} \frac{\ddot{\eta}(t)}{\dot{\eta}(t)} - \frac{\dot{A}(t)}{A(t)} \right). \quad (7)$$

Hence, the operator [18]

$$\mathbf{N}\mu(s) := \int_J N(s, t)\mu(t)dt, \quad s \in J, \quad (8)$$

is a Fredholm integral operator and the operator

$$\mathbf{M}\mu(s) := \int_J M(s, t)\mu(t)dt, \quad s \in J, \quad (9)$$

is a singular integral operator.

Theorem 1. [10, 18] *If F is a solution of the Dirichlet boundary value problem $\operatorname{Re} [F(\eta(t))] = \gamma(t)$ with boundary values*

$$F = \gamma + i\mu \quad \gamma \in H \quad (10)$$

and $\operatorname{Im} F(0) = 0$, then the imaginary part of F satisfies the integral equation

$$\mu - \mathbf{N}\mu = -\mathbf{M}\gamma. \quad (11)$$

2.2. Dirichlet Problem with Discontinuous Coefficients

Suppose that γ is a given function with finite number of jump discontinuities at the points $t_k \in [0, 2\pi)$ for $k = 1, 2, \dots, l$. Denote the limiting values of $\gamma(t)$ at the points of discontinuities by

$$a_k^- = \gamma(t_k - 0), \quad a_k^+ = \gamma(t_k + 0), \quad k = 1, 2, \dots, l. \quad (12)$$

Find a function u harmonic in G^+ such that its boundary values on smooth boundary Γ satisfy [4, 8, 9]

$$u(\eta(t)) = \gamma(t). \quad (13)$$

The unique solution u of the Dirichlet problem (13) can be regarded as a real part of an analytic function in G , $F(z)$. We allow the boundary Γ to have corner points at $\eta(t_k)$ for $k = 1, 2, \dots, l$. Let

$$b_k^- = \arg_{t_k}(\eta(t_k - 0) - \eta(t_k)), \quad b_k^+ = \arg_{t_k}(\eta(t_k + 0) - \eta(t_k)), \quad (14)$$

$$k = 1, 2, \dots, l,$$

where each value of the argument $\arg_{t_k}$ is chosen such that the branch cut from the point $\eta(t_k)$ to the point at ∞ is outside of the domain G . Then $b_k^+ - b_k^-$ represents the interior angle between the left and right tangents to the boundary Γ at $\eta(t_k)$. We assume that $0 < b_k^+ - b_k^- < 2\pi$. If the boundary Γ is smooth at $\eta(t_k)$, then $b_k^+ - b_k^- = \pi$.

Let the function $f(z)$ be defined by

$$f(z) = F(z) + i \sum_{k=1}^l \frac{a_k^- - a_k^+}{b_k^- - b_k^+} \log_{t_k}(z - \eta(t_k)). \quad (15)$$

The representation (36) is different from the one used in [15]. The branch cut of $\log_{t_k}$ is chosen to be outside of the domain G from the point $\eta(t_k)$ to the point at ∞ . Then $f(z)$ is analytic in G with

$$\operatorname{Re}[f(\eta(t))] = \phi(t) \quad (16)$$

where

$$\phi(t) = \gamma(t) - \sum_{k=1}^l \frac{a_k^- - a_k^+}{b_k^- - b_k^+} \gamma_k(t) \quad (17)$$

and

$$\gamma_k(t) = \operatorname{Im} \log_{t_k}(\eta(t) - \eta(t_k)) = \arg_{t_k}(\eta(t) - \eta(t_k)). \quad (18)$$

It is clear that the function $\phi(t)$ is continuous on $[0, 2\pi] - \{t_1, \dots, t_l\}$. The function $\phi(t)$ is also continuous at the each point t_j for $j = 1, 2, \dots, m$. To show that, we should show that

$$\phi(t_j - 0) - \phi(t_j + 0) = 0.$$

Since $\gamma_k(t)$ is continuous at t_j for each $j \neq k$. Hence, we have

$$\phi(t_j - 0) - \phi(t_j + 0) = \gamma(t_j - 0) - \frac{a_j^- - a_j^+}{b_j^- - b_j^+} \gamma_j(t_j - 0)$$

$$\begin{aligned}
& -\gamma(t_j + 0) + \frac{a_j^- - a_j^+}{b_j^- - b_j^+} \gamma_j(t_j + 0) \\
& = a_j^- - \frac{a_j^- - a_j^+}{b_j^- - b_j^+} b_j^- - a_j^+ + \frac{a_j^- - a_j^+}{b_j^- - b_j^+} b_j^+ \\
& = a_j^- - a_j^+ - \frac{a_j^- - a_j^+}{b_j^- - b_j^+} (b_j^- - b_j^+) \\
& = a_j^- - a_j^+ - (a_j^- - a_j^+) \\
& = 0.
\end{aligned}$$

Thus we have succeeded in reducing the problem of solving the Dirichlet problem with discontinuous coefficients (13) to a problem of solving Dirichlet problem with continuous coefficients (37). Without lose the generality, we assume that $\text{Im}[f(0)] = 0$. Let $\psi = \text{Im}[f]$, i.e.,

$$f(\eta(t)) = \phi(t) + i\psi(t) \quad (19)$$

then by Theorem (1) we have

$$\psi - \mathbf{N}\psi = -\mathbf{M}\phi. \quad (20)$$

By solving (41), we obtain ψ , then we obtain $f(z)$ from (40), so we can obtain $F(z)$ from (36) and our final answer $u(z)$ will be the real part of the $F(z)$.

3. Dirichlet Problem with Discontinuous Coefficients in Bounded Multiply Connected Region

3.1. Notations and Auxiliary Material

Let G be a Bounded multiply connected region, of connectivity $m + 1 \geq 1$, in the extended complex plane $\overline{\mathbb{C}}$ whit boundary $\Gamma = \cup_{j=0}^m \Gamma_j$ consisting of $m + 1$ smooth closed Jordan curves Γ_j , $j = 0, 1, 2, \dots, m$. The curves $\Gamma_1, \dots, \Gamma_m$ are inside the curve Γ_0 . The complement G^- is defined as $G^- := \overline{\mathbb{C}} \setminus \overline{G}$ (see Figure 2). Suppose that α is a fixed point in G .

Figure 2. A bounded multiply connected region in the extended complex plane

The curve Γ_0 has counterclockwise orientations and the curves $\Gamma_1, \dots, \Gamma_m$ always have clockwise orientations (in fact the orientation of boundary Γ is such that G be on the left of Γ). A 2π -periodic twice continuously differentiable complex function $\eta_j(t)$ with non-vanishing first derivative

$$\dot{\eta}_j(t) = d\eta_j(t)/dt \neq 0, \quad t \in J_j := [0, 2\pi], \quad (21)$$

$j = 0, 1, 2, \dots, m$ is a parameterization for the curve Γ_j . we will represent the disjoint union of the intervals J_j by J as a total parameter domain. Automatically, the whole boundary Γ will be parameterized by complex function η defined on J by

$$\eta(t) := \begin{cases} \eta_0(t), & t \in J_0, \\ \eta_1(t), & t \in J_1, \\ \vdots & \\ \eta_m(t), & t \in J_m. \end{cases} \quad (22)$$

Assume that H be the space of all real Hölder continuous functions on the boundary Γ . Since η is smooth, a function $\phi \in H$ can be explained by $\hat{\phi}(t) := \phi(\eta(t))$, $t \in J$, as a real Hölder continuous 2π -periodic functions $\hat{\phi}(t)$ of

the parameter $t \in J$, i.e.,

$$\hat{\phi}(t) := \begin{cases} \hat{\phi}_0(t), & t \in J_0, \\ \hat{\phi}_1(t), & t \in J_1, \\ \vdots & \\ \hat{\phi}_m(t), & t \in J_m, \end{cases} \quad (23)$$

with real Hölder continuous 2π -periodic functions $\hat{\phi}_j$ defined on J_j ; and vice versa.

From now on, for complex-valued or real-valued functions ψ defined on the boundary Γ and for $t \in J$, we will not differentiate between $\psi(\eta(t))$ and $\psi(t)$. For $t \in J_k$, the values $\psi(t)$ will be indicated by $\psi_k(t)$.

Suppose that $\gamma \in H$ is a given function, the Dirichlet problem can be described as follows (see e.g. [1, p.307], [8, p. 145] and [9, p. 164]):

Suppose that $\chi^{[j]}$ be the piecewise constant function defined on J by

$$\chi^{[j]}(t) := \begin{cases} 1, & \text{if } t \in J_j, \\ 0, & \text{if } t \notin J_j, \end{cases} \quad (24)$$

for $j = 0, 1, \dots, m$. Then, we define the space S by

$$S = \text{span}\{\chi^{[0]}, \chi^{[1]}, \dots, \chi^{[m]}\}. \quad (25)$$

We define also the space $\mathring{S}$ by

$$\mathring{S} = \text{span}\{\chi^{[1]}, \dots, \chi^{[m]}\}. \quad (26)$$

Then

$$\dim(S) = m + 1, \quad \dim(\mathring{S}) = m. \quad (27)$$

It follows from the definition of the space S that a function $h \in S$ if and only if h can be written as

$$h(t) = \begin{cases} h_0, & t \in J_0, \\ h_1, & t \in J_1, \\ \vdots & \\ h_m, & t \in J_m, \end{cases} \quad (28)$$

with real constants $h_0, h_1, \dots, h_m$. We define an operator $\mathbf{R} : S \rightarrow \mathring{S}$ by

$$\mathbf{R}h = h - h_0.$$

Theorem 2 ([15]). *Let γ be a given function. Then, there exists a unique function $\mathring{h} \in \mathring{S}$ and a unique function μ such that*

$$f = \gamma + \mathring{h} + i\mu \quad (29)$$

are boundary values of a single-valued analytic function f in G with $\text{Im } f(\alpha) = 0$. The function μ is the unique solution of the integral equation

$$(\mathbf{I} - \mathbf{N})\mu = -\mathbf{M}\gamma, \quad (30)$$

$\mathring{h} = \mathbf{R}h$, $f(\alpha) = -h_0$ where h is given by

$$h = [\mathbf{M}\mu - (\mathbf{I} - \mathbf{N})\gamma]/2. \quad (31)$$

3.2. Dirichlet Problem with Discontinuous Coefficients

Suppose that $\gamma(t)$ is a given function in the following form

$$\gamma(t) := \begin{cases} \gamma_0(t), & t \in J_0, \\ \gamma_1(t), & t \in J_1, \\ \vdots \\ \gamma_m(t), & t \in J_m, \end{cases} \quad (32)$$

where for at least one $j = 0, 1, \dots, m$ the function $\gamma_j(t)$ has finite number of first kind discontinuities at $t_k^j \in J_i, k = 1, \dots, l_j$. Find a function u continuous on the closure $\overline{G}$ (except probably at t_k^j), harmonic in G , such that its boundary values satisfy on Γ

$$u(\eta(t)) = \gamma(t). \quad (33)$$

In the other hand, the discontinuity points of the functions $\gamma_j(t)$ on Γ are in the following form:

$$\begin{aligned} \Gamma_0 &: t_1^0, t_2^0, t_3^0, \dots, t_{l_0}^0 \\ \Gamma_1 &: t_1^1, t_2^1, t_3^1, \dots, t_{l_1}^1 \\ \Gamma_2 &: t_1^2, t_2^2, t_3^2, \dots, t_{l_2}^2 \\ &\vdots \\ \Gamma_m &: t_1^m, t_2^m, t_3^m, \dots, t_{l_m}^m \end{aligned}$$

it means we have l_0 discontinuity points on Γ_0 , l_1 discontinuity points on Γ_1 , $\dots$, l_m discontinuity points on Γ_m .

The unique solution u of the Dirichlet problem (33) can be regarded as a real part of an analytic function $F(z)$ in G . Suppose that the discontinuity points of the function $\gamma_j(t)$ are at the points $t_k^j \in [0, 2\pi)$ for $k = 1, 2, \dots, l_j$. Denote the limiting values of $\gamma_j(t)$ at the points of discontinuities by

$$a_{j,k}^- = \gamma_j(t_k^j - 0), \quad a_{j,k}^+ = \gamma_j(t_k^j + 0), \quad k = 1, 2, \dots, l_j. \quad (34)$$

Let

$$b_{j,k}^- = \arg_{t_k^j} (\eta_j(t_k^j - 0) - \eta_j(t_k^j)), \quad b_{j,k}^+ = \arg_{t_k^j} (\eta_j(t_k^j + 0) - \eta_j(t_k^j)), \quad (35)$$

$$k = 1, 2, \dots, l_j,$$

where each value of the argument $\arg_{t_k^j}$ is chosen such that the branch cut from the point $\eta_j(t_k^j)$ is outside of the domain G . Then $(b_{j,k}^+ - b_{j,k}^-)$ represents the interior angle between the left and right tangents to the boundary Γ at $\eta_j(t_k^j)$. We assume that $0 < b_{j,k}^+ - b_{j,k}^- < 2\pi$. If the boundary Γ is smooth at $\eta_j(t_k^j)$, then $b_{j,k}^+ - b_{j,k}^- = \pi$.

Let the function $f(z)$ be defined by

$$f(z) = F(z) + i \sum_{j=0}^m \sum_{k=1}^{l_j} \frac{a_{j,k}^- - a_{j,k}^+}{b_{j,k}^- - b_{j,k}^+} \log_{t_k^j} (z - \eta_j(t_k^j)). \quad (36)$$

Then $f(z)$ is analytic in G with

$$\operatorname{Re}[f(\eta(t))] = \phi(t) \quad (37)$$

where

$$\phi(t) = \gamma(t) - \sum_{j=0}^m \sum_{k=1}^{l_j} \frac{a_{j,k}^- - a_{j,k}^+}{b_{j,k}^- - b_{j,k}^+} \gamma_k^j(t) \quad (38)$$

and

$$\gamma_k^j(t) = \operatorname{Im} \log_{t_k^j} (\eta(t) - \eta_j(t_k^j)) = \arg_{t_k^j} (\eta(t) - \eta_j(t_k^j)). \quad (39)$$

It is clear that the function $\phi(t)$ is continuous on $[0, 2\pi] - \{t_k^j : j = 0, \dots, m \text{ \& } k = 1, \dots, l_j\}$. The function $\phi(t)$ is also continuous at the each point t_k^j . To prove that, we should show that

$$\phi(t_k^j - 0) - \phi(t_k^j + 0) = 0.$$

Since $\gamma_k^j(t)$ is continuous at t_i^s for each $i \neq k$ and $s \neq j$. Hence, we have

$$\begin{aligned}
 \phi(t_k^j - 0) - \phi(t_k^j + 0) &= \gamma(t_k^j - 0) - \frac{a_{j,k}^- - a_{j,k}^+}{b_{j,k}^- - b_{j,k}^+} \gamma_k^j(t_j - 0) \\
 &\quad - \gamma(t_k^j + 0) + \frac{a_{j,k}^- - a_{j,k}^+}{b_{j,k}^- - b_{j,k}^+} \gamma_k^j(t_j + 0) \\
 &= a_{k,j}^- - \frac{a_{k,j}^- - a_{k,j}^+}{b_{k,j}^- - b_{k,j}^+} b_{k,j}^- - a_{k,j}^+ + \frac{a_{k,j}^- - a_{k,j}^+}{b_{k,j}^- - b_{k,j}^+} b_{k,j}^+ \\
 &= a_{k,j}^- - a_{k,j}^+ - \frac{a_{k,j}^- - a_{k,j}^+}{b_{k,j}^- - b_{k,j}^+} (b_{k,j}^- - b_{k,j}^+) \\
 &= a_{k,j}^- - a_{k,j}^+ - (a_{k,j}^- - a_{k,j}^+) \\
 &= 0.
 \end{aligned}$$

Thus we have succeeded in reducing the problem of solving the Dirichlet problem with discontinuous coefficients (33) to a problem of solving Dirichlet problem with continuous coefficients (37). Without lose the generality, we assume that $\text{Im}[f(0)] = 0$. Let $\psi = \text{Im}[f]$, i.e.,

$$f(\eta(t)) = \phi(t) + i\psi(t) \quad (40)$$

then by Theorem (2) we have

$$\psi - \mathbf{N}\psi = -\mathbf{M}\phi. \quad (41)$$

By solving (41), we obtain ψ , then we obtain $f(z)$ from (40), so we can obtain $F(z)$ from (36) and our final answer $u(z)$ will be the real part of the $F(z)$.

4. Numerical Examples

Since the functions A_j and η_j are 2π -periodic, a reliable procedure for solving the integral equation (41) numerically is by using the Nyström method with the trapezoidal rule [1]. Using the Nyström method with trapezoidal rule reduces solving the integral equation (41) to solving a linear system [1]

$$(I - B)\mathbf{x} = -C\mathbf{y}, \quad (42)$$

where B is the $n \times n$ discretizing matrix of the operator $\mathbf{N}$, C is the $n \times n$ discretizing matrix of the operator $\mathbf{M}$, I is the identity matrix, and $\mathbf{x}$ is an $n \times 1$ vector of the approximate values of the unknown function ψ at the nodes and $\mathbf{y}$ is an $n \times 1$ vector of the values of the function ϕ at the nodes. Since the integral equation (41) is uniquely solvable, then for sufficiently large n , the linear system (42) is uniquely solvable [1]. See [11, 12, 13, 14, 15, 17] for more details. For some ideas on how to solve numerically boundary integral equations with the generalized Neumann kernel on regions with corners, see [17].

In this paper, the linear system (42) will be solved by the MATLAB function `gmres`. The function `gmres` can be used with a matrix-vector product function, i.e., it is not necessary to form the matrices B and C explicitly. The matrix-vector product functions for the matrices B and C are defined using the function `zfmm2dpart` in the MATLAB toolbox `FMMLIB2D` developed by Greengard and Gimbutas [5]. Using the Fast Multipole Method (FMM), the product of the matrix B by a vector can be computed in $O(n)$ operations. Computing the matrix-vector product $C\mathbf{y}$ in the right-hand side of the linear system (42) requires using both the FMM and the FFT which requires $O(n \ln n)$ operations. Thus, the linear systems (42) will be solved in $O(n \ln n)$ operations. See [14, 13] for more details. For comparisons, we use MATLAB toolbox `pdetool` to obtain the solutions of examples by finite element method. In the following numerical examples, $u(z)$ is the exact solution of the Dirichlet boundary value problem with discontinuous coefficients, $u_n(z)$ is the approximate solution obtained with n node points on the boundary obtained by the presented method, and $u_{fem}(z)$ is the approximate solution obtained by the finite element method.

Example 1. Find a function u harmonic in the unit disk having the boundary value 1 in the right half plane and 0 in the left half plane.

This example has been considered in [3, Example 4, p. 248] using the conformal mapping and Poisson's integral formula method. The exact solution is given by

$$u(z) = \frac{1}{\pi} \arg \left(i \cdot \frac{i - z}{i + z} \right).$$

Figure 3. The region for Example 1

This function u has two points of discontinuities at $t_1 = \frac{\pi}{2}$ ($z_1 = A$) and $t_2 = \frac{3\pi}{2}$ ($z_2 = B$). By using the explanations in Section 2.2 we have:

$$\begin{aligned}
 a_1^- &= u(t_1 - 0) = u\left(\frac{\pi}{2} - 0\right) = 1, \\
 a_1^+ &= u(t_1 + 0) = u\left(\frac{\pi}{2} + 0\right) = 0, \\
 a_2^- &= u(t_2 - 0) = u\left(\frac{3\pi}{2} - 0\right) = 0, \\
 a_2^+ &= u(t_2 + 0) = u\left(\frac{3\pi}{2} + 0\right) = 1, \\
 \eta(t_1) &= \eta\left(\frac{\pi}{2}\right) = A \quad , \quad \eta(t_2) = \eta\left(\frac{3\pi}{2}\right) = B, \\
 \frac{a_1^- - a_1^+}{\pi} &= \frac{1}{\pi} \quad , \quad \frac{a_2^- - a_2^+}{\pi} = -\frac{1}{\pi}.
 \end{aligned}$$

So we have,

$$\begin{aligned}
 \phi(t) &= u(t) - \sum_{k=1}^2 \frac{a_k^- - a_k^+}{\pi} \arg(\eta(t) - \eta(t_k)) \\
 &= u(t) - \frac{1}{\pi} \arg_{t_1}(\eta(t) - i) + \frac{1}{\pi} \arg_{t_2}(\eta(t) + i).
 \end{aligned}$$

Figure 4. The surface plot of the solution for Example 1
that is obtained by our method

Figure 5. The absolute error $|u_n(z) - u(z)|$ for the entire
region obtained with $n = 1024$ by our method

Figure 6. The absolute error $|u_{fem}(z) - u(z)|$ for the entire region obtained with 130561 points by finite element method

The surface plot of the solution for Example 1 obtained by our method is shown in Figure 4 and the absolute error $|u_n(z) - u(z)|$ for the entire region obtained with $n = 1024$ by our method is shown in Figure 5. In Figure 6 we can see the absolute error $|u_{fem}(z) - u(z)|$ for the entire region obtained by finite element method with 130561 points. As it is clear in the Figure 5 and Figure 6, our method is more accurate than finite element method, especially at the points near the discontinuity points.

Example 2. Solve the following boundary-value problem(see Figure 7).

$$\text{P.D.E.: } u_{xx} + u_{yy} = 0, \quad 0 < x < \pi, \quad 0 < y < 1;$$

$$\text{B.C.: } u(0, y) = u(\pi, y) = 0, \quad 0 < y < 1,$$

$$u(x, 1) = 0, \quad u(x, 0) = 100, \quad 0 < x < \pi.$$

This example has been considered in[6, Example 3.2-3] using the separation of variable method. The exact solution is given by

$$u(x, y) = \frac{400}{\pi} \sum_{m=1}^{\infty} \frac{\sin((2m-1)x) \sinh((2m-1)(1-y))}{(2m-1) \sinh(2m-1)}.$$