APPROXIMATION OF BINOMIAL DISTRIBUTION BY AN IMPROVED POISSON DISTRIBUTION

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Abstract: This paper derives an approximation of the binomial distribution with parameters n and p by an improved Poisson distribution with mean $\lambda = np$. The approximation is more accurate than the Poisson approximation when n is sufficiently large.

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1. Introduction

It is well-known that a discrete random variable X has the binomial distribution with parameters $n \in \mathbb{N}$ and $p \in (0, 1)$ if and only if its probability function is of the form

$$\mathbf{b}_{n,p}(x) = \binom{n}{x} p^x q^{n-x}, \quad x = 0, 1, \dots, n, \quad (1.1)$$

where $E(X) = \frac{np}{p}$ and variance $Var(X) = \frac{np}{p^2}$ are the mean and variance of this random variable, respectively. We know that if $n \rightarrow \infty$ and $p \rightarrow 0$ while $\lambda = np$ remains fixed, then $\mathbf{b}_{n,p}(x) \rightarrow \mathbf{p}_\lambda(x) = \frac{e^{-\lambda} \lambda^x}{x!}$ for every $x \in$

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$\{0, 1, \dots, n\}$. Therefore, the Poisson distribution with mean $\lambda = np$ can be used as an approximation of the binomial distribution when n is large and p is small. In this case, there has been much methodological research related to the Poisson approximation to the binomial distribution. For example, [1] gave some inequalities of binomial and Poisson probabilities, [4] and [6] gave inequalities for the ratio of Poisson and binomial probabilities and in terms of the relative error of two such probabilities can be found in [3], [2] and [7] – [10]. However, there has no research related to improved Poisson approximation to the binomial distribution. In this paper, we are interested to derive an improved Poisson probability function, $\widehat{\mathbf{p}}_\lambda(x)$, for approximating the binomial probability function. The accuracy of the approximation is measured in the form of $|\mathbf{b}_{n,p}(x) - \widehat{\mathbf{p}}_\lambda(x)|$ for $x \in \{0, 1, \dots, n\}$. The result of this study is in Section 2. In Section 3, some numerical examples are given to illustrate the improved approximation and the conclusion of this study is presented in the last section.

2. Result

We use the same method in [5] to derive an improved Poisson distribution that can be obtained from the binomial distribution. The following lemma follows from [5].

Lemma 2.1. *For $x, n \in \mathbb{N}$, then the following property holds:*

$$\prod_{i=0}^{x-1} \left(1 - \frac{i}{n}\right) = \frac{1}{1 + \frac{x(x-1)}{2n} + O\left(\frac{1}{n^2}\right)}. \quad (2.1)$$

Theorem 2.1. *For $x \in \{0, 1, \dots, n\}$ and $\lambda = np$, if n is large then we have the following:*

$$\mathbf{b}_{n,p}(x) \approx \widehat{\mathbf{p}}_\lambda(x), \quad (2.2)$$

where $\widehat{\mathbf{p}}_\lambda(x) = \frac{\mathbf{p}_\lambda(x)}{e^{\frac{\lambda^2}{2n}} q^x \left\{1 + \frac{x(x-1)}{2n}\right\}}$.

Proof. For $x = 0$, $\mathbf{b}_{n,p}(0) = (1-p)^n = \left(1 - \frac{\lambda}{n}\right)^n = e^{n \log(1 - \frac{\lambda}{n})} = e^{-\lambda - \frac{\lambda^2}{2n} - \dots}$. Thus, when n is large, we have $\mathbf{b}_{n,p}(0) = \frac{e^{-\lambda}}{e^{\frac{\lambda^2}{2n}}} = \frac{\mathbf{p}_\lambda(0)}{e^{\frac{\lambda^2}{2n}}}$.

For $x \in \{1, \dots, n\}$,

$$\mathbf{b}_{n,p}(x) = \frac{n!}{x!(n-x)!} \frac{(np)^x q^n}{n^x q^x}$$

$$\begin{aligned} &= \frac{\lambda^x}{x!q^x} \left[\frac{n}{n} \cdots \left(1 - \frac{x-1}{n} \right) \right] \left(1 - \frac{\lambda}{n} \right)^n \\ &= \frac{\lambda^x}{x!q^x} \prod_{i=0}^{x-1} \left(1 - \frac{i}{n} \right) \left(e^{-\lambda - \frac{\lambda^2}{2n} - \cdots} \right) \\ &= \frac{\mathbf{p}_\lambda(x) e^{-\frac{\lambda^2}{2n} - \cdots}}{q^x \left\{ 1 + \frac{x(x-1)}{2n} + O\left(\frac{1}{n^2}\right) \right\}} \text{ (by Lemma 2.1).} \end{aligned}$$

For large n , we obtain

$$\mathbf{b}_{n,p}(x) \approx \frac{\mathbf{p}_\lambda(x)}{e^{\frac{\lambda^2}{2n}} q^x \left\{ 1 + \frac{x(x-1)}{2n} \right\}},$$

which yields (2.2). □

3. Numerical Examples

The following examples have been given to illustrate how well an improved Poisson distribution approximates a binomial distribution.

3.1. Let $n = 100$ and $p = 0.01$, then $\lambda = 1.0$ and the numerical results are as follows:

x	$\mathbf{b}_{n,p}(x)$	$\widehat{\mathbf{p}}_\lambda(x)$	$\mathbf{p}_\lambda(x)$	$ \mathbf{b}_{n,p}(x) - \widehat{\mathbf{p}}_\lambda(x) $	$ \mathbf{b}_{n,p}(x) - \mathbf{p}_\lambda(x) $
0	0.36603234	0.36604463	0.36787944	0.00001229	0.00184710
1	0.36972964	0.36974206	0.36787944	0.00001242	0.00185020
2	0.18486482	0.18488952	0.18393972	0.00002470	0.00092510
3	0.06099917	0.06104358	0.06131324	0.00004441	0.00031407
4	0.01494171	0.01497877	0.01532831	0.00003706	0.00038660
5	0.00289779	0.00291598	0.00306566	0.00001819	0.00016787
6	0.00046345	0.00046956	0.00051094	0.00000611	0.00004749
7	0.00006286	0.00006440	0.00007299	0.00000153	0.00001013
8	0.00000738	0.00000769	0.00000912	0.00000030	0.00000174
9	0.00000076	0.00000081	0.00000101	0.00000005	0.00000025
10	0.00000007	0.00000008	0.00000010	0.00000001	0.00000003

3.2. Let $n = 1000$ and $p = 0.005$, then $\lambda = 5.0$ and the numerical results are as follows:

x	$\mathbf{b}_{n,p}(x)$	$\widehat{\mathbf{p}}_{\lambda}(x)$	$\mathbf{p}_{\lambda}(x)$	$ \mathbf{b}_{n,p}(x) - \widehat{\mathbf{p}}_{\lambda}(x) $	$ \mathbf{b}_{n,p}(x) - \mathbf{p}_{\lambda}(x) $
0	0.00665397	0.00665425	0.00673795	0.00000028	0.00008398
1	0.03343703	0.03343843	0.03368973	0.00000140	0.00025271
2	0.08392862	0.08393221	0.08422434	0.00000359	0.00029572
3	0.14030279	0.14030964	0.14037390	0.00000685	0.00007111
4	0.17573100	0.17574274	0.17546737	0.00001173	0.00026363
5	0.17590762	0.17592636	0.17546737	0.00001874	0.00044025
6	0.14658968	0.14661618	0.14622281	0.00002650	0.00036687
7	0.10460168	0.10463358	0.10444486	0.00003190	0.00015682
8	0.06524464	0.06527707	0.06527804	0.00003243	0.00003340
9	0.03613774	0.03616583	0.03626558	0.00002809	0.00012784
10	0.01799623	0.01801726	0.01813279	0.00002103	0.00013656
11	0.00813900	0.00815280	0.00824218	0.00001380	0.00010318
12	0.00337080	0.00337884	0.00343424	0.00000804	0.00006344
13	0.00128734	0.00129155	0.00132086	0.00000421	0.00003352
14	0.00045607	0.00045806	0.00047174	0.00000199	0.00001567
15	0.00015065	0.00015151	0.00015725	0.00000086	0.00000660
16	0.00004660	0.00004695	0.00004914	0.00000034	0.00000253
17	0.00001356	0.00001368	0.00001445	0.00000013	0.00000090
18	0.00000372	0.00000376	0.00000401	0.00000004	0.00000029
19	0.00000097	0.00000098	0.00000106	0.00000001	0.00000009
20	0.00000024	0.00000024	0.00000026	0.00000000	0.00000003
21	0.00000006	0.00000006	0.00000006	0.00000000	0.00000001
22	0.00000001	0.00000001	0.00000001	0.00000000	0.00000000

The improved approximation is better than the Poisson approximation.

4. Conclusion

The result of this study is an improved Poisson distribution with mean $\lambda = np$. It can be used as a good approximation of the binomial distribution with parameters n and p when n is large. Additionally, the improved approximation is better than the Poisson approximation.

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