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ON (m, n)-IDEALS OF AN ORDERED SEMIGROUP

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Abstract: In this paper we introduce the concept of (m, n)-simple ordered semigroups extending the notion of simple ordered semigroups, and prove that an ordered semigroup $(S, \cdot, \leq)$ does not contain proper (m, n)-ideals if and only if it is both $(m, 0)$-simple and $(0, n)$-simple.

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1. Preliminaries

Given a semigroup (without order) S, Lajos [5] proved that S does not contain proper bi-ideals if and only if it is both left and right simple. This result was extended to ordered semigroups by Kehayopulu, Ponizovskii, Tsingelis [3]; the authors gave an example to show that an ordered semigroup without proper bi-ideals need not to be an ordered group.

Using the notion of (m, n)-ideals in semigroups (without order) defined by Lajos [4], Sanborisoot and Changphas [6] introduced the concept of (m, n)-ideals in ordered semigroups. Based on this concept, we introduce (m, n)-simple ordered semigroups, and prove that an ordered semigroup $(S, \cdot, \leq)$ does not contain proper (m, n)-ideals if and only if it is both $(m, 0)$-simple and $(0, n)$-simple. The results obtained are more general than that of the result proved in [3].

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In [1], an *ordered semigroup* $(S,\cdot,\leq)$ is a semigroup $(S,\cdot)$ together with a partial order $\leq$ that is *compatible* with the semigroup operation, meaning that, for any x,y,z in S,

$$x \leq y \text{ implies } zx \leq zy \text{ and } xz \leq yz.$$

If A,B are nonempty subsets of S, we write the set product AB for the set of all elements xy of S with $x \in A$ and $y \in B$, and write $(A]$ for the set of all elements x of S such that $x \leq a$ for some a in A, i.e.,

$$(A] = \{x \in S \mid x \leq a \text{ for some } a \in A\}.$$

It is known (see [2]) that the following are true:

(1) $A \subseteq (A]$;

(2) $A \subseteq B \Rightarrow (A] \subseteq (B]$;

(3) $(A](B] \subseteq (AB]$;

(4) $(A \cup B] = (A] \cup (B]$;

(5) $((A]] = (A]$.

A nonempty subset A of an ordered semigroup $(S,\cdot,\leq)$ is called a *subsemigroup* of S if $AA \subseteq A$.

Let m,n be nonnegative integers. A subsemigroup A of an ordered semigroup $(S,\cdot,\leq)$ is called an (m,n)*-ideal* [6] of S if it satisfies the following conditions:

(i) $A^mSA^n \subseteq A$;

(ii) for any $x \in A$ and $y \in S$, $y \leq x$ implies $y \in A$.

Here, $A^0S = S$ and $SA^0 = S$. If A is an (m,n)-ideal of S such that $A \subset S$ (the symbol $\subset$ stands for proper subset), then it is called a *proper* (m,n)*-ideal* of S.

Note that proper $(1,1)$-ideals of an ordered semigroup $(S,\cdot,\leq)$ are called proper bi-ideals of S.

In [6], the authors proved the following lemma:

Lemma 1. *For any element a of an ordered semigroup $(S,\cdot,\leq)$,*

$$(a \cup a^2 \cup \cdots \cup a^{m+n} \cup a^mSa^n]$$

is the smallest (m,n)-ideal of S containing a.

2. Main Results

We define (m,n)-simple ordered semigroups as follows:

Definition 2. Let m,n be nonnegative integers. An ordered semigroup $(S,\cdot,\leq)$ is said to be (m,n)-*simple* if it does not contain proper (m,n)-ideal.

Note that a left simple (right simple) ordered semigroup is an $(0,1)$-simple ($(1,0)$-simple).

Lemma 3. *Let $(S,\cdot,\leq)$ be an ordered semigroup, and let m,n be nonnegative integers. The following statements hold*:

(1) *S is $(m,0)$-simple if and only if $S=(a^mS]$ for all $a\in S$;*

(2) *S is $(0,n)$-simple if and only if $S=(Sa^n]$ for all $a\in S$.*

Proof. (1) Assume that S is $(m,0)$-simple, and let $a\in S$. By

$$(a^mS](a^mS]\subseteq(a^mSa^mS]\subseteq(a^mS]$$

and

$$(a^mS]^mS\subseteq(a^mS]$$

it follows that $(a^mS]$ is an (m,n)-ideal of S. Hence $S=(a^mS]$ by assumption.

Conversely, assume that $S=(a^mS]$ for all $a\in S$. Let A be an $(m,0)$-ideal of S. To show that $S=A$, let $a\in S$. We have

$$S=(a^mS]\subseteq(A^mS]\subseteq(A]=A.$$

Thus S is $(m,0)$-simple.

(2) This can be proved similarly. □

Corollary 4. (see [3]) *The following statements hold for an ordered semigroup $(S,\cdot,\leq)$:*

(1) *S is left simple if and only if $S=(aS]$ for all $a\in S$;*

(2) *S is right simple if and only if $S=(Sa]$ for all $a\in S$.*

Now, we prove the main result:

Theorem 5. *Let m, n be nonnegative integers. An ordered semigroup $(S, \cdot, \leq)$ does not contain proper (m, n)-ideal if and only if it is both $(m, 0)$-simple and $(0, n)$-simple.*

Proof. Assume that S does not contain (m, n)-ideal. If A is an $(m, 0)$-ideal of S, then A is an (m, n)-ideal of S because

$$A^m S A^n \subseteq A^m S \subseteq A.$$

Hence $A = S$. Similarly, if A is an $(0, n)$-ideal of S, then A is an (m, n)-ideal of S; hence $A = S$.

Conversely, we assume that S is both $(m, 0)$-simple and $(0, n)$-simple. Let A be an (m, n)-ideal of S. To show that $S \subseteq A$, let $a \in S$ and $b \in A$. By Lemma 1, we have

$$S = (b \cup b^2 \cup \cdots \cup b^m \cup b^m S].$$

There are two cases to consider.

Case 1: $a \leq b^k$ for some $1 \leq k \leq m$. By $b^k \in A$, it follows that $a \in A$.

Case 2: $a \leq b^m x$ for some $x \in S$. We have

$$S = (b \cup b^2 \cup \cdots \cup b^n \cup S b^n].$$

There are two subcases to consider.

Case 2.1: $x \leq b^k$ for some $1 \leq k \leq n$. Since

$$a \leq b^m x \leq b^m b^k \in A,$$

it follows that $a \in A$.

Case 2.2: $x \leq y b^n$ for some $y \in S$. We have

$$a \leq b^m x \leq b^m y b^n \in A^m S A^n \subseteq A.$$

Thus $a \in A$. □

Corollary 6. (see [3]) *An ordered semigroup $(S, \cdot, \leq)$ does not contain proper bi-ideal if and only if it is both left and right simple.*

Corollary 7. (see [5]) *A semigroup $(S, \cdot)$ does not contain proper bi-ideal if and only if it is both left and right simple.*

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