International Journal of Pure and Applied Mathematics
Volume 100 No. 1 2015, 109-125
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v100i1.10

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APPROXIMATE SOLUTIONS OF SINGULAR DIFFERENTIAL EQUATIONS WITH ESTIMATION ERROR BY USING BERNSTEIN POLYNOMIALS

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Abstract: We present an approximate solution depending on collocation method and Bernstein polynomials for numerical solution of a singular nonlinear differential equations with the mixed conditions. The method is given with two different priori error estimates. By using the residual correction procedure, the absolute error might be estimated and obtained more accurate results. Illustrative examples are included to demonstrate the validity and applicability of the presented technique.

AMS Subject Classification: 15B99, 14F10
Key Words: singular differential equations, Bernstein polynomials, error estimate

1. Introduction

In this work, we consider the singular problems of the type

$$y''(x) + p(x)y'(x) + q(x)y(x) = g(x) \quad 0 < x \leq 1, \tag{1}$$

subject to the conditions

Received: January 28, 2015

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$$y(0) = \alpha_1, \quad y(1) = \beta_1, \quad \text{or} \quad y'(0) = \alpha_2, \quad y(0) = \beta_2, \tag{2}$$

where $x = 0$ is a singular point in $p(x)$, also $p(x)$, $q(x)$ and $g(x)$ are continuous functions on $(0, 1]$ and the parameters $\alpha_1, \alpha_2, \beta$ are real constants.

Problems of form (1)–(2) have been studied in many areas of science, chemistry and physics for example, equilibrium of isothermal gas sphere, reaction-diffusion process, geophysics, etc. Exact/approximate solutions of these problems are of great importance due to its wide application in scientific research. Nasab and Kilicman employed Wavelet analysis method for solving linear and nonlinear singular boundary value problems [1]. Bataineh and Ishak Hashim used Legendre Operational matrix to approximate solution of two points BVPs [2].

Bernstein operational matrix of differentiation, proposed by Bhatti used Bernstein polynomial Basis to solve Differential equation [3]. Pandey and kumar solved Lane-Emden type equations by Bernstein operational matrix [4]. Yousefi employed operational matrices of Bernstein polynomials and their applications to solve Bessel differential equation [5]. Yuzbasi used Bernstein polynomials to solve fractional riccati type differential equations [6]. Isik and Sezer employed Bernstein series to solve class of Lane-Emden equation [7]. Recently, Yiming Chen used Bernstein polynomials to find Numerical solution for the variable order linear cable equation [8]. Rostamy used a new operational matrix method based on the Bernstein polynomials for solving the backward inverse heat conduction problems [9].

In the present paper, we use Bernstein operational matrix to solve a linear and nonlinear singular boundary value problems it was solved by wavelet analysis method by [1]. We report our numerical solutions its become more accurate as we can see only small number of Bernstein polynomial basis functions are needed to get the approximate solution in full agreement with the exact solution up to 10 digits. This article is structured as follows. In Section 2, we describe the basic formulation of Bernstein polynomials and its operational matrix differentiation. In Section 3, we explained the applications of the operational matrix of derivative. In Section 4, we give an error analysis of the method and estimation of the error, also corrected absolute error is given. In Section 5, we report our numerical finding, exact solution and demonstrate the validity, accuracy and applicability of the operational matrices by considering numerical examples. Section 6, consists of brief summary and conclusion.

2. Bernstein Polynomials and its Operational Matrix of Differentiation

The Bernstein polynomials of degree m are defined by

$$B_{i,m}(x) = \binom{m}{i} x^i (1-x)^{m-i}, \quad i = 0, 1, \ldots, m$$

where the binomial coefficient is

$$\binom{m}{i} = \frac{m!}{i!\,(m-i)!}.$$

There are $m+1$ nth-degree Bernstein polynomials. For mathematical convenience, we usually set $B_{i,m} = 0$, if $i < 0$ or $i > m$.

In general, we approximate any function $y(x)$ with the first $(m+1)$ Bernstein polynomials as

$$y_m(x) = \sum_{i=0}^{m} c_i B_{i,m}(x) = \mathrm{C}^{\mathrm{T}} \phi(x), \tag{3}$$

where

$$\begin{aligned} \mathrm{C}^{\mathrm{T}} &= [\mathrm{c}_0, \mathrm{c}_1, \ldots, \mathrm{c}_m], \\ \phi(x) &= [B_{0,m}(x), B_{1,m}(x), \ldots, B_{m,m}(x)]^{\mathrm{T}}. \end{aligned}$$

The derivatives of the vector $\phi(x)$ can be expressed as

$$\frac{\mathrm{d}\phi(x)}{\mathrm{d}x} = \mathrm{D}^1 \phi(x) \tag{4}$$

where D^1 is the $(m+1) \times (m+1)$ operational matrix of derivative given as. Which is satisfied Eq. (4), also Eq. (4) can be generalized as

$$\frac{\mathrm{d}^2\phi(x)}{\mathrm{d}x^2} = (\mathrm{D}^1)^2 \phi(x), \ldots, \frac{\mathrm{d}^n\phi(x)}{\mathrm{d}x^n} = (\mathrm{D}^1)^n \phi(x). \tag{5}$$

For example with $m = 4$

$$D = \begin{pmatrix} -4 & -1 & 0 & 0 & 0 \\ 4 & -2 & -2 & 0 & 0 \\ 0 & 3 & 0 & -3 & 0 \\ 0 & 0 & 2 & 2 & -4 \\ 0 & 0 & 0 & 1 & 4 \end{pmatrix},$$

and if $m = 5$

$$D = \begin{pmatrix} -5 & -1 & 0 & 0 & 0 & 0 \\ 5 & -3 & -2 & 0 & 0 & 0 \\ 0 & 4 & -1 & -3 & 0 & 0 \\ 0 & 0 & 3 & 1 & -4 & 0 \\ 0 & 0 & 0 & 2 & 3 & -5 \\ 0 & 0 & 0 & 0 & 1 & 5 \end{pmatrix}.$$

3. Applications of the Operational Matrix of Derivative

To solve (1)–(2), by means of the operational matrix of derivative approximate $y(x)$ and $g(x)$ by Bernstein Polynomials as

$$y(x) \simeq \left(\mathrm{C}^{\mathrm{T}}\phi(x)\right), \tag{6}$$

$$g(x) \simeq \mathrm{G}^{\mathrm{T}}\phi(x), \tag{7}$$

we have

$$y''(x) \simeq \mathrm{C}^{\mathrm{T}}(\mathrm{D}^1)^2\phi(x), \tag{8}$$

$$y'(x) \simeq \mathrm{C}^{\mathrm{T}}\mathrm{D}^1\phi(x), \tag{9}$$

employing equations (6)–(9) the residual $\Re(x)$ for Eq. (1) can be written as

$$\Re(x) \simeq \mathrm{C}^{\mathrm{T}}(\mathrm{D}^1)^2\phi(x) + p(x)\mathrm{C}^{\mathrm{T}}\mathrm{D}^1\phi(x) + q(x)\left(\mathrm{C}^{\mathrm{T}}\phi(x)\right) - \mathrm{G}^{\mathrm{T}}\phi(x). \tag{10}$$

We find an approximate solution, namely Bernstein series solution, of (1) as

$$y_m(x) = \sum_{i=0}^{m} c_i B_{i,m}(x) = \mathrm{C}^{\mathrm{T}}\phi(x), \tag{11}$$

such that $y_m(x)$ satisfies (1) on the collocation nodes $0 < x_0 < x_1 <, \ldots, < x_m < R$.
Here, $B_{i,m}$, $0 \leq i \leq m$ are Bernstein polynomial, we substitute this collocation nodes in Eq. (10).
We can find the collocation nodes by applying Chebyshev roots

$$x_i = \frac{1}{2} + \frac{1}{2\cos((2i+1)\frac{\pi}{2n})}, \quad i = 0, 1, \ldots, m-1. \tag{12}$$

Also we use another one as a special, by intersect between $B_{i,m}(x)$ and $B_{i-2,m}(x)$, $i = 2, 3, \ldots, m-2$, then we have $(m-2)$ points call collocation points also substitute this points wich is also substituted in Eq. (10).

4. Error Analysis and Estimation of the Absolute Error

In this Section, we will give error analysis of the method. First, residual correction procedure which may estimate absolute error will be given for the problem. Second, an error analysis will be introduced to get bound the absolute error. Let $y_m(x)$ and $y(x)$ be the approximate solution and the exact solution of (1), respectively. The following procedure, residual correction First, we defined the error e_m as $e_m = y(x) - y_m(x)$, so $|e_m| = |y(x) - y_m(x)|$ as an absolute error. Now, we substitute our approximate solution $y_m(x)$ in Eq. (1) then we have the term

$$R := y''_m(x) + p(x)y'_m(x) + q(x)y_m(x) - g(x), \tag{13}$$

by subtracting R from both side of Eq. (1), we obtained the equation:

$$y''(x) + p(x)y'(x) + q(x)y(x) - R = g(x) - R, \tag{14}$$

when we substitute R in Eq. (14), we have

$$y''(x) - y''_m(x) + p(x)(y'(x) - y'_m(x)) + q(x)(y(x) - y_m(x)) + g(x) = g(x) - R, \tag{15}$$

then we obtain this equation and solve it by Bernstein polynomials of degree n, where $n > m$.

$$e''_m(x) + p(x)e'_m(x) + q(x)e_m(x) = -R, \tag{16}$$

with the conditions

$$e(0) = 0, \quad e(1) = 0, \quad \text{or} \quad e'(0) = 0, \quad e(0) = 0. \tag{17}$$

Let E_n be an approximate solution of (16), then $|E_n|$ is estimation of absolute error.

Corollary 1. *If y_m is Bernstein series solution to (1) and E_n also an approximate solution for (16) then, $y_m(x) + E_n$ is the corrected approximate solution for (1). Moreover, we call for $|y(x) - (y_m(x) + E_n)|$ is corrected of absolute error.*

Corollary 2. *Let y_m and y_{m+1} be approximate solutions of (1), then we can find second analysis of error by using the triangle inequality,*

$$||y(x) - y_{m_1}(x)| - |y(x) - y_{m_2}(x)|| \leq |y_{m_1}(x) - y_{m_2}(x)|. \tag{18}$$

If the errors are not too close together, we can find a rough upper bound for the errors. We can test the upper bound as follows. If the error sequence is decreasing (or increasing), then

$$\begin{aligned} ||y(x) - y_{m+1}(x)| - |y(x) - y_m(x)|| &= (1-C)\,|y(x) - y_m(x)| \\ &\leq |y_{m+1}(x) - y_m(x)|, \quad 0 \leq C < 1. \end{aligned}$$

or

$$|y(x) - y_{m+1}(x)| < |y(x) - y_m(x)| \leq \frac{1}{1-C}\,|y_{m+1}(x) - y_m(x)| \qquad (19)$$

where

$$|y(x) - y_{m+1}(x)| = C\,|y(x) - y_m(x)|\,.$$

5. Numerical Experiments

To illustrate the effectiveness of the presented method, we shall consider the following examples of singular equations. We used the collocation nodes Chebyshev roots and the other one as a special to find the result.

Example 1.

Consider the singular boundary value problem, which has been considered by [1],

$$y''(x) + \mathrm{e}^{1/x} y'(x) + y(x) = 6x + x^3 + 3x^2\mathrm{e}^{1/x}, \qquad (20)$$

$$y(0) = 0, \quad y(1) = 1. \qquad (21)$$

The exact solution of this problem is $y(x) = x^3$. We solve this problem by using Bernstein polynomials with $m = 3$, we have nonlinear system of four equations. Solving this system we get

$$c_0 = 0, \quad c_1 = 0, \quad c_2 = 0, \quad c_3 = 1.$$

Thus, we can write

$$y_(x) = \mathrm{C}^{\mathrm{T}}\phi(x) = x^3.$$

Which is the exact solution.

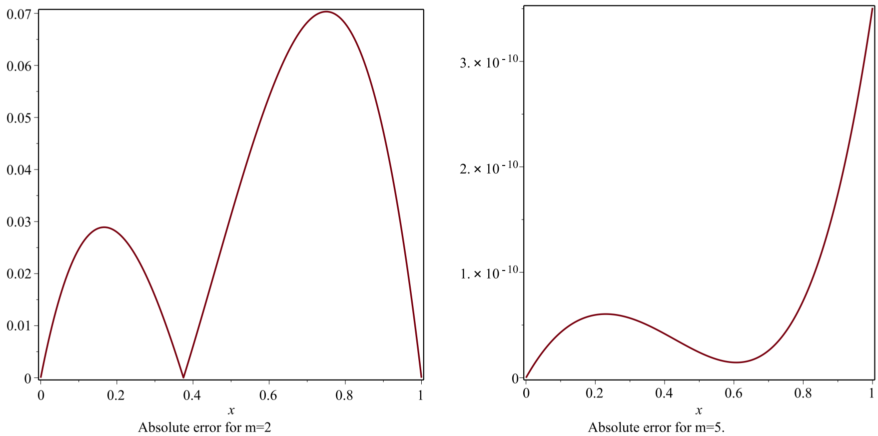

Figure 1: The absolute error to example 1, for the case $m = 2$ and $m = 5$.

If we apply the method with $m = 2$, we obtain $[c_0 = 0, c_1 = -.187439, c_2 = 1]$. The absolute errors for $m = 2$ are plotted in Figure 1.
Also we apply the method with $m = 5$, we obtain $[c_0 = 0, c_1 = 0.11661 \times 10^{-9}, c_2 = 0.7 \times 10^{-10}, c_3 = 0.10, c_4 = 0.40, c_5 = 1]$. The absolute errors for $m = 5$ are plotted in Figure 1. .

Example 2.

Consider the singular initial value problem BVPs, wich has been considered by [1, 11], arising in chemistry

$$y''(x) + \frac{1}{x}y'(x) - \frac{64}{49}\mathrm{e}^{y(x)} = 0, \quad 0 < x \leq 1, \tag{22}$$

$$y'(0) = 0, \quad y(1) = 0. \tag{23}$$

the exact solution of this problem is

$$y(x) = 2\ln 7 - 2\ln(8 - x^2).$$

Firstly, we can expand the non-linear term e^y in (22) by using the Taylor series as follows:

$$e^y = 1 + +y + \frac{y^2}{2!} + \frac{y^3}{3!} + \frac{y^4}{4!} + \frac{y^5}{5!}.$$

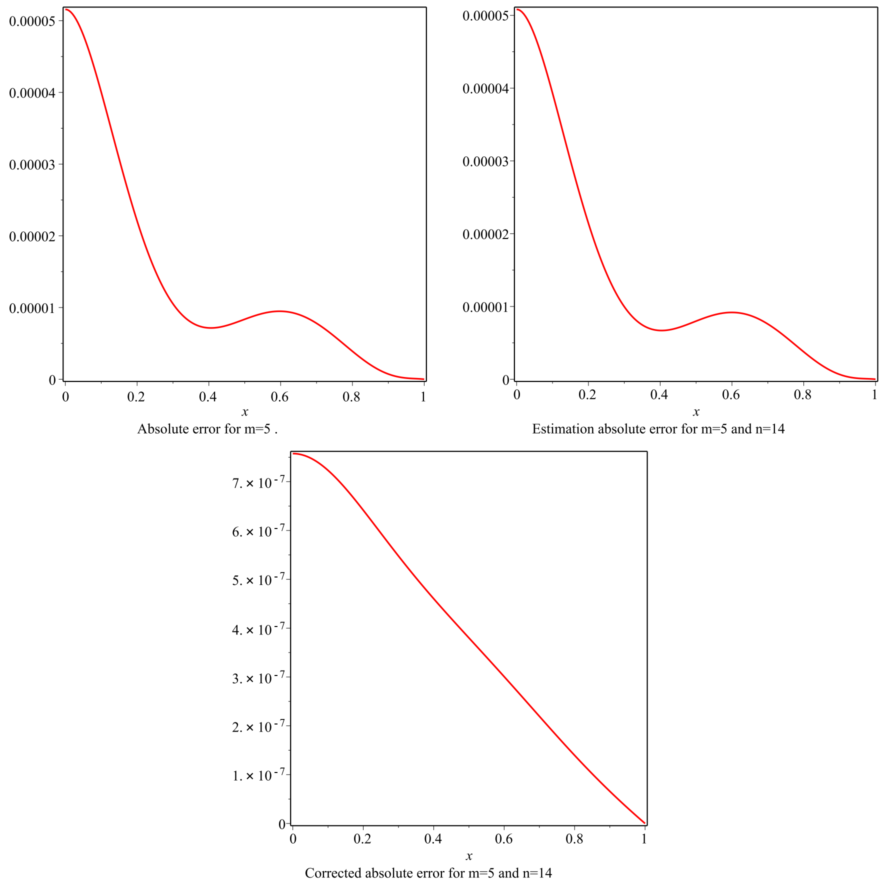

Figure 2: The absolute error, estimated absolute error and corrected absolute error to Example 2, for the case $m = 5, n = 14$.

We solve the problem for $m = 5$ and $m = 9$. The absolute errors and their estimations which are obtained by residual correction procedure for for $n = 14$ are plotted in Figure 2 and Figure 3. Also, the corrected approximate solutions $y_m(x) + E_n^*(x)$ are given in the same figures. We can see from the figures that residual correction procedure works well and the corrected approximate solutions are better than the approximate solutions. To find an upper bound by using (19), let us calculate the error sequences. Since the sequence is a decreasing sequence, the difference of the consecutive approximate solutions bounds both absolute errors approximately as in Figure 4

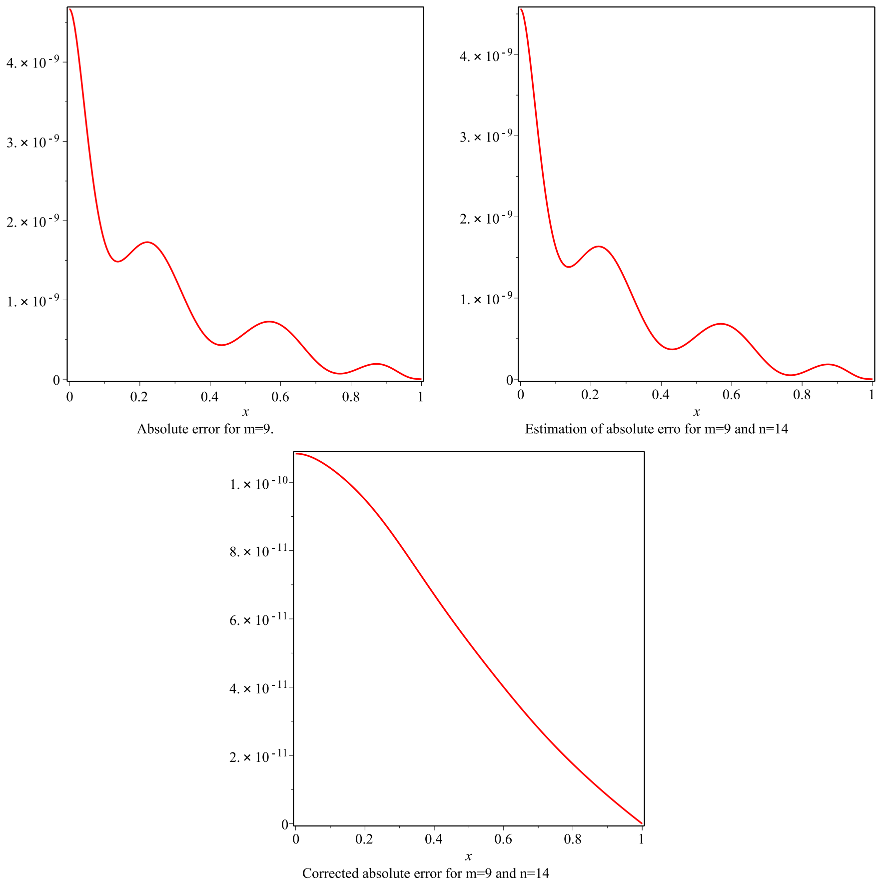

Figure 3: The absolute error, estimated absolute error and corrected absolute error to Example 2, for the case $m = 9, n = 14$.

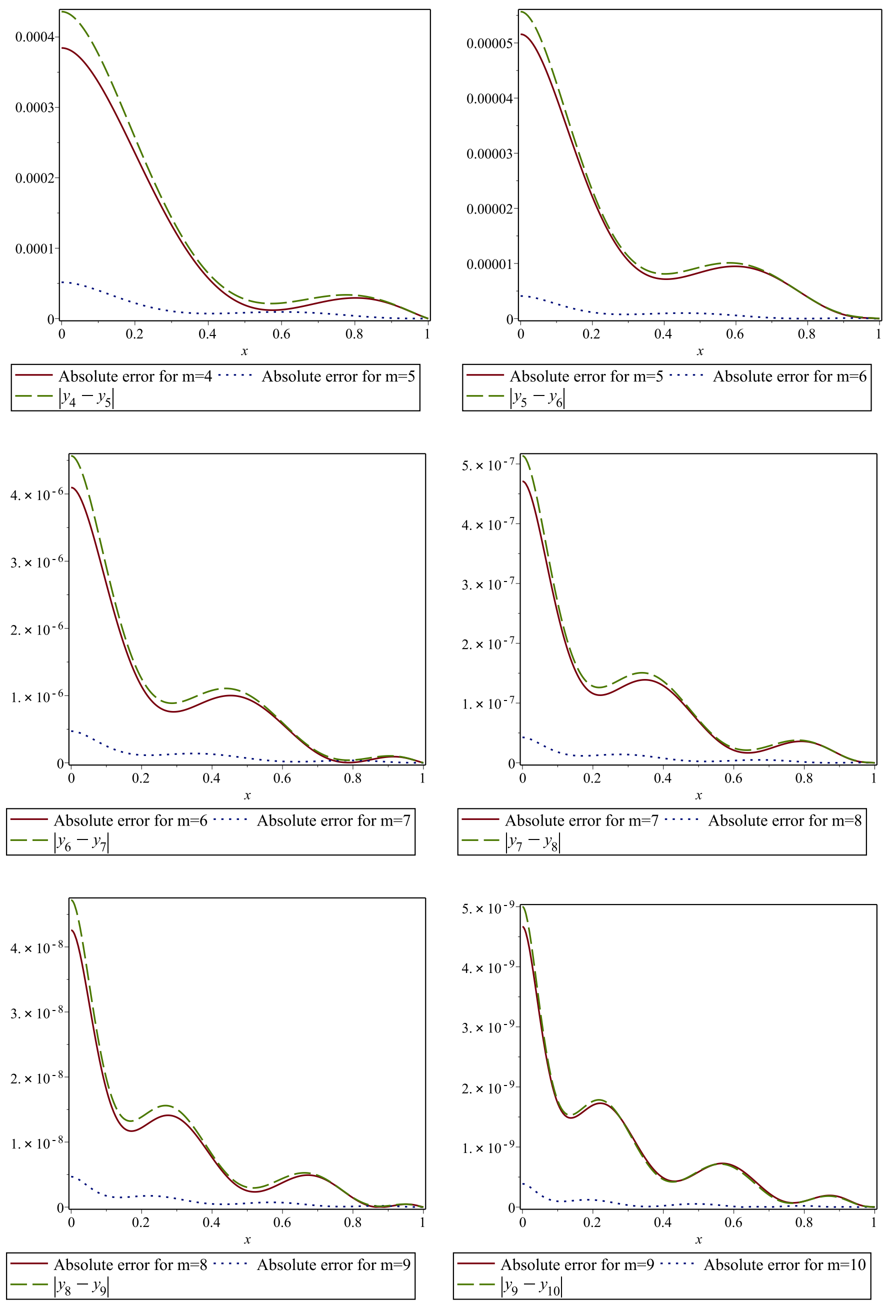

Figure 4: The absolute error of m, $m+1$ and $|y_m - y_{m+1}|$ of Example 2.

Example 3.

Consider the singular initial value problem, which has been considered by [1, 10],

$$y''(x) + \frac{2}{x}y'(x) + y^5(x) = 0, \quad x \in (0, 1], \tag{24}$$

$$y'(0) = 0, \quad y(1) = \frac{\sqrt{3}}{2}. \tag{25}$$

The exact solution of this problem is

$$y(x) = \frac{1}{\sqrt{1 + \frac{x^2}{3}}}$$

which describes the equilibrium of isothermal gas sphere, we solve this problem by applying the method with different value of $m = 5, 9$, we approximate the solution as

$$y_(x) = \sum_{i=0}^{10} c_i P_i(x) = \mathrm{C}^{\mathrm{T}}\phi(x).$$

Here, we have D^1 and D^2. By using Eq. (10), we have

$$\mathrm{C}^{\mathrm{T}}(\mathrm{D}^1)^2\phi(x) + \frac{2}{x}\,\mathrm{C}^{\mathrm{T}}\mathrm{D}^1\phi(x) + \left(\mathrm{C}^{\mathrm{T}}\phi(x)\right)^5 = 0. \tag{26}$$

By applying the method, we obtain the solutions for $m = 5$ and $m = 9$. Fig 5 and Fig 6 represent the absolute error and estimations of absolute error for $m = 5, 9$ and $n = 14$. Also, we plotted the corrected approximate solutions in the figures. Again, the procedure estimates the absolute errors very well and the corrected approximate solutions are more accurate than the approximate solutions. As a second error analysis, we obtain the error sequences as in Figure 7 for consecutive numbers. As seen in Figure 7, the absolute of e_m and e_{m+1} are bounded by $|y_m - y_{m+1}|$ approximately.

Example 4.

Consider this problem that is coincided by heat conduction model of the human head,

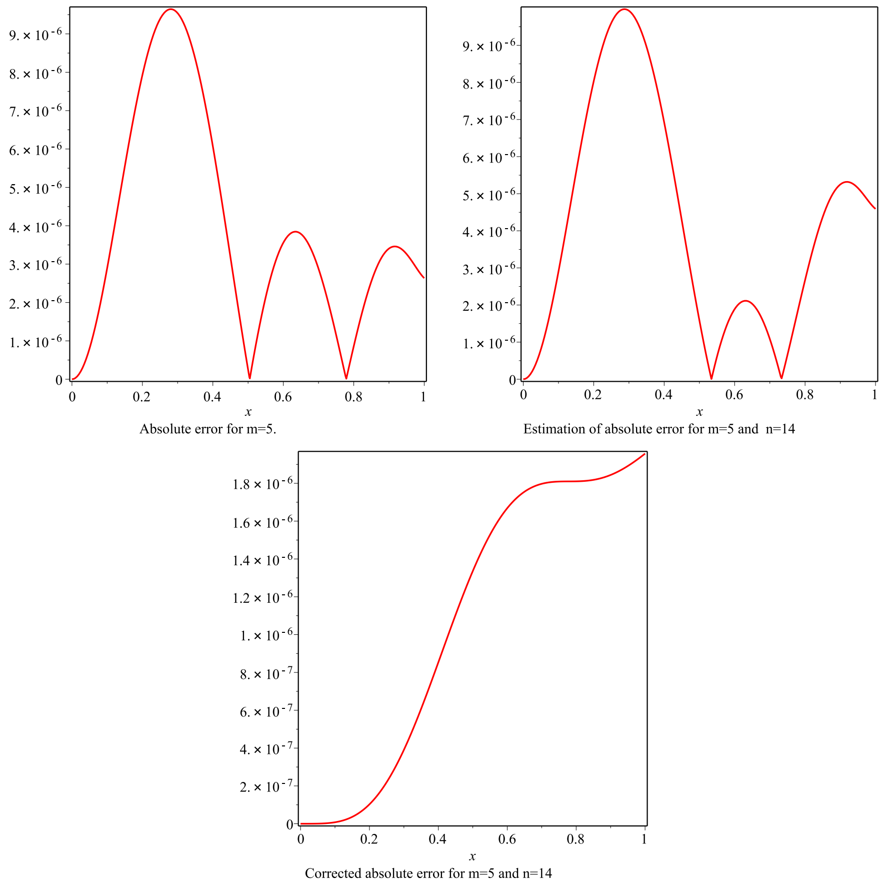

Figure 5: The absolute error, estimated absolute error and corrected absolute error to Example 3, for the case $n = 5, m = 14$.

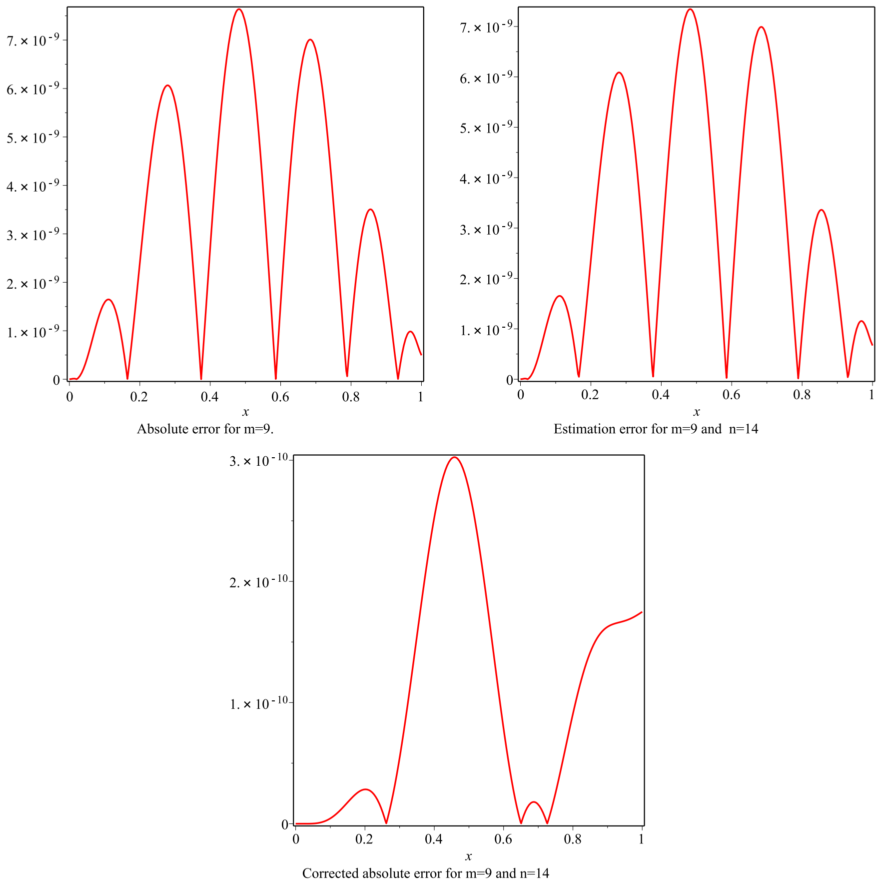

Figure 6: The absolute error, estimated absolute error and corrected absolute error to Example 3, for the case $n = 9, m = 14$.

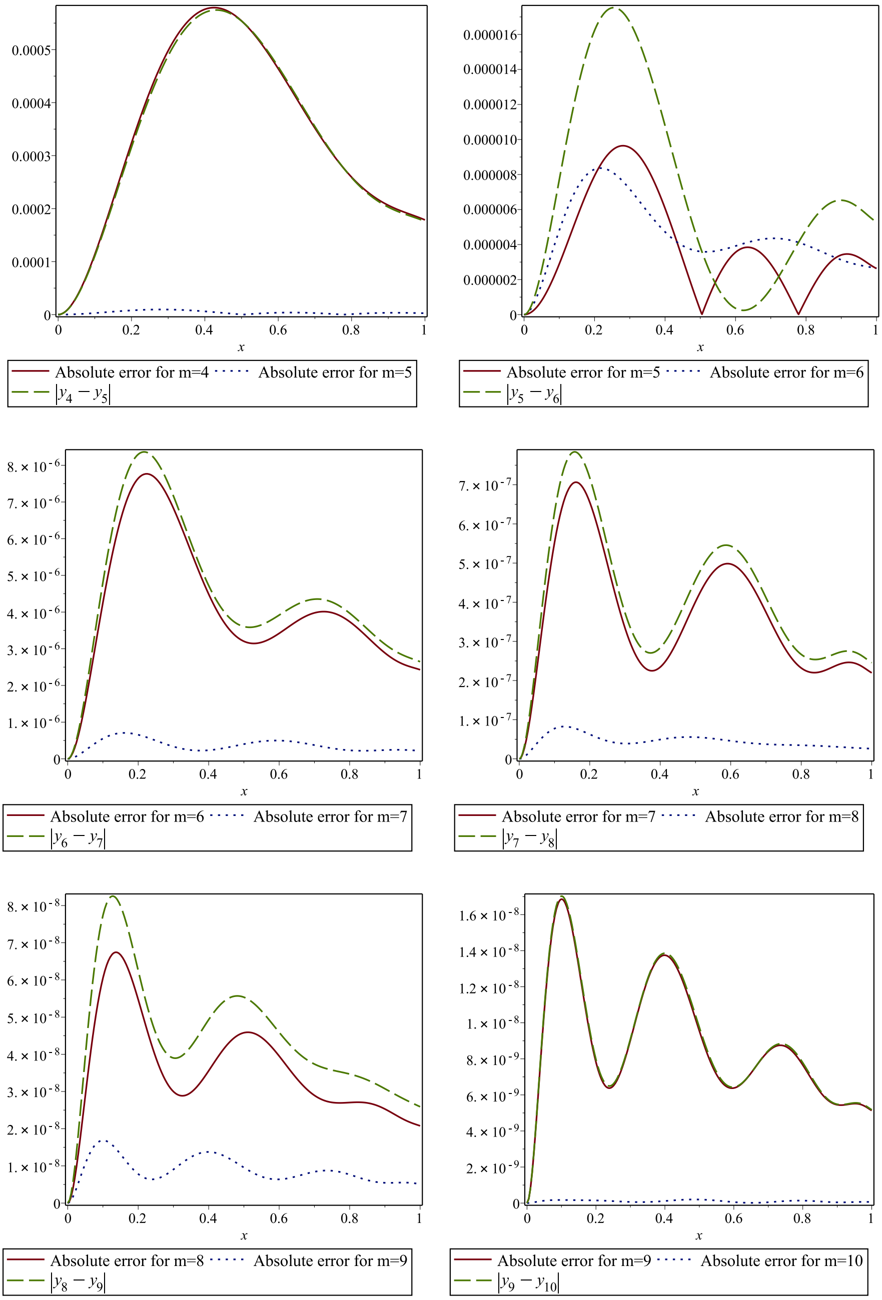

Figure 7: The absolute error of m, $m+1$ and $|y_m - y_{m+1}|$ of Example 3.

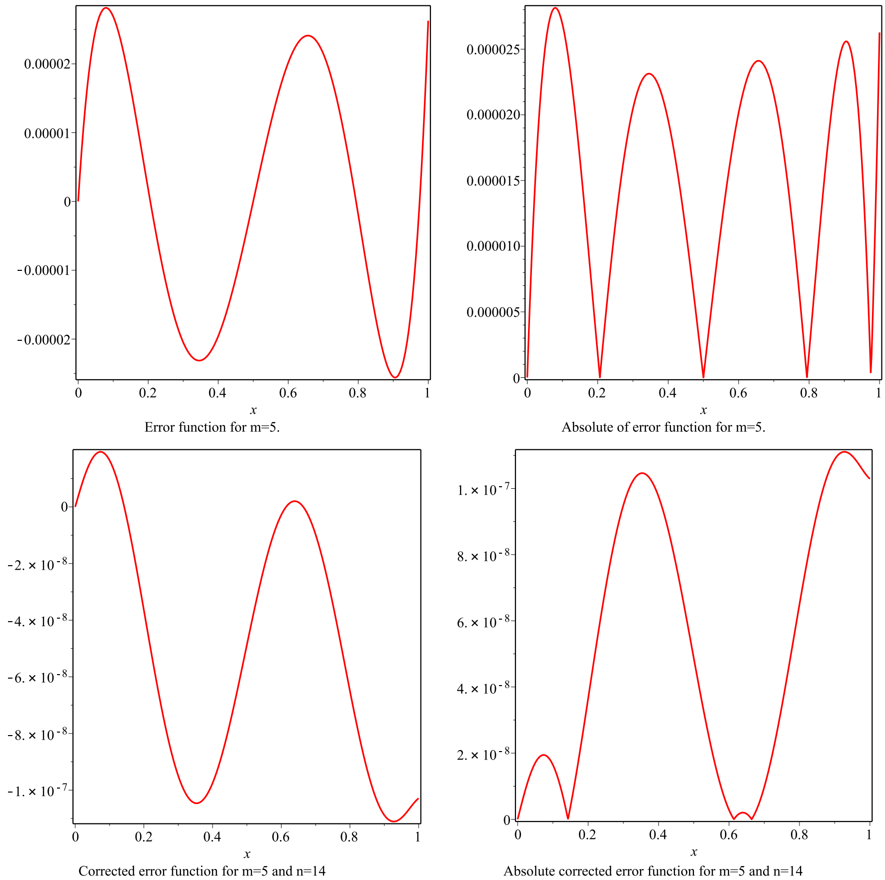

Figure 8: The error function, absolute of error function, corrected error function and absolute corrected error function of Example 4.

$$y''(x) + \frac{2}{x}y'(x) = -e^{-y}. \tag{27}$$

We consider the solution of this problem with conditions as follows:

$$y'(0) = 0, \quad y(1) + y'(1) = 0. \tag{28}$$

In this example, we don't have exact solution so we have to find error function, also we find corrected error function and absolute corrected error function, all results are shown in Fig 8.

6. Conclusions

In this paper, the Bernstein operational matrix of derivative is applied to solve a class of singular nonlinear differential equations. The method is given with some error analysis. Residual correction procedure is extended for the problem. In case of the error sequence is monotone decreasing (or increasing), one can specify an upper bound for the absolute error by (18) even if the exact solution is unknown. As seen from the examples, if the exact solution is a polynomial or piecewise polynomial, the method offers the exact solution. Residual correction procedure works very well for all examples. Moreover, more accurate results are obtained by using the procedure.

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