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POSITIVE SOLUTIONS FOR A COUPLED SYSTEM OF NONLINEAR SECOND ORDER EIGENVALUE PROBLEMS

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Abstract: In this work, we consider a system of coupled nonlinear second order eigenvalue problems. Under suitable conditions, existence of positive solutions are established, for determined eigenvalues, by the use of abstract fixed-point.

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1. Introduction

The existence and multiplicity of positive solutions for nonlinear second order BVP of ordinary differential equations have attracted many authors' attention and concern.

Johnny Henderson and H. Wang [6] considered a nonlinear second order eigenvalue problem

$$\left.\begin{array}{rcl} u''(t)+\lambda a(t)f(u(t)) &=& 0, \quad 0<t<1, \\ u(0)=u(1) &=& 0. \end{array}\right\} \qquad (1.)$$

They determined the value of λ (eigenvalue) for which there exist positive solutions to the BVP(1).

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Ling Hu and Lianglong Wang [7] studied the existence of multiple positive solutions for systems of nonlinear second order BVP.

$$\left.\begin{array}{rcl} -u''(x) &=& f(x,v), \\ -u''(x) &=& g(x,u), \\ \alpha u(0)-\beta u'(0) &=& 0, \ \ \gamma u(1)+\delta u'(1)=0, \\ \alpha v(0)-\beta v'(0) &=& 0, \ \ \gamma v(1)+\delta v'(1)=0. \end{array}\right\} \qquad (2.)$$

By the application of Krasnosel'skii [8] fixed-point theorem, the existence of positive solutions of BVP (2) is established. Motivated by the works of [6] and [7], this paper is concerned with the existence of positive solutions for the coupled system of nonlinear second order eigenvalue problem

$$\left.\begin{array}{l} u''(t)+\lambda a(t)f(v(t))=0, \\ v''(t)+\mu b(t)g(u(t))=0, \\ \alpha u(0)-\beta u'(0)=0, \ \ \gamma u(1)+\delta u'(1)=0, \\ \alpha v(0)-\beta v'(0)=0, \ \ \gamma v(1)+\delta v'(1)=0, \end{array}\right\} \qquad (3)$$

where $f,g\in C([0,1],I\!R_+)$, $a,b\in C([0,1],I\!R_+)$, $\alpha,\beta,\gamma,\delta\geq 0$ and $\rho=\alpha\gamma+\beta\gamma+\alpha\delta>0$.
A fixed-point theorem due to Krasnosel'skil [8] is applied to obtain positive solution x s of the BVP(3), for each λ,μ belonging to an open interval.

2. Preliminary Notes

Obviously, $(u,v)\in C^2[0,1]\times C^2[0,1]$ is the solution of the $BVP(3)$ if and only if $(u,v)\in C[0,1]\times C[0,1]$ is the solution of the system of integral equations

$$\left.\begin{array}{l} u(t)=\lambda \displaystyle\int_0^1 G(t,s)a(s)f(v(s))ds \\ v(t)=\mu \displaystyle\int_0^1 G(t,s)b(s)g(u(s))ds \end{array}\right\} \qquad (4)$$

where $G(t,s)$ is the Green's function defined as follows:

$$G(t,s)=\begin{cases} \dfrac{1}{\rho}(\gamma+\delta-\gamma t)(\beta+\alpha s), & 0\leq s\leq t\leq 1, \\ \dfrac{1}{\rho}(\beta+\alpha t)(\gamma+\delta-\gamma s), & 0\leq t\leq s\leq 1, \end{cases}$$

The integral equation (4) can be transferred to the nonlinear integral equation

$$u(t)=\lambda\int_0^1 G(t,s)a(s)f\left(\mu\int_0^1 G(s,r)b(r)g(u(r))dr\right)ds,\ t\in(0,1) \qquad (5)$$

Lemma 2.1. *(see [1], [3], [4], [7]): - The Green's function $G(t,s)$ satisfies*

(i) $G(t,s)\le G(s,s)$, *for* $0\le t,\ \ s\le 1$,

(ii) $G(t,s)\ge M\cdot G(s,s)$, *for* $\frac{1}{4}\le t\le\frac{3}{4}$, $0\le s\le 1$,

where

$$M=\min\left\{\frac{\gamma+4\delta}{4(\gamma+\delta)}\ ,\ \frac{\alpha+4\beta}{4(\alpha+\beta)}\right\}<1.$$

The proof of this lemma is standard and omitted.

Definition 2.2. The values of λ,μ for which there exist positive solutions to the BVP(3) are called eigenvalues and the corresponding solutions $u(t)>0,\ \ v(t)>0$ are called eigenfunctions.

Let $B=C[0,1]$ be a Banach space with norm $\|u\|=\max\limits_{0\le t\le 1}|u(t)|$. Define a cone K in B by

$$K=\left\{u\in B: u(t)\ge 0 \text{ and } \min_{\frac{1}{4}\le t\le\frac{3}{4}} u(t)\ge M\|u\|.\right\}.$$

Define an integral operator $A:K\longrightarrow B$ by

$$Au(t)=\lambda\int_0^1 G(t,s)a(s)f\left(\mu\int_0^1 G(s,r)b(r)g(u(r))dr\right)ds,\ u\in K \qquad (6)$$

Lemma 2.3. *(see [7]) If the operator A is defined as in (6), then $A: K\longrightarrow K$ is completely continuous.*

Proof. : For each $u\in K,\ Au\ge 0$ since the functions G,a,b,f and g are non-negative. Hence $Au(t)\ge 0$. From lemma (1) and for $u\in K$,

$$\begin{aligned} Au(t) &= \lambda\int_0^1 G(t,s)a(s)f\left(\mu\int_0^1 G(s,r)b(r)g(u(r))dr\right)ds \\ &\le \lambda\int_0^1 G(s,s)a(s)f\left(\mu\int_0^1 G(s,r)b(r)g(u(r))dr\right)ds \end{aligned}$$

By the non-negativity of the functions G, a, b, f and g, we have

$$\|Au\| \leq \lambda \int_0^1 G(s,s)a(s)f\left(\mu \int_0^1 G(s,r)b(r)g(u(r))dr\right)ds \tag{7}$$

Also, for $u \in K$ and for $\frac{1}{4} \leq t \leq \frac{3}{4}$, we have

$$\begin{aligned}
\min_{\frac{1}{4}\leq t\leq \frac{3}{4}} Au &= \min_{\frac{1}{4}\leq t\leq \frac{3}{4}} \lambda \int_0^1 G(t,s)a(s)f\left(\mu \int_0^1 G(s,r)b(r)g(u(r))dr\right)ds \\
&\geq \lambda M \int_0^1 G(s,s)a(s)f\left(\mu \int_0^1 G(s,r)b(r)g(u(r))dr\right)ds \\
&\geq M\|Au\|.
\end{aligned}$$

Hence, $Au \in K$ and consequently $A(K) \subset K$.
Since the functions G, a, b, f and g are continuous, it follows that $A : K \longrightarrow K$ is completely continuous. This completes the proof.

□

From the above arguments, we know that the existence of positive solutions of the BVP(3) is equivalent to the existence of positive fixed points of the operator A in the cone K.

3. Main Results

We begin this section by stating the Krasnosel'skii fixed-point theorem which is also given in ([2], [5], [8]) for it important in establishing our main result.

Theorem 3.1. *Let B be a Banach Space and $K \subset B$ be a cone in B. Assume Ω_1, Ω_2 are open subsets of B such that $0 \in \Omega_1, \overline{\Omega}_1 \subset \Omega_2$. If $A : K \cap \left(\overline{\Omega}_2 \backslash \Omega_1\right) \longrightarrow K$ is a completely continuous operator such that either*

(i) $\|Au\| \leq \|u\|,\ u \in K \cap \partial\Omega_1$ *and* $\|Au\| \geq \|u\|,\ u \in K \cap \partial\Omega_2$, *or*

(ii) $\|Au\| \geq \|u\|,\ u \in K \cap \partial\Omega_1$ *and* $\|Au\| \leq \|u\|,\ u \in K \cap \partial\Omega_2$,

then A has a fixed point in $K \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

Next, the following conditions are assumed true:

C_1. $f : [0,\infty) \longrightarrow [0,\infty)$ and $g : [0,\infty) \longrightarrow [0,\infty)$ are continuous.

C_2. $a : [0,1] \longrightarrow [0,\infty)$ and $b : [0,1] \longrightarrow [0,\infty)$ are continuous and $a(t) \neq 0,\ b(t) \neq 0$ on any subinterval of $[0,1]$.

C_3. $\lim\limits_{u\to 0^+}\dfrac{f(u)}{u}=f_0$ and $\lim\limits_{u\to 0^+}\dfrac{g(u)}{u}=g_0$.

C_4. $\lim\limits_{u\to \infty}\dfrac{f(u)}{u}=f_\infty$ and $\lim\limits_{u\to \infty}\dfrac{g(u)}{u}=g_\infty$.

Theorem 3.2. *Assume that conditions C_1, C_2, C_3 and C_4 are satisfied and let*

$$\left(M\int_{1/4}^{3/4}G\left(\frac{1}{2},s\right)a(s)ds\right)f_\infty>\left(\int_0^1 G(s,s)a(s)ds\right)f_0$$

and

$$\left(M\int_{1/4}^{3/4}G(r,r)b(r)dr\right)g_\infty>\left(\int_0^1 G(r,r)b(r)dr\right)g_0.$$

Then for each λ,μ satisfying

$$\frac{1}{\left(M\int_{1/4}^{3/4}G(\frac{1}{2},s)a(s)ds\right)f_\infty}<\lambda<\frac{1}{\left(\int_0^1 G(s,s)a(s)ds\right)f_0} \tag{8}$$

and

$$\frac{1}{\left(M\int_{1/4}^{3/4}G(r,r)b(r)dr\right)g_\infty}<\mu<\frac{1}{\left(\int_0^1 G(r,r)b(r)dr\right)g_0}, \tag{9}$$

there exists at least one positive solution (u,v) of the BVP (3) in K.

Proof. : Let λ,μ be given as in (8) and (9). Choose $\varepsilon>0$ such that

$$\frac{1}{\left(M\int_{1/4}^{3/4}G(\frac{1}{2},s)a(s)ds\right)(f_\infty-\varepsilon)}\le\lambda\le\frac{1}{\left(\int_0^1 G(s,s)a(s)ds\right)(f_0+\varepsilon)} \tag{10}$$

and

$$\frac{1}{\left(M\int_{1/4}^{3/4}G(r,r)b(r)dr\right)(g_\infty-\varepsilon)}\le\mu\le\frac{1}{\left(\int_0^1 G(r,r)b(r)dr\right)(g_0+\varepsilon)} \tag{11}$$

Now consider f_0 and g_0: There exists a constant $H_1 > 0$ such that
$f(u) \leq (f_0 + \varepsilon)u$, $g(u) \leq (g_0 + \varepsilon)u$, for $0 < u \leq H_1$.
For $u \in K$ with $\|u\| = H_1$, we have

$$Au(t) = \lambda \int_0^1 G(t,s)a(s)f\left(\mu \int_0^1 G(s,r)b(r)g(u(r))dr\right)ds.$$

$$\|Au\| \leq \lambda \int_0^1 G(s,s)a(s)(f_0+\varepsilon)\mu \int_0^1 G(s,r)b(r)g(u(r))drds.$$

$$\begin{aligned}
&\leq \lambda \int_0^1 G(s,s)a(s)(f_0+\varepsilon)ds \cdot \mu \int_0^1 G(r,r)b(r)g(u(r))dr. \\
&\leq \lambda \int_0^1 G(s,s)a(s)(f_0+\varepsilon)ds \cdot \mu \int_0^1 G(r,r)b(r)(g_0+\varepsilon)udr. \\
&\leq \lambda \int_0^1 G(s,s)a(s)(f_0+\varepsilon)ds \cdot \mu \int_0^1 G(r,r)b(r)(g_0+\varepsilon)\cdot H_1dr \\
&\leq \lambda \int_0^1 G(s,s)a(s)(f_0+\varepsilon)ds \cdot \mu \int_0^1 G(r,r)b(r)(g_0+\varepsilon)\|u\|dr.
\end{aligned}$$

Using (10) and (11), we have

$$\|Au\| \leq \|u\|.$$

If we set $\Omega_1 = \{u \in B : \|u\| < H_1\}$, then

$$\|Au\| \leq \|u\|, \text{ for } u \in (K \cap \partial\Omega_1).$$

Next, consider f_∞ and g_∞: There exists a constant $H_{2*} > 0$ such that
$f(u) \geq (f_\infty - \varepsilon)u$ and $g(u) \geq (g_\infty - \varepsilon)u$, for all $u \geq H_{2*}$.
Let $H_2 = \max\{2H_1, H_{2*}/M\}$.
Then for $u \in K$ with $\|u\| = H_2$, we have

$$\min_{\frac{1}{4}\leq t\leq\frac{3}{4}} u(t) \geq M\|u\| \geq H_{2*} \text{ and}$$

$$\begin{aligned}
Au(\tfrac{1}{2}) &= \lambda \int_0^1 G(\frac{1}{2},s)a(s)f\left(\mu \int_0^1 G(s,r)b(r)g(u(r))dr\right)ds. \\
&\geq \lambda \int_{1/4}^{3/4} G(\frac{1}{2},s)a(s)(f_\infty-\varepsilon)\mu \int_{1/4}^{3/4} G(s,r)b(r)g(u(r))drds. \\
&\geq \lambda \int_{1/4}^{3/4} G(\frac{1}{2},s)a(s)(f_\infty-\varepsilon)ds \cdot \mu m \int_{1/4}^{3/4} G(r,r)b(r)(g_\infty-\varepsilon)udr.
\end{aligned}$$

$$\geq \lambda \int_{1/4}^{3/4} G(\frac{1}{2},s)a(s)(f_\infty - \varepsilon)ds \cdot \mu M^2 \int_{1/4}^{3/4} G(r,r)b(r)(g_\infty - \varepsilon)\|u\|dr.$$
$$\geq \lambda M \int_{1/4}^{3/4} G(\frac{1}{2},s)a(s)(f_\infty - \varepsilon)ds \cdot \mu M \int_{1/4}^{3/4} G(r,r)b(r)(g_\infty - \varepsilon)\|u\|dr.$$

Using (10) and (11), we have

$$\left| Au\left(\frac{1}{2}\right)\right| \geq \|u\|.$$

Thus, $\|Au\| \geq \left| Au\left(\frac{1}{2}\right)\right| \geq \|u\| \Longrightarrow \|Au\| \geq \|u\|.$

If we set $\Omega_2 = \{u \in B : \|u\| < H_2\}$, then $\|Au\| \geq \|u\|$, for $u \in (K \cap \partial\Omega_2)$. By the first part of Theorem 1, it follows that the operator A has a fixed point in $K \cap (\overline{\Omega}_2 \backslash \Omega_1)$. □

Theorem 3.3. *Assume that conditions C_1, C_2, C_3 and C_4 are satisfied and let*

$$\left(M\int_{1/4}^{3/4} G\left(\frac{1}{2},s\right)a(s)ds\right) f_0 > \left(\int_0^1 G(s,s)a(s)ds\right) f_\infty$$

and

$$\left(M\int_{1/4}^{3/4} G(r,r)b(r)dr\right) g_0 > \left(\int_0^1 G(r,r)b(r)dr\right) g_\infty.$$

Then for each λ, μ satisfying

$$\frac{1}{\left(M\int_{1/4}^{3/4} G(\frac{1}{2},s)a(s)ds\right) f_0} < \lambda < \frac{1}{\left(\int_0^1 G(s,s)a(s)ds\right) f_\infty} \tag{12}$$

and

$$\frac{1}{\left(M\int_{1/4}^{3/4} G(r,r)b(r)dr\right) g_0} < \mu < \frac{1}{\left(\int_0^1 G(r,r)b(r)dr\right) g_\infty}, \tag{13}$$

there exists at least one positive solution (u, v) of the bvp (3) in K.

Proof. Let λ, μ be given as in (12) and (13). Choose $\varepsilon > 0$ such that

$$\frac{1}{\left(M\int_{1/4}^{3/4} G(\frac{1}{2},s)a(s)ds\right) (f_0 - \varepsilon)} \leq \lambda \leq \frac{1}{\left(\int_0^1 G(s,s)a(s)ds\right) (f_\infty + \varepsilon)} \tag{14}$$

and

$$\frac{1}{\left(M\int_{1/4}^{3/4} G(r,r)b(r)dr\right)(g_0-\varepsilon)} \le \mu \le \frac{1}{\left(\int_0^1 G(r,r)b(r)dr\right)(g_\infty+\varepsilon)}. \tag{15}$$

Consider f_0 and g_0: There exists a constant $H_1 > 0$ such that $f(u) \ge (f_0 - \varepsilon)u$ and $g(u) \ge (g_0 - \varepsilon)u, \quad \text{for } 0 < u \le H_1$.
For $u \in K$ with $\|u\| = H_1$, we have

$$\begin{aligned}
Au(\tfrac{1}{2}) &= \lambda \int_0^1 G(\frac{1}{2}, s)a(s)f\left(\mu\int_0^1 G(s,r)b(r)g(u(r))dr\right)ds. \\
&\ge \lambda \int_{1/4}^{3/4} G\left(\frac{1}{2}, s\right)a(s)(f_0-\varepsilon)\mu\int_{1/4}^{3/4} G(s,r)b(r)g(u(r))drds. \\
&\ge \lambda \int_{1/4}^{3/4} G\left(\frac{1}{2}, s\right)a(s)(f_0-\varepsilon)ds\cdot \mu M\int_{1/4}^{3/4} G(r,r)b(r)(g_0-\varepsilon)udr. \\
&\ge \lambda \int_{1/4}^{3/4} G\left(\frac{1}{2}, s\right)a(s)(f_0-\varepsilon)ds\cdot \mu M^2\int_{1/4}^{3/4} G(r,r)b(r)(g_0-\varepsilon)\|u\|dr. \\
&\ge \lambda M\int_{1/4}^{3/4} G\left(\frac{1}{2}, s\right)a(s)(f_0-\varepsilon)ds\cdot \mu M\int_{1/4}^{3/4} G(r,r)b(r)(g_0-\varepsilon)\|u\|dr.
\end{aligned}$$

Using (14) and (15), we have

$$\left|Au\left(\frac{1}{2}\right)\right| \ge \|u\|.$$

Thus, $\|Au\| \ge \left|Au\left(\frac{1}{2}\right)\right| \ge \|u\| \Longrightarrow \|Au\| \ge \|u\|$.

If we set $\Omega_1 = \{u \in B : \|u\| < H_1\}$, we have $\|Au\| \ge \|u\|$, for $u \in (K \cap \partial\Omega_1)$.

Next, consider f_∞ and g_∞: Then there exists a constant $H_{2*} > 0$ such that $f(u) \le (f_\infty + \varepsilon)u$ and $g(u) \le (g_\infty + \varepsilon)u$, for all $u \ge H_{2*}$.
There are two cases:

Case 1: Suppose f and g are bounded. Then there exists a constant $N > 0, \;\; N_0 > 0$ such that $f(u) \le N$ and $g(u) \le N_0$, for $0 < u < \infty$.
Let $H_2 = \max.\left\{2H_1, \;\; \lambda N\int_0^1 G(s,s)a(s)ds\right\}$. Then for $u \in K$ and $\|u\| = H_2$,

we have

$$\begin{aligned}
Au(t) &= \lambda\int_0^1 G(t,s)a(s)f\left(\mu\int_0^1 G(s,r)b(r)g(u(r))dr\right)ds.\\
\|Au\| &\leq \lambda\int_0^1 G(s,s)a(s)f\left(\mu\int_0^1 G(s,r)b(r)g(u(r))dr\right)ds.\\
&\leq \lambda\int_0^1 G(s,s)a(s)f\left(\mu\int_0^1 G(s,r)b(r)\cdot N_0dr\right)ds.\\
&\leq \lambda\int_0^1 G(s,s)a(s)f\left(\mu\int_0^1 G(s,r)b(r)\cdot N_0dr\right)ds.\\
&\leq \lambda\int_0^1 G(s,s)a(s)\cdot Nds\\
&\leq \lambda N\int_0^1 G(s,s)a(s)ds.\\
&\leq H_2=\|u\|.
\end{aligned}$$

$\Longrightarrow \|Au\|\leq\|u\|$.
If we set $\Omega_2=\{u\in B:\|u\|<H_2\}$, then $\|Au\|\leq\|u\|$, for $u\in(K\cap\partial\Omega_2)$.

Case 2: Suppose f and g are not bounded and let $H_2\geq\max\{2H_1,\ \ H_{2*}\}$ be chosen such that $H_{2*}\leq u\leq H_2$. Then for $u\in K$ with $\|u\|=H_2$, we have

$$Au(t)=\lambda\int_0^1 G(t,s)a(s)f\left(\mu\int_0^1 G(s,r)b(r)g(u(r))dr\right)ds.$$

$$\begin{aligned}
\|Au\| &\leq \lambda\int_0^1 G(s,s)a(s)f\left(\mu\int_0^1 G(s,r)b(r)g(u(r))dr\right)ds.\\
&\leq \lambda\int_0^1 G(s,s)a(s)\left(f_\infty+\varepsilon\right)\mu\int_0^1 G(s,r)b(r)g(u(r))drds.\\
&\leq \lambda\int_0^1 G(s,s)a(s)\left(f_\infty+\varepsilon\right)ds\cdot\mu\int_0^1 G(r,r)b(r)(g_\infty+\varepsilon)u\cdot dr.\\
&\leq \lambda\int_0^1 G(s,s)a(s)\left(f_\infty+\varepsilon\right)ds\cdot\mu\int_0^1 G(r,r)b(r)(g_\infty+\varepsilon)\|u\|dr.
\end{aligned}$$

Using (14) and (15), we have $\|Au\|\leq\|u\|$.
If we set $\Omega_2=\{u\in B:\|u\|<H_2\}$, then $\|Au\|\leq\|u\|\quad$ for $u\in(K\cap\partial\Omega_2)$.
Therefore, in either case,

$$\|Au\|\leq\|u\|,\ \text{ for }\ u\in(K\cap\partial\Omega_2).$$

By the second part of Theorem 3.1, the operator A has a fixed point in $K\cap(\overline{\Omega}_2\backslash\Omega_1)$. □

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