POSITIVE SOLUTIONS FOR A COUPLED SYSTEM OF NONLINEAR SECOND ORDER EIGENVALUE PROBLEMS

Moses B. Akorede¹, Peter O. Arawomo² §
¹,²Department of Mathematics
University of Ibadan
Ibadan, NIGERIA

Abstract: In this work, we consider a system of coupled nonlinear second order eigenvalue problems. Under suitable conditions, existence of positive solutions are established, for determined eigenvalues, by the use of abstract fixed-point.

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1. Introduction

The existence and multiplicity of positive solutions for nonlinear second order BVP of ordinary differential equations have attracted many authors’ attention and concern.


\[ \begin{align*}
    u''(t) + \lambda a(t)f(u(t)) &= 0, \quad 0 < t < 1, \\
    u(0) &= u(1) = 0.
\end{align*} \] (1)

They determined the value of \( \lambda \) (eigenvalue) for which there exist positive solutions to the BVP(1).
Ling Hu and Lianglong Wang [7] studied the existence of multiple positive solutions for systems of nonlinear second order BVP.

\[
\begin{aligned}
-u''(x) &= f(x, v), \\
-u''(x) &= g(x, u), \\
\alpha u(0) - \beta u'(0) &= 0, \quad \gamma u(1) + \delta u'(1) = 0, \\
\alpha v(0) - \beta v'(0) &= 0, \quad \gamma v(1) + \delta v'(1) = 0.
\end{aligned}
\]

(2)

By the application of Krasnosel’skii [8] fixed-point theorem, the existence of positive solutions of BVP (2) is established. Motivated by the works of [6] and [7], this paper is concerned with the existence of positive solutions for the coupled system of nonlinear second order eigenvalue problem

\[
\begin{aligned}
-u''(t) + \lambda a(t)f(v(t)) &= 0, \\
v''(t) + \mu b(t)g(u(t)) &= 0, \\
\alpha u(0) - \beta u'(0) &= 0, \quad \gamma u(1) + \delta u'(1) = 0, \\
\alpha v(0) - \beta v'(0) &= 0, \quad \gamma v(1) + \delta v'(1) = 0.
\end{aligned}
\]

(3)

where \( f, g \in C([0, 1], \mathbb{R}^+), \ a, b \in C([0, 1], \mathbb{R}^+), \ \alpha, \beta, \gamma, \delta \geq 0 \) and \( \rho = \alpha \gamma + \beta \gamma + \alpha \delta > 0 \).

A fixed-point theorem due to Krasnosel’skii [8] is applied to obtain positive solution \( x \) s of the BVP(3), for each \( \lambda, \mu \) belonging to an open interval.

2. Preliminary Notes

Obviously, \((u, v) \in C^2[0, 1] \times C^2[0, 1]\) is the solution of the BVP(3) if and only if \((u, v) \in C[0, 1] \times C[0, 1]\) is the solution of the system of integral equations

\[
\begin{aligned}
u(t) &= \lambda \int_0^1 G(t, s)a(s)f(v(s))ds \\
v(t) &= \mu \int_0^1 G(t, s)b(s)g(u(s))ds
\end{aligned}
\]

(4)

where \( G(t, s) \) is the Green’s function defined as follows:

\[
G(t, s) = \begin{cases}
\frac{1}{\rho} (\gamma + \delta - \gamma t)(\beta + \alpha s), & 0 \leq s \leq t \leq 1, \\
\frac{1}{\rho} (\beta + \alpha t)(\gamma + \delta - \gamma s), & 0 \leq t \leq s \leq 1,
\end{cases}
\]
The integral equation (4) can be transferred to the nonlinear integral equation

\[ u(t) = \lambda \int_0^1 G(t, s)a(s)f \left( \mu \int_0^1 G(s, r)b(r)g(u(r))dr \right) ds, \quad t \in (0, 1) \]  

(5)

**Lemma 2.1.** (see [1], [3], [4], [7]): - The Green’s function \(G(t, s)\) satisfies

(i) \(G(t, s) \leq G(s, s)\), for \(0 \leq t, s \leq 1\),

(ii) \(G(t, s) \geq M \cdot G(s, s)\), for \(\frac{1}{4} \leq t \leq \frac{3}{4}, \ 0 \leq s \leq 1\),

where \(M = \min \left\{ \frac{\gamma + 4\delta}{4(\gamma + \delta)}, \frac{\alpha + 4\beta}{4(\alpha + \beta)} \right\} < 1\).

The proof of this lemma is standard and omitted.

**Definition 2.2.** The values of \(\lambda, \mu\) for which there exist positive solutions to the BVP(3) are called eigenvalues and the corresponding solutions \(u(t) > 0, \ v(t) > 0\) are called eigenfunctions.

Let \(B = C[0, 1]\) be a Banach space with norm \(\|u\| = \max_{0 \leq t \leq 1} |u(t)|\). Define a cone \(K\) in \(B\) by

\[ K = \left\{ u \in B : u(t) \geq 0 \text{ and } \min_{\frac{1}{4} \leq t \leq \frac{3}{4}} u(t) \geq M\|u\| \right\}. \]

Define an integral operator \(A : K \rightarrow B\) by

\[ Au(t) = \lambda \int_0^1 G(t, s)a(s)f \left( \mu \int_0^1 G(s, r)b(r)g(u(r))dr \right) ds, \quad u \in K \]  

(6)

**Lemma 2.3.** (see [7]) If the operator \(A\) is defined as in (6), then \(A : K \rightarrow K\) is completely continuous.

**Proof.** : For each \(u \in K\), \(Au \geq 0\) since the functions \(G, a, b, f\) and \(g\) are non-negative. Hence \(Au(t) \geq 0\). From lemma (1) and for \(u \in K\),

\[ Au(t) = \lambda \int_0^1 G(t, s)a(s)f \left( \mu \int_0^1 G(s, r)b(r)g(u(r))dr \right) ds \]

\[ \leq \lambda \int_0^1 G(s, s)a(s)f \left( \mu \int_0^1 G(s, r)b(r)g(u(r))dr \right) ds \]
By the non-negativity of the functions $G, a, b, f$ and $g$, we have

$$\|Au\| \leq \lambda \int_{0}^{1} G(s, s)a(s)f \left( \mu \int_{0}^{1} G(s, r)b(r)g(u(r))dr \right) ds \quad (7)$$

Also, for $u \in K$ and for $\frac{1}{4} \leq t \leq \frac{3}{4}$, we have

$$\min_{\frac{1}{4} \leq t \leq \frac{3}{4}} Au = \min_{\frac{1}{4} \leq t \leq \frac{3}{4}} \lambda \int_{0}^{1} G(t, s)a(s)f \left( \mu \int_{0}^{1} G(s, r)b(r)g(u(r))dr \right) ds \geq \lambda M \int_{0}^{1} G(s, s)a(s)f \left( \mu \int_{0}^{1} G(s, r)b(r)g(u(r))dr \right) ds \geq M \|Au\|.$$ 

Hence, $Au \in K$ and consequently $A(K) \subset K$.

Since the functions $G, a, b, f$ and $g$ are continuous, it follows that $A : K \rightarrow K$ is completely continuous. This completes the proof.

From the above arguments, we know that the existence of positive solutions of the BVP(3) is equivalent to the existence of positive fixed points of the operator $A$ in the cone $K$.

### 3. Main Results

We begin this section by stating the Krasnosel’skii fixed-point theorem which is also given in ( [2], [5], [8] ) for it important in establishing our main result.

**Theorem 3.1.** Let $B$ be a Banach Space and $K \subset B$ be a cone in $B$. Assume $\Omega_1, \Omega_2$ are open subsets of $B$ such that $0 \in \Omega_1, \overline{\Omega_1} \subset \Omega_2$. If $A : K \cap (\overline{\Omega_2} \setminus \Omega_1) \rightarrow K$ is a completely continuous operator such that either

(i) $\|Au\| \leq \|u\|$, $u \in K \cap \partial \Omega_1$ and $\|Au\| \geq \|u\|$, $u \in K \cap \partial \Omega_2$, or

(ii) $\|Au\| \geq \|u\|$, $u \in K \cap \partial \Omega_1$ and $\|Au\| \leq \|u\|$, $u \in K \cap \partial \Omega_2$,

then $A$ has a fixed point in $K \cap (\overline{\Omega_2} \setminus \Omega_1)$.

Next, the following conditions are assumed true:

- $C_1. f : [0, \infty) \rightarrow [0, \infty)$ and $g : [0, \infty) \rightarrow [0, \infty)$ are continuous.

- $C_2. a : [0, 1] \rightarrow [0, \infty)$ and $b : [0, 1] \rightarrow [0, \infty)$ are continuous and $a(t) \neq 0$, $b(t) \neq 0$ on any subinterval of $[0, 1]$. 

\[ C_3. \lim_{u \to 0^+} \frac{f(u)}{u} = f_0 \text{ and } \lim_{u \to 0^+} \frac{g(u)}{u} = g_0. \]

\[ C_4. \lim_{u \to \infty} \frac{f(u)}{u} = f_\infty \text{ and } \lim_{u \to \infty} \frac{g(u)}{u} = g_\infty. \]

**Theorem 3.2.** Assume that conditions \( C_1, C_2, C_3 \) and \( C_4 \) are satisfied and let

\[
\left( M \int_{1/4}^{3/4} G\left( \frac{1}{2}, s \right) a(s) ds \right) f_\infty > \left( \int_0^1 G(s, s) a(s) ds \right) f_0
\]

and

\[
\left( M \int_{1/4}^{3/4} G(r, r) b(r) dr \right) g_\infty > \left( \int_0^1 G(r, r) b(r) dr \right) g_0.
\]

Then for each \( \lambda, \mu \) satisfying

\[
\frac{1}{\left( M \int_{1/4}^{3/4} G\left( \frac{1}{2}, s \right) a(s) ds \right) f_\infty } < \lambda < \frac{1}{\left( \int_0^1 G(s, s) a(s) ds \right) f_0 } \quad (8)
\]

and

\[
\frac{1}{\left( M \int_{1/4}^{3/4} G(r, r) b(r) dr \right) g_\infty } < \mu < \frac{1}{\left( \int_0^1 G(r, r) b(r) dr \right) g_0 }, \quad (9)
\]

there exists at least one positive solution \((u, v)\) of the BVP (3) in \( K \).

**Proof.** Let \( \lambda, \mu \) be given as in (8) and (9). Choose \( \varepsilon > 0 \) such that

\[
\frac{1}{\left( M \int_{1/4}^{3/4} G\left( \frac{1}{2}, s \right) a(s) ds \right) (f_\infty - \varepsilon) } \leq \lambda \leq \frac{1}{\left( \int_0^1 G(s, s) a(s) ds \right) (f_0 + \varepsilon) } \quad (10)
\]

and

\[
\frac{1}{\left( M \int_{1/4}^{3/4} G(r, r) b(r) dr \right) (g_\infty - \varepsilon) } \leq \mu \leq \frac{1}{\left( \int_0^1 G(r, r) b(r) dr \right) (g_0 + \varepsilon) } \quad (11)
\]
Now consider $f_0$ and $g_0$: There exists a constant $H_1 > 0$ such that $f(u) \leq (f_0 + \varepsilon)u$, $g(u) \leq (g_0 + \varepsilon)u$, for $0 < u \leq H_1$.

For $u \in K$ with $\|u\| = H_1$, we have

$$Au(t) = \lambda \int_0^1 G(t, s)a(s)f \left( \mu \int_0^1 G(s, r)b(r)g(u(r))dr \right) ds.$$  

$$\|Au\| \leq \lambda \int_0^1 G(s, s)a(s)(f_0 + \varepsilon)ds \cdot \mu \int_0^1 G(r, r)b(r)g(u(r))dr.$$  

$$\leq \lambda \int_0^1 G(s, s)a(s)(f_0 + \varepsilon)ds \cdot \mu \int_0^1 G(r, r)b(r)(g_0 + \varepsilon)dr.$$  

$$\leq \lambda \int_0^1 G(s, s)a(s)(f_0 + \varepsilon)ds \cdot \mu \int_0^1 G(r, r)b(r)(g_0 + \varepsilon)\cdot H_1dr.$$  

Using (10) and (11), we have

$$\|Au\| \leq \|u\|.$$  

If we set $\Omega_1 = \{u \in B : \|u\| < H_1\}$, then

$$\|Au\| \leq \|u\|, \text{ for } u \in (K \cap \partial \Omega_1).$$  

Next, consider $f_\infty$ and $g_\infty$: There exists a constant $H_2* > 0$ such that $f(u) \geq (f_\infty - \varepsilon)u$ and $g(u) \geq (g_\infty - \varepsilon)u$, for all $u \geq H_2*$.

Let $H_2 = \max\{2H_1, H_2*/M\}$.

Then for $u \in K$ with $\|u\| = H_2$, we have

$$\min_{\frac{1}{4} \leq t \leq \frac{3}{4}} u(t) \geq M\|u\| \geq H_2* \text{ and}$$

$$Au(\frac{1}{2}) = \lambda \int_{1/4}^{3/4} G(\frac{1}{2}, s)a(s)f \left( \mu \int_{1/4}^{3/4} G(s, r)b(r)g(u(r))dr \right) ds.$$  

$$\geq \lambda \int_{1/4}^{3/4} G(\frac{1}{2}, s)a(s)(f_\infty - \varepsilon)\mu \int_{1/4}^{3/4} G(s, r)b(r)g(u(r))drds.$$  

$$\geq \lambda \int_{1/4}^{3/4} G(\frac{1}{2}, s)a(s)(f_\infty - \varepsilon)ds \cdot \mu \int_{1/4}^{3/4} G(r, r)b(r)(g_\infty - \varepsilon)dr.$$
\[
\begin{align*}
\geq & \lambda \int_{1/4}^{3/4} G\left(\frac{1}{2}, s\right) a(s)(f_\infty - \varepsilon) ds \cdot \mu M^2 \int_{1/4}^{3/4} G(r, r) b(r)(g_\infty - \varepsilon) \|u\| dr.
\geq & \lambda M \int_{1/4}^{3/4} G\left(\frac{1}{2}, s\right) a(s)(f_\infty - \varepsilon) ds \cdot \mu \int_{1/4}^{3/4} G(r, r) b(r)(g_\infty - \varepsilon) \|u\| dr.
\end{align*}
\]

Using (10) and (11), we have
\[
\left| Au \left(\frac{1}{2}\right)\right| \geq \|u\|.
\]
Thus, \( \|Au\| \geq \left| Au \left(\frac{1}{2}\right)\right| \geq \|u\| \implies \|Au\| \geq \|u\| \).

If we set \( \Omega_2 = \{ u \in B : \|u\| < H_2 \} \), then \( \|Au\| \geq \|u\| \), for \( u \in (K \cap \partial \Omega_2) \).

By the first part of Theorem 1, it follows that the operator \( A \) has a fixed point in \( K \cap (\overline{\Omega_2} \setminus \Omega_1) \).

**Theorem 3.3.** Assume that conditions \( C_1, C_2, C_3 \) and \( C_4 \) are satisfied and let
\[
\left( M \int_{1/4}^{3/4} G\left(\frac{1}{2}, s\right) a(s) ds \right) f_0 > \left( \int_0^1 G(s, s) a(s) ds \right) f_\infty
\]
and
\[
\left( M \int_{1/4}^{3/4} G(r, r) b(r) dr \right) g_0 > \left( \int_0^1 G(r, r) b(r) dr \right) g_\infty.
\]
Then for each \( \lambda, \mu \) satisfying
\[
\frac{1}{\left( M \int_{1/4}^{3/4} G\left(\frac{1}{2}, s\right) a(s) ds \right) f_0} < \lambda < \frac{1}{\left( \int_0^1 G(s, s) a(s) ds \right) f_\infty}
\]
and
\[
\frac{1}{\left( M \int_{1/4}^{3/4} G(r, r) b(r) dr \right) g_0} < \mu < \frac{1}{\left( \int_0^1 G(r, r) b(r) dr \right) g_\infty},
\]
there exists at least one positive solution \( (u, v) \) of the bvp (3) in \( K \).

**Proof.** Let \( \lambda, \mu \) be given as in (12) and (13). Choose \( \varepsilon > 0 \) such that
\[
\frac{1}{\left( M \int_{1/4}^{3/4} G\left(\frac{1}{2}, s\right) a(s) ds \right) (f_0 - \varepsilon)} \leq \lambda \leq \frac{1}{\left( \int_0^1 G(s, s) a(s) ds \right) (f_\infty + \varepsilon)}
\]
(14)
and

\[ \frac{1}{M \int_{1/4}^{3/4} G(r,r)b(r)dr} (g_0 - \varepsilon) \leq \mu \leq \frac{1}{\int_0^1 G(r,r)b(r)dr} (g_\infty + \varepsilon). \]

(15)

Consider \( f_0 \) and \( g_0 \): There exists a constant \( H_1 > 0 \) such that \( f(u) \geq (f_0 - \varepsilon)u \)
and \( g(u) \geq (g_0 - \varepsilon)u \), for \( 0 < u \leq H_1 \).

For \( u \in K \) with \( \|u\| = H_1 \), we have

\[
Au\left(\frac{1}{2}\right) = \lambda \int_0^1 G\left(\frac{1}{2},s\right)a(s)f\left(\mu \int_0^1 G(s,r)b(r)g(u(r))dr\right)ds.
\]

\[ \geq \lambda \int_0^{3/4} G\left(\frac{1}{2},s\right)a(s)(f_0 - \varepsilon)ds \cdot \mu M \int_{1/4}^{3/4} G(r,r)b(r)(g_0 - \varepsilon)udr. \]

\[ \geq \lambda M \int_{1/4}^{3/4} G\left(\frac{1}{2},s\right)a(s)(f_0 - \varepsilon)ds \cdot \mu M \int_{1/4}^{3/4} G(r,r)b(r)(g_0 - \varepsilon)\|u\|dr. \]

Using (14) and (15), we have

\[ \left| Au\left(\frac{1}{2}\right) \right| \geq \|u\|. \]

Thus, \( \|Au\| \geq \left| Au\left(\frac{1}{2}\right) \right| \geq \|u\| \implies \|Au\| \geq \|u\|. \)

If we set \( \Omega_1 = \{u \in B: \|u\| < H_1\} \), we have \( \|Au\| \geq \|u\| \), for \( u \in (K \cap \partial\Omega_1) \).

Next, consider \( f_\infty \) and \( g_\infty \): Then there exists a constant \( H_{2*} > 0 \) such that \( f(u) \leq (f_\infty + \varepsilon)u \) and \( g(u) \leq (g_\infty + \varepsilon)u \), for all \( u \geq H_{2*} \).

There are two cases:

**Case 1:** Suppose \( f \) and \( g \) are bounded. Then there exists a constant \( N > 0, \ N_0 > 0 \) such that \( f(u) \leq N \) and \( g(u) \leq N_0 \), for \( 0 < u < \infty \).

Let \( H_2 = \max \left\{ 2H_1, \ \lambda N \int_0^1 G(s,s)a(s)ds \right\} \). Then for \( u \in K \) and \( \|u\| = H_2 \),
Using (14) and (15), we have

\[ Au(t) = \lambda \int_0^1 G(t,s) a(s) f \left( \mu \int_0^1 G(s,r) b(r) g(u(r))dr \right) ds. \]

\[ \|Au\| \leq \lambda \int_0^1 G(s,s) a(s) f \left( \mu \int_0^1 G(s,r) b(r) g(u(r))dr \right) ds. \]

\[ \leq \lambda \int_0^1 G(s,s) a(s) f \left( \mu \int_0^1 G(s,r) b(r) \cdot N_0 dr \right) ds. \]

\[ \leq \lambda \int_0^1 G(s,s) a(s) \cdot N ds \]

\[ \leq \lambda N \int_0^1 G(s,s) a(s) ds. \]

\[ \leq H_2 = \|u\|. \]

\[ \Longrightarrow \|Au\| \leq \|u\|. \]

If we set \( \Omega_2 = \{ u \in B : \|u\| < H_2 \} \), then \( \|Au\| \leq \|u\| \), for \( u \in (K \cap \partial \Omega_2) \).

**Case 2:** Suppose \( f \) and \( g \) are not bounded and let \( H_2 \geq \max\{2H_1, H_{2*} \} \) be chosen such that \( H_{2*} \leq u \leq H_2 \). Then for \( u \in K \) with \( \|u\| = H_2 \), we have

\[ Au(t) = \lambda \int_0^1 G(t,s) a(s) f \left( \mu \int_0^1 G(s,r) b(r) g(u(r))dr \right) ds. \]

\[ \|Au\| \leq \lambda \int_0^1 G(s,s) a(s) f \left( \mu \int_0^1 G(s,r) b(r) g(u(r))dr \right) ds. \]

\[ \leq \lambda \int_0^1 G(s,s) a(s) (f_\infty + \varepsilon) \mu \int_0^1 G(s,r) b(r) g(u(r))dr ds. \]

\[ \leq \lambda \int_0^1 G(s,s) a(s) (f_\infty + \varepsilon) ds \cdot \mu \int_0^1 G(r,r) b(r) (g_\infty + \varepsilon) u \cdot dr. \]

\[ \leq \lambda \int_0^1 G(s,s) a(s) (f_\infty + \varepsilon) ds \cdot \mu \int_0^1 G(r,r) b(r) (g_\infty + \varepsilon) \|u\| dr. \]

Using (14) and (15), we have \( \|Au\| \leq \|u\| \).

If we set \( \Omega_2 = \{ u \in B : \|u\| < H_2 \} \), then \( \|Au\| \leq \|u\| \) for \( u \in (K \cap \partial \Omega_2) \). Therefore, in either case,

\[ \|Au\| \leq \|u\|, \quad \text{for } u \in (K \cap \partial \Omega_2). \]

By the second part of Theorem 3.1, the operator A has a fixed point in \( K \cap (\overline{\Omega_2} \setminus \Omega_1) \).
References


