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STABILITY OF LIE HOMOMORPHISMS ON BANACH ALGEBRAS

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Abstract: In this paper we prove the Hyers-Ulam stability of special homomorphisms on some Banach algebras associated to a generalized functional equation.

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1. Introduction

A classical question in the theory of functional equations is: When is it true that a function which approximately satisfies a functional equation must be close to an exact solution of the equation? If the problem accepts a solution, we say that the equation is *stable*. For some sources see [1–4].

Definition 1.1. An n-ary algebra A is a complex linear space, endowed with an n-array product $(x_1, ..., x_n) \to [x_1...x_n]_A$ from A^n into A such that

$$\begin{aligned} [\,[x_1...x_n]_A\; y_1...y_{n-1}\,]_A &= [\,x_1\,[x_2...x_n\,y_1]_A\,y_2...y_{n-1}\,]_A = ... \\ &= [\,x_1...x_{n-1}\;[x_n\,y_1...y_{n-1}]_A\,]_A \end{aligned}$$

for all $x_1, ..., x_n, y_1, ..., y_{n-1}$ in A.

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Definition 1.2. A Banach algebra A, endowed with the Lie product $[x,y] := (xy - yx)/2$ on A, is called a Lie Banach algebra.

Definition 1.3. Assume that the algbras A and B are complex n-ary algebras. A linear mapping $H : A \to B$ is said to be an n-ary Lie homomorphism if

$$H(\,[x_1\ ...\ x_n]_A\,) = [\,H(x_1)...H(x_n)\,]_B,$$

$$H[\,[x_1\ ...\ x_n]_A, [y_1\ ...\ y_n]_A\,] = [\,H[x_1\ ...\ x_n]_A, H[y_1\ ...\ y_n]_A\,]$$

hold for all $x_1, ..., x_n$ in A.

We want to investigate the Hyers-Ulam stability of n-ary Lie homomorphisms.

2. Hyers-Ulam Stability of n-ary Homomorphisms

In this section, we want to investigate the Hyers-Ulam stability of n-ary Lie homomorphisms acting on n-ary Lie Banach algebras associated with the following generalized functional equation:

$$\mathrm{f}\left(\frac{\sum_{i=1}^{n} x_i}{n+1}\right) + \mathrm{f}\left(\frac{nx_1 - \sum_{i=2}^{n-1} x_i - (n+1)x_n}{n+1}\right) + \mathrm{f}\left(\frac{(n+1)x_1 + nx_n}{n+1}\right) - 2\,\mathrm{f}(x_1) = 0. \quad (1)$$

For a given mapping $g : A \to B$, we define

$$\begin{aligned} C_\mu \quad g \quad & (x_1, ..., x_n, y_1, ..., y_n, z_1, ..., z_n, w_1, ..., w_n\ ) \\ = \quad & \frac{1}{\mu}\, g\left(\frac{\mu\ x_2 + \sum_{i=1, i\neq 2}^{n} x_i}{n+1}\right) + g\left(\frac{n\, x_1 - \sum_{i=2}^{n-1} x_i - (n+1)\, x_n}{n+1}\right) \\ + \quad & g\left(\frac{(n+1)\, x_1 + n\, x_n}{n+1}\right) - 2g(x_1) + g\left([y_1\ ...\ y_n]_A\right) - [g(y_1)\ ...\ g(y_n)]_B \\ + \quad & g\big[[z_1\ ...\ z_n]_A, [w_1\ ...\ w_n]_a\big] - \big[g[z_1\ ...\ z_n]_A, g[w_1\ ...\ w_n]_A\big]. \end{aligned}$$

Theorem 2.1. *Assume that $(A, [\ ]_A)$ and $(B, [\ ]_B)$ are n-ary Lie Banach algebras and $n \geq 3$. Suppose that $f : A \longrightarrow B$ be an odd function satisfying:*

$$\|C_\mu f(x_1,...,x_n,y_1,...,y_n,z_1,...,z_n,w_1,...,w_n)\| \leq \varphi(x_1,...,x_n,y_1,...,y_n,z_1,...,z_n,w_1,...,w_n), \quad (2)$$

for all $\mu \in \mathbb{C}$ *with* $|\mu| = 1$ *and all* $x_1,...,x_n,y_1,...,y_n,z_1,...,z_n,w_1,...,w_n$ *in* A, *where* $\varphi : A^{4n} \longrightarrow [0,\infty)$ *is a function such that*

$$\varphi(2x_1,...,2x_n,2y_1,...,2y_n,2z_1,...,2z_n,2w_1,...,2w_n) \leq 2r\ \varphi(x_1,...,x_n,y_1,...,y_n,z_1,...,z_n,w_1,...,w_n), \quad (3)$$

for some $0 < r < 1$ *and all* $x_1,...,x_n,y_1,...,y_n,z_1,...,z_n,w_1,...,w_n \in A$. *If*

$$\lim_{m,k\to\infty} \frac{1}{2^m}\sum_{i=1}^{k}\frac{1}{2^i}\varphi(0,...,0,2^{i+m-1}x_2,...,2^{i+m-1}x_n,0,...,0) = 0, \quad (4)$$

for all $x_2,...,x_n \in A$, *then there exists a unique an n-ary Lie homomorphism* $H : A \longrightarrow B$ *such that*

$$\|f(x) - H(x)\| \leq \sum_{i=1}^{\infty}\frac{1}{2^i}\varphi(0,2^{i-1}(\frac{n+1}{n})x_2,...,2^{i-1}(\frac{n+1}{n})x_{n-1},2^{i-1}(\frac{n+1}{n})x,0,...,0), \quad (5)$$

for all $x,x_2,...,x_{n-1} \in A$ *satisfying* $(1+2n)\,x = -(x_2+...+x_{n-1})$.

Proof. In (2), put $\mu = 1$, $x_1 = y_1 = ... = y_n = z_1 = ... = z_n = w_1 = ... = w_n = 0$. Then we get

$$\|C_1 \mathrm{f}\,(0,x_2,...,x_{n-1},x_n,0,...,0)\| \leq \varphi\,(0,x_2,...,x_{n-1},x_n,0,...,0)$$

for all $x_2,...,x_{n-1},x_n \in A$ satisfying $(1+2n)\,x_n = -(x_2+...+x_{n-1})$. Hence

$$\|\mathrm{f}\,(\frac{n}{n+1}x_n) - \frac{1}{2}\mathrm{f}\,(2\,(\frac{n}{n+1}x_n))\| \leq \frac{1}{2}\varphi\,(0,x_2,...,x_n,0,...,0)$$

for all $x_2,...,x_{n-1},x_n \in A$ satisfying $(1+2n)\,x_n = -(x_2+...+x_{n-1})$. By proceeding in this way, we obtain

$$\|f(\frac{n}{n+1}x_n) - \frac{1}{2^k}\mathrm{f}(2^k(\frac{n}{n+1}x_n))\| \leq \sum_{i=1}^{k}\frac{1}{2^i}\varphi(0,2^{i-1}x_2,...,2^{i-1}x_n,0,...,0), \quad (6)$$

for all $x_2, ..., x_{n-1}, x_n \in A$ satisfying $(1+2\,n)\,x_n = -\,(x_2 + ... + x_{n-1})$. Replace x_n by $2^m x_n$ in (6) and then divide by 2^m, we get

$$\|\frac{1}{2^m}\mathrm{f}\,(2^m\,(\frac{n}{n+1}\,x_n)\,) - \frac{1}{2^{m+k}}\mathrm{f}\,(2^{m+k}\,(\frac{n}{n+1}\,x_n))\| \leq \frac{1}{2^m}\sum_{i=1}^{k}\frac{1}{2^i}\varphi\,(0, 2^{i+m-1}x_2, ..., 2^{i+m-1}x_n, 0, ..., 0), \quad (7)$$

for all integers m, k and all $x_2, ..., x_{n-1}, x_n \in A$ satisfying $(1+2\,n)\,x_n = -\,(x_2 + ... + x_{n-1})$. By the relations (4) and (7), the sequence $\{\frac{1}{2^m}\mathrm{f}\,(2^m\,(\frac{n}{n+1})\,x_n)\}_m$ is a Cauchy sequence in B, for all $x_n \in A$, so it is convergent. Thus there exists $H : A \to B$ such that $H(x) = \lim_{m\to\infty}\frac{1}{2^m}f(2^m x)$ for all $x \in A$. In (6), let $k \to \infty$, then we get

$$\|\mathrm{f}(\frac{n}{n+1}\,x_n) - H(\frac{n}{n+1}\,x_n)\| \leq \sum_{i=1}^{\infty}\frac{1}{2^i}\,\varphi(0, 2^{i-1}\,x_2, ..., 2^{i-1}\,x_n, 0, ..., 0)$$

for all $x_2, ..., x_{n-1}, x_n \in A$ satisfying $(1+2\,n)\,x_n = -\,(x_2 + ... + x_{n-1})$. Since $(1+2\,n)(\frac{n+1}{n})\,x_n = -\,((\frac{n+1}{n})\,x_2 + ... + (\frac{n+1}{n})\,x_{n-1})$, thus (5) holds. Now we show that $C_\mu\,H(x_1, ..., x_n, 0, ..., 0) = 0$ for all $x_1, ..., x_{n-1}, x_n \in A$ and all $\mu \in \mathbb{C}$ with $|\,\mu| = 1$. Note that $H(0) = 0$ and by using (3), we get

$$\|C_\mu\,H\,(x_1, ..., x_n, 0, ..., 0)\| \leq \lim_{m\to\infty} r^m \varphi(x_1, ..., x_n, 0, ..., 0) = 0$$

for all $X \in A^n$. In the relation $C_\mu\,H(x_1, ..., x_n, 0, ..., 0) = 0$, put $\mu = 1$, $x_1 = 0$, $w = \frac{n}{n+1}\,x_n$ and $z = \frac{1}{n+1}\,\sum_{i=2}^{n} x_i$, then we get $H(z+w) = H(z)+H(w)$. So, H is additive. Also, it is now easy to see that $H(\mu\,x) = \mu\,H(x)$ for all $\mu \in \mathbb{C}$ with $|\,\mu| = 1$ and all x in A. Thus H is indeed $\mathbb{C}$ -linear. To show that H is an n-ary homomorphism, by using (3), we prove that $C_\mu H(0, ..., 0, y_1, ..., y_n, 0, ..., 0) = 0$ for all $y_1, ..., y_n$ in A. We have

$$\|C_\mu\,H(0, ..., 0, y_1, ..., y_n, 0, ..., 0)\| \leq \lim_{m\to\infty} 2^{m-nm}\,r^m \varphi(0, ..., 0, y_1, ..., y_n, 0, ..., 0) = 0,$$

for all $Y \in A^n$. Similarly, $\|C_\mu\,H\,(0, ..., 0, z_1, ..., z_n, w_1, ..., w_n)\| = 0$, so indeed H is an n-ary Lie homomorphism. □

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