ON OPTIMAL DERIVATIVE ERROR BOUNDS
FOR LAGRANGE INTERPOLATION

F. Dubeau
Mathematics Department
University of Sherbrooke
2500 University Boul., Sherbrooke (Qc), CANADA, J1K 2R1

Abstract: We analyze the approximation of the derivative of a function by considering the derivative of its Lagrange interpolation polynomial. Optimal truncation errors bounds are established by a direct approach using Peano’s kernels and depend on the regularity of the function.

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1. Introduction

Let \( I = [a, b] \) and \( C^l(I) \) be the set of continuously differentiable functions up to order \( l \) on \( I \). Let \( p \in [1, +\infty] \), and let the set of absolutely continuous function on \( I \) be defined by

\[
AC^{l+1,p}(I) = \left\{ f \in C^l(I) \mid \begin{array}{l}
(a) \ f^{(l+1)} \in L^p(I), \ \text{and} \\
(b) \ f^{(l)}(s) = f^{(l)}(r) + \int_{r}^{s} f^{(l+1)}(y)dy, \ \forall r, s \in I
\end{array} \right\}.
\]

Let us assume that a set of nodes \( \{x_i\}_{i=0}^{n} \) such that \( a \leq x_0 < \cdots < x_i < \cdots < x_n \leq b \) is given. For a continuous function \( f(\cdot) \) on \( I \), let \( L(x) \) be its...
Lagrange interpolation polynomial satisfying
\[ L(x_i) = f(x_i) \quad \text{for} \quad i = 0, \ldots, n. \]

Using the Lagrange interpolation formula, we have
\[ f(x) \approx L(x) = \sum_{i=0}^{n} f(x_i)l_{n,i}(x) \quad (1) \]

where \( l_{n,i}(x) \) is the \( i \)th Lagrange interpolating polynomial of degree \( n \) given by
\[ l_{n,i}(x) = \prod_{j=0}^{n} \frac{(x-x_j)}{(x_i-x_j)} \quad \text{for} \quad i = 0, \ldots, n. \]

To approximate the \( k \)th derivative, we use
\[ f^{(k)}(x) \approx L^{(k)}(x) = \sum_{i=0}^{n} f(x_i)l_{n,i}^{(k)}(x) \quad (2) \]

for \( k = 1, \ldots, n \). The truncature error for the \( k \)th derivative process is defined by
\[ R^{(k)}_{n,l}(f;x) = f^{(k)}(x) - L^{(k)}(x) = f^{(k)}(x) - \sum_{i=0}^{n} f(x_i)l_{n,i}^{(k)}(x). \]

Since (1) and (2) are based on the Lagrange interpolation polynomial, they are exact for polynomials of degree \( \leq n \).

In this paper we would like to present optimal error bounds for \( R^{(k)}_{n,l}(f;x) \) which depends on the regularity of \( f(\cdot) \). Our results extend results presented in [4], [5], [3], and [6] for regular functions \( f(\cdot) \in AC^{n+1,\infty}(I) \).

2. Taylor’s Expansion

The Taylor’s expansion of \( f \in AC^{l+1,p}(I) \) of order \( l + 1 \) around \( c \in I \), see [2] and [7], is
\[ f(x) = \sum_{j=0}^{l} \frac{f^{(j)}(c)}{j!}(x-c)^j + \int_{a}^{b} f^{(l+1)}(y)K_{T,l}(y;x,c)dy, \quad (3) \]
where $K_{T,l}(y; x, c)$ is the kernel given by
\[
K_{T,l}(y; x, c) = \frac{1}{l!} \left[ (x - y)^l 1_{[c,b]}(y) + (-1)^{l+1} (y - x)^l 1_{[a,c]}(y) \right],
\]
This kernel is a piecewise polynomial function of degree $l$, and consequently $K_{T,l}(\cdot; x, c) \in L^\infty(I)$. In this expression, if $E$ is a set, then
\[
1_E(y) = \begin{cases} 
1 & \text{if } y \in E, \\
0 & \text{if } y \notin E.
\end{cases}
\]

3. Truncation Error

For any integer $l$ such that $k \leq l \leq n$, let $f(x) \in AC^{l+1,p}(I)$. Since the process (2) is exact for polynomials of degree $\leq n$, using a Taylor’s expansion (3) of order $l + 1$ for $f(\cdot)$, the truncation error becomes
\[
R^{(k)}_{n,l}(f; x) = \int_a^b f^{(l+1)}(y) K^{(k)}_{n,l}(y; x) dy,
\]
where $K^{(k)}_{n,l}(y; x, c)$ is the Peano kernel associated to the process, given by
\[
K^{(k)}_{n,l}(y; x, c) = R^{(k)}_{n,l}(K_{T,l}(\cdot; c); x) = K_{T,l-n}(y; x, c) - \sum_{i=0}^n L^{(k)}_{n,i}(x) K_{T,l}(y; x_i, c).
\]
Let $q \in [1, +\infty]$ be the conjugate of $p$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Using Holder’s inequality, we get from (4)
\[
|R^{(k)}_{n,l}(f; x)| \leq \left\| f^{(l+1)}(\cdot) \right\|_{p,I} \left\| K^{(k)}_{n,l}(\cdot; x, c) \right\|_{q,I}.
\]
Hence we have proved our main result which point out the dependance of the error in terms of the regularity of the function.

**Theorem 1.** Since $R^{(k)}_{n,l}(f; x) = 0$ for any polynomial of degree $\leq n$, then (5) holds for any $f \in AC^{l+1,p}(I)$ for $k \leq l \leq n$.

**Theorem 2.** For each $x \in I$, the bound given by (5) is the best one.
Proof. To show we can have equality in (4) and (5), we use a standard construction for $f^{(l+1)}(\cdot)$ given in [1]. Indeed, for $1 < p \leq \infty$ ($1 \leq q < \infty$), let $f(\cdot) \in AC^{l+1,p}(I)$ be such that

$$f^{(l+1)}(y) = \begin{cases} \left| K^{(k)}_{n,l}(y; x, c) \right|^{q-2} K^{(k)}_{n,l}(y; x, c) & \text{for } K^{(k)}_{n,l}(y; x, c) \neq 0, \\ 0 & \text{for } K^{(k)}_{n,l}(y; x, c) = 0. \end{cases}$$

Then we get an equality

$$\int_a^b f^{(l+1)}(y) K^{(k)}_{n,l}(y; x, c) dy = \left\| f^{(l+1)}(\cdot) \right\|_{p,I} \left\| K^{(k)}_{n,l}(\cdot; x, c) \right\|_{q,I},$$

because

$$\left\| f^{(l+1)}(\cdot) \right\|_{p,I} = \left\| K^{(k)}_{n,l}(\cdot; x, c) \right\|_{q,I}^{q-1},$$

and

$$\int_a^b f^{(l+1)}(y) K^{(k)}_{n,l}(y; x, c) dy = \left\| K^{(k)}_{n,l}(\cdot; x, c) \right\|_{q,I}^q.$$

For $p = 1$ ($q = \infty$), we suppose that $\left\| K^{(k)}_{n,l}(\cdot; x, c) \right\|_{\infty,I} > 0$. Let us consider $0 < \epsilon < \left\| K^{(k)}_{n,l}(\cdot; x, c) \right\|_{\infty,I}$, and let us define the set $A_\epsilon$ by

$$A_\epsilon = \left\{ y \in I \mid \left| K^{(k)}_{n,l}(y; x, c) \right| \geq \left\| K^{(k)}_{n,l}(\cdot; x, c) \right\|_{\infty,I} - \epsilon \right\}.$$

This set is such that its Lebesque’s measure is $0 < \mu(A_\epsilon) \leq b - a$. If we set

$$f^{(l+1)}(y) = \begin{cases} \left| K^{(k)}_{n,l}(y; x, c) \right|^{-1} K^{(k)}_{n,l}(y; x, c) & \text{for } y \in A_\epsilon, \\ 0 & \text{elsewhere}, \end{cases}$$

then

$$\left\| f^{(l+1)}(\cdot) \right\|_{1,I} \left( \left\| K^{(k)}_{n,l}(\cdot; x, c) \right\|_{\infty,I} - \epsilon \right) \leq \int_a^b f^{(l+1)}(y) K^{(k)}_{n,l}(y; x, c) dy \leq \left\| f^{(l+1)}(\cdot) \right\|_{1,I} \left\| K^{(k)}_{n,l}(\cdot; x, c) \right\|_{\infty,I},$$

because

$$\left\| f^{(l+1)}(\cdot) \right\|_{1,I} = \mu(A_\epsilon).$$

Since $\epsilon > 0$ can be arbitrary small, the result follows. \qed
Theorem 3. The kernel $K_{n,l}^{(k)}(y; x, c)$ does not depend on $c$

Proof. For $c$ and $\tilde{c}$ we have

$$\int_{a}^{b} f^{(l+1)}(y) \left[ K_{n,l}^{(k)}(y; x, c) - K_{n,l}^{(k)}(y; x, \tilde{c}) \right] dy = 0,$$

for any $f^{(l+1)}(\cdot) \in L^p(I)$ and $p \in [1, \infty]$. It follows that for each given fixed $x$

$$K_{n,l}^{(k)}(y; x, c) = K_{n,l}^{(k)}(y; x, \tilde{c})$$

almost everywhere with respect to $y$. Moreover, the two expression can be different only at a finite number of $y$ values because they are two piecewise polynomial functions.

Finally, since $\|K_{n,l}^{(k)}(\cdot; x, c)\|_{q,I}$ is a continuous function with respect to $x \in I$, its maximum value exists, is finite, and there exists at least one $x^* \in I$ such that

$$\max_{x \in I} \|K_{n,l}^{(k)}(\cdot; x, c)\|_{q,I} = \|K_{n,l}^{(k)}(\cdot; x^*, c)\|_{q,I}.$$

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References


