International Journal of Pure and Applied Mathematics
Volume 100 No. 1 2015, 75-80
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v100i1.7

ijpam.eu

ON OPTIMAL DERIVATIVE ERROR BOUNDS FOR LAGRANGE INTERPOLATION

F. Dubeau

Mathematics Department
University of Sherbrooke
2500 University Boul., Sherbrooke (Qc), CANADA, J1K 2R1

Abstract: We analyze the approximation of the derivative of a function by considering the derivative of its Lagrange interpolation polynomial. Optimal truncation errors bounds are established by a direct approach using Peano's kernels and depend on the regularity of the function.

AMS Subject Classification: 65D25, 65D05, 26A46
Key Words: numerical differentiation, Lagrange interpolation, Taylor's expansion, Peano's kernel

1. Introduction

Let $I = [a, b]$ and $C^l(I)$ be the set of continuously differentiable functions up to order l on I. Let $p \in [1, +\infty]$, and let the set of absolutely continuous function on I be defined by

$$AC^{l+1,p}(I) = \left\{ f \in C^l(I) \,\middle|\, \begin{array}{l} (a)\ f^{(l+1)} \in L^p(I), \quad and \\ (b)\ f^{(l)}(s) = f^{(l)}(r) \\ \qquad\qquad + \int_r^s f^{(l+1)}(y)dy,\ \forall r, s \in I \end{array} \right\}.$$

Let us assume that a set of nodes $\{x_i\}_{i=0}^n$ such that $a \le x_0 < \cdots < x_i < \cdots < x_n \le b$ is given. For a continuous function $f(\cdot)$ on I, let $L(x)$ be its

Received: December 3, 2014

Lagrange interpolation polynomial satisfying

$$L(x_i) = f(x_i) \quad \text{for} \quad i = 0, \ldots, n.$$

Using the Lagrange interpolation formula, we have

$$f(x) \approx L(x) = \sum_{i=0}^{n} f(x_i) l_{n,i}(x) \tag{1}$$

where $l_{n,i}(x)$ is the ith Lagrange interpolating polynomial of degree n given by

$$l_{n,i}(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{(x - x_j)}{(x_i - x_j)} \quad \text{for} \quad i = 0, \ldots, n.$$

To approximate the kth derivative, we use

$$f^{(k)}(x) \approx L^{(k)}(x) = \sum_{i=0}^{n} f(x_i) l_{n,i}^{(k)}(x) \tag{2}$$

for $k = 1, \ldots, n$. The troncature error for the kth derivative process is defined by

$$R_{n,l}^{(k)}(f; x) = f^{(k)}(x) - L^{(k)}(x) = f^{(k)}(x) - \sum_{i=0}^{n} f(x_i) l_{n,i}^{(k)}(x).$$

Since (1) and (2) are based on the Lagrange interpolation polynomial, they are exact for polynomials of degree $\leq n$.

In this paper we would like to present optimal error bounds for $R_{n,l}^{(k)}(f; x)$ which depends on the regularity of $f(\cdot)$. Our results extend results presented in [4], [5], [3], and [6] for regular functions $f(\cdot) \in AC^{n+1,\infty}(I)$.

2. Taylor's Expansion

The Taylor's expansion of $f \in AC^{l+1,p}(I)$ of order $l + 1$ around $c \in I$, see [2] and [7], is

$$f(x) = \sum_{j=0}^{l} \frac{f^{(j)}(c)}{j!}(x - c)^j + \int_a^b f^{(l+1)}(y) K_{T,l}(y; x, c) dy, \tag{3}$$

where $K_{T,l}(y;x,c)$ is the kernel given by

$$K_{T,l}(y;x,c) = \frac{1}{l!}\left[(x-y)_+^l \mathbf{1}_{[c,b]}(y) + (-1)^{l+1}(y-x)_+^l \mathbf{1}_{[a,c]}(y)\right],$$

This kernel is a piecewise polynomial function of degree l, and consequently $K_{T,l}(\cdot;x,c) \in L^\infty(I)$. In this expression, if E is a set, then

$$\mathbf{1}_E(y) = \begin{cases} 1 & \text{if} \quad y \in E, \\ 0 & \text{if} \quad y \notin E. \end{cases}$$

3. Truncation Error

For any integer l such that $k \le l \le n$, let $f(x) \in AC^{l+1,p}(I)$. Since the process (2) is exact for polynomials of degree $\le n$, using a Taylor's expansion (3) of order $l+1$ for $f(\cdot)$, the truncation error becomes

$$R_{n,l}^{(k)}(f;x) = \int_a^b f^{(l+1)}(y) K_{n,l}^{(k)}(y;x,c)dy, \tag{4}$$

where $K_{n,l}^{(k)}(y;x,c)$ is the Peano kernel associated to the process, given by

$$\begin{aligned} K_{n,l}^{(k)}(y;x,c) &= R_{n,l}^{(k)}(K_{T,l}(y;\cdot,c);x) \\ &= K_{T,l-k}(y;x,c) - \sum_{i=0}^{n} l_{n,i}^{(k)}(x) K_{T,l}(y;x_i,c). \end{aligned}$$

Let $q \in [1,+\infty]$ be the conjugate of p such that $\frac{1}{p}+\frac{1}{q}=1$. Using Holder's inequality, we get from (4)

$$\left|R_{n,l}^{(k)}(f;x)\right| \le \left\|f^{(l+1)}(\cdot)\right\|_{p,I} \left\|K_{n,l}^{(k)}(\cdot;x,c)\right\|_{q,I}. \tag{5}$$

Hence we have proved our main result which point out the dependance of the error in terms of the regularity of the function.

Theorem 1. *Since $R_{n,l}^{(k)}(f;x) = 0$ for any polynomial of degree $\le n$, then (5) holds for any $f \in AC^{l+1,p}(I)$ for $k \le l \le n$.* □

Theorem 2. *For each $x \in I$, the bound given by (5) is the best one.*

Proof. To show we can have equality in (4) and (5), we use a standard construction for $f^{(l+1)}(\cdot)$ given in [1]. Indeed, for $1 < p \leq \infty$ $(1 \leq q < \infty)$, let $f(\cdot) \in AC^{l+1,p}(I)$ be such that

$$f^{(l+1)}(y) = \begin{cases} \left|K_{n,l}^{(k)}(y;x,c)\right|^{q-2} K_{n,l}^{(k)}(y;x,c) & \text{for} \quad K_{n,l}^{(k)}(y;x,c) \neq 0, \\ \\ 0 & \text{for} \quad K_{n,l}^{(k)}(y;x,c) = 0. \end{cases}$$

Then we get an equality

$$\int_a^b f^{(l+1)}(y) K_{n,l}^{(k)}(y;x,c)dy = \left\|f^{(l+1)}(\cdot)\right\|_{p,I} \left\|K_{n,l}^{(k)}(\cdot;x,c)\right\|_{q,I},$$

because

$$\left\|f^{(l+1)}(\cdot)\right\|_{p,I} = \left\|K_{n,l}^{(k)}(\cdot;x,c)\right\|_{q,I}^{q-1},$$

and

$$\int_a^b f^{(l+1)}(y) K_{n,l}^{(k)}(y;x,c)dy = \left\|K_{n,l}^{(k)}(\cdot;x,c)\right\|_{q,I}^{q}.$$

For $p = 1$ $(q = \infty)$, we suppose that $\left\|K_{n,l}^{(k)}(\cdot;x,c)\right\|_{\infty,I} > 0$. Let us consider $0 < \epsilon < \left\|K_{n,l}^{(k)}(\cdot;x,c)\right\|_{\infty,I}$, and let us define the set A_ϵ by

$$A_\epsilon = \left\{ y \in I | \left|K_{n,l}^{(k)}(y;x,c)\right| \geq \left\|K_{n,l}^{(k)}(\cdot;x,c)\right\|_{\infty,I} - \epsilon \right\}.$$

This set is such that its Lebesque's measure is $0 < \mu(A_\epsilon) \leq b - a$. If we set

$$f^{(l+1)}(y) = \begin{cases} \left|K_{n,l}^{(k)}(y;x,c)\right|^{-1} K_{n,l}^{(k)}(y;x,c) & \text{for} \quad y \in A_\epsilon, \\ \\ 0 & \text{elsewhere}, \end{cases}$$

then

$$\begin{aligned} \left\|f^{(l+1)}(\cdot)\right\|_{1,I} \left(\left\|K_{n,l}^{(k)}(\cdot;x,c)\right\|_{\infty,I} - \epsilon \right) &\leq \int_a^b f^{(l+1)}(y) K_{n,l}^{(k)}(y;x,c)dy \\ &\leq \left\|f^{(l+1)}(\cdot)\right\|_{1,I} \left\|K_{n,l}^{(k)}(\cdot;x,c)\right\|_{\infty,I}, \end{aligned}$$

because

$$\left\|f^{(l+1)}(\cdot)\right\|_{1,I} = \mu(A_\epsilon).$$

Since $\epsilon > 0$ can be arbitrary small, the result follows. $\square$

Theorem 3. *The kernel* $K_{n,l}^{(k)}(y;x,c)$ *does not depend on* c

Proof. For c and $\tilde{c}$ we have

$$\int_a^b f^{(l+1)}(y)\left[K_{n,l}^{(k)}(y;x,c)-K_{n,l}^{(k)}(y;x,\tilde{c})\right]dy=0,$$

for any $f^{(l+1)}(\cdot)\in L^p(I)$ and $p\in[1,\infty]$. It follows that for each given fixed x

$$K_{n,l}^{(k)}(y;x,c)=K_{n,l}^{(k)}(y;x,\tilde{c})$$

almost everywhere with respect to y. Moreover, the two expression can be different only at a finite number of y values because they are two piecewise polynomial functions. □

Finally, since $\left\|K_{n,l}^{(k)}(\cdot;x,c)\right\|_{q,I}$ is a continuous function with respect to $x\in I$, its maximum value exists, is finite, and there exists at least one $x^*\in I$ such that

$$\max_{x\in I}\left\|K_{n,l}^{(k)}(\cdot;x,c)\right\|_{q,I}=\left\|K_{n,l}^{(k)}(\cdot;x^*,c)\right\|_{q,I}.$$

Acknowledgments

This work has been financially supported by an individual discovery grant from NSERC (Natural Sciences and Engineering Research Council of Canada).

References

[1] R.A. Adams, *Sobolev Spaces*, Academic Press, New York (1975).

[2] E. Asplund and L. Bungart, *A first course in integration*, Holt, Rinehart and Winston, New York (1966).

[3] M.S. Floater, Error formulas for divided difference expansions and numerical differentiation, *Journal of Approximation Theory*, **122** (2003), 1-9, **doi:** 10.1016/S0021-9045(03)00025-X.

[4] G.W. Howell, Derivative error bounds for Lagrange interpolation: An extension of Cauchy‘s bound for the error of Lagrange interpolation, *Journal of Approximation Theory*, **67** (1991), 164-173, **doi:** 10.1016/0021-9045(91)90015-3.

[5] A. Shadrin, Error bounds for Lagrange interpolation, *Journal of Approximation Theory*, **80** (1995), 25-49, **doi:** 10.1006/jath.1995.1003.

[6] A. Spitzbart and N. Macon, Numerical differentiation formulas, *The American Mathematical Monthly*, **64** (1957), 721-723, **doi:** 10.2307/2309751.

[7] L.L. Schumaker, *Spline functions basic theory*, Wiley, New York (1981).