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σ-β-OPEN SETS ON σ-STRUCTURES

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Abstract: The purpose of this paper is to study σ-β-open sets. Firstly, we define the σ-β-open sets and some basic properties of them. Secondly, we introduce the notions of σ-β-continuous functions and σ-β-open(σ-β-closed) functions, and investigate characterizations for such functions by using the σ-interior, σ-closure, σ-β-interior and σ-β-closure operators. Finally, we investigate the relationships among σ-continuity and σ-semicontinuity, σ-precontinuity and σ-β-continuity.

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1. Introduction

Császár [1] introduced the notions of generalized topology and generalized open sets as the following: Let X be a nonempty set and μ be a collection of subsets of X. Then μ is called a *generalized topology* (briefly GT) on X iff $\emptyset \in \mu$ and

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$G_i \in \mu$ for $i \in I \neq \emptyset$ implies $G = \cup_{i\in I} G_i \in \mu$. The elements of μ are called *μ-open* sets and the complements are called *μ-closed* sets. Kim and Min [2] introduced the notion of σ-structures which is an extended notion of generalized topology defined by Császár, and investigated the notion of σ-semiopen sets [3] in the spaces with σ-structures analogous to semi-open sets [5] introduced by Levine on a given topological space. We introduced the notions of σ-semi-interior and σ-semi-closure operators on a space with a σ-structure, and investigated the notions of σ-preopen sets [4], σ-pre-interior and σ-pre-closure operators on a space with a σ-structure. In particular, in [4] we showed that the family $\sigma PO(X)$ of all σ-preopen sets in (X, σ) is a generalized topology in sense of Császár and the family $\sigma SO(X)$ of all σ-semiopen subsets is strong but $\sigma PO(X)$ may not be strong in (X, σ). We also introduced the notion of σ-continuity [2], σ-semicontinuity [3] and σ-precontinuity [4] on sets with σ-structures, and studied some basic properties of such functions [2,3,4]. The purpose of this paper is to study σ-β-open sets which are the generalized σ-open sets of σ-semiopen sets and σ-preopen sets. Firstly, we define the σ-β-open sets and study some basic properties including the following things: (1) The family $\sigma\beta O(X)$ of all σ-β-open sets in (X, σ) is a strong σ-structure; (2) $\sigma\beta O(X)$ is a supratopology [6] but it is not always topology. Then we introduce the notions of σ-β-continuous functions and σ-β-open(σ-β-closed) functions, and investigate characterizations for such functions. Finally, we study the relationships among σ-continuity, σ-semicontinuity, σ-precontinuity and σ-β-continuity.

2. Preliminaries

Definition 2.1 ([2]). Let X be a nonempty set and $\sigma \subseteq 2^X$. Then σ is called a *σ-structure* on X if for $i \in I \neq \emptyset$, $U_i \in \sigma$ implies $\cup_{i\in I} U_i \in \sigma$. The elements of σ are called *σ-open sets* and the complements are called *σ-closed sets.*

Let X be a nonempty set, and let σ be a σ-structure on X. Then the two operators I_σ and C_σ [2] are defined as the following:

$I_\sigma(A) = \cup\{S \subseteq X : S \subseteq A, \quad S \text{ is } \sigma\text{-open }\}$;

$C_\sigma(A) = \cap\{F \subseteq X : A \subseteq F, \quad F \text{ is } \sigma\text{-closed }\}$.

Theorem 2.2 ([2]). *Let σ be a σ-structure on a nonempty set X and $A, B \subseteq X$. Then we have the following statements:*

(1) $I_\sigma(\emptyset) = \emptyset$; $\quad C_\sigma(X) = X$;

(2) $I_\sigma(A) \subseteq A \subseteq C_\sigma(A)$*;*

(3) if $A \subseteq B$*, then* $I_\sigma(A) \subseteq I_\sigma(B)$ *and* $C_\sigma(A) \subseteq C_\sigma(B)$*;*

(4) $I_\sigma(I_\sigma(A)) = I_\sigma(A)$*;* $C_\sigma(C_\sigma(A)) = C_\sigma(A)$*;*

(5)$I_\sigma(A) = X - C_\sigma(X - A)$*;* $C_\sigma(A) = X - I_\sigma(X - A)$*;*

(6) A *is* σ*-open iff* $A = I_\sigma(A)$ *for* $A \neq \emptyset$*;*

(7) A *is* σ*-closed iff* $A = C_\sigma(A)$ *for* $A \neq X$*.*

Let σ be a σ-structure on a nonempty set X and $A \subseteq X$. A subset A of X is called a *σ-semiopen set* [3] (resp., *σ-preopen set* [4]) if $A \subseteq C_\sigma(I_\sigma(A))$ (resp., $A \subseteq I_\sigma(C_\sigma(A))$). The complement of a σ-semiopen set is called a *σ-semiclosed set* (resp., *σ-preclosed set*). The family of all σ-semiopen (resp., *σ-preopen*) sets in X will be denoted by $\sigma SO(X)$ (resp., $\sigma PO(X)$).

3. Main Results

Definition 3.1. Let σ be a σ-structure on a nonempty set X and $A \subseteq X$. A subset A of X is called a *σ-β-open set* if $A \subseteq C_\sigma(I_\sigma(C_\sigma(A)))$. The complement of a σ-β-open set is called a *σ-β-closed set.* The family of all σ-β-open sets in X will be denoted by $\sigma\beta O(X)$.

In [4], we have shown that every σ-open set is both σ-preopen and σ-preopen but the converse is not be true always; moreover, there is no any relation of implication between σ-preopen sets and σ-semiopen sets. From these facts and Definition 3.1, obviously we have the following diagram:

$$\begin{array}{ccccc} & & \sigma\text{-semiopen} & & \\ & \nearrow & & \searrow & \\ \sigma\text{-open} & & & & \sigma\text{-}\beta\text{-open} \\ & \searrow & & \nearrow & \\ & & \sigma\text{-preopen} & & \end{array}$$

In the next example, we can show that a σ-β-open set may not be σ-preopen or σ-semiopen:

Example 3.2. (1) Let $X = \{a, b, c\}$ and $\sigma = \{\{a\}, \{b\}, \{a, b\}, X\}$ a σ-structure in X. For $A = \{b, c\}$, we know that $C_\sigma(I_\sigma(C_\sigma(A)) = \{b, c\}$ and $I_\sigma(C_\sigma(A)) = \{b\}$. So A is σ-β-open but it is not σ-preopen.

(2) Let $X = \{a, b, c\}$ and $\sigma = \{\{b, c\}, \{a, b\}, X\}$ a σ-structure in X. For $A = \{a, c\}$, $C_\sigma(I_\sigma(C_\sigma(A))) = X$ and $C_\sigma(I_\sigma(A)) = \emptyset$. So A is σ-β-open but not σ-semiopen.

Theorem 3.3. *Let σ be a σ-structure on a nonempty set X. Then we have the following:*

(1) $\emptyset$ is σ-β-open;

(2) X is σ-β-open;

(3) any union of σ-β-open sets is σ-β-open.

Proof. (1) Obvious.

(2) In case $\emptyset$ or $X \in \sigma$: We have $C_\sigma(X) = X$ or $I_\sigma(X) = X$. It implies that $X \subseteq C_\sigma(I_\sigma(C_\sigma(X)))$, and so X is σ-β-open.

In case $\emptyset, X \notin \sigma$: Obviously $C_\sigma(X) = X$ and $S = \cup_{U \in \sigma} U \neq X$; then $I_\sigma(C_\sigma(X)) = I_\sigma(X) = S \subset X$. Since there is no any σ-closed subset containing S, $C_\sigma(I_\sigma(C_\sigma(X))) = C_\sigma(S) = X$, that is, $X \subseteq C_\sigma(I_\sigma(C_\sigma(X)))$. Hence X is σ-β-open.

(3) Obvious. □

Let X be a nonempty set. A subclass $\sigma \subseteq 2^X$ is called *a supratopology* on X [6] if $\emptyset, X \in \sigma$ and σ is closed under arbitrary union. (X, σ) is called *a supratopological space*. Let σ be a σ-structure on a nonempty set X. The σ-structure σ is said to be *strong* [4] if $X \in \sigma$. Clearly, topology and supratopology are a kind of strong σ-structures. In [4], we showed that the family $\sigma SO(X)$ of all σ-semiopen subsets is strong but $\sigma PO(X)$ may not be strong in (X, σ).

Remark 3.4. Let X be any space with a σ-structure.

(1) From Theorem 3.3, it follows that the family $\sigma\beta O(X)$ of all σ-β-open sets is a strong σ-structure; moreover, it is supratopology:

(2) The intersection of any two σ-β-open subsets may not be σ-β-open as shown in the next example, so the family $\sigma\beta O(X)$ is not always a topology.

Example 3.5. In (2) of Example 3.2, consider two σ-β-open sets $A = \{a, c\}$ and $B = \{b, c\}$. Note that for $A \cap B = \{c\}$, $C_\sigma(I_\sigma(C_\sigma(A \cap B))) = C_\sigma(I_\sigma(C_\sigma(\{c\}))) = C_\sigma(I_\sigma(\{c\})) = C_\sigma(\emptyset) = \emptyset$. So the intersection $A \cap B$ of σ-β-open sets A and B is not σ-β-open.

Lemma 3.6. *Let σ be a σ-structure on a nonempty set X and $A \subseteq X$. Then A is a σ-β-closed set if and only if $I_\sigma(C_\sigma(I_\sigma(A))) \subseteq A$.*

Proof. Firstly, by Theorem 2.2, $X - I_\sigma(C_\sigma(I_\sigma(A))) = C_\sigma(X - C_\sigma(I_\sigma(A))) = C_\sigma(I_\sigma(X - I_\sigma(A))) = C_\sigma(I_\sigma(C_\sigma(X - A)))$. Let $I_\sigma(C_\sigma(I_\sigma(A))) \subseteq A$. Then $X - A \subseteq X - I_\sigma(C_\sigma(I_\sigma(A))) = C_\sigma(I_\sigma(C_\sigma(X - A)))$ and so $X - A$ is σ-β-open. Finally, we have that A is a σ-β-closed set if and only if $I_\sigma(C_\sigma(I_\sigma(A))) \subseteq A$. □

Definition 3.7. Let σ be a σ-structure on a nonempty set X. For a subset A of X, the σ-β-*closure* and the σ-β-*interior* of A, denoted by $\beta C_\sigma(A)$ and $\beta I_\sigma(A)$, respectively, are defined as the following:

$$\beta C_\sigma(A) = \cap\{F : A \subseteq F, F \text{ is } \sigma\text{-}\beta\text{-closed in } X\};$$

$$\beta I_\sigma(A) = \cup\{U : U \subseteq A, U \text{ is } \sigma\text{-}\beta\text{-open in } X\}.$$

Theorem 3.8. *Let σ be a σ-structure on a nonempty set X and $A, B \subseteq X$. Then we have the following statements:*

(1) $I_\sigma(A) \subseteq \beta I_\sigma(A) \subseteq A$ and $A \subseteq \beta C_\sigma(A) \subseteq C_\sigma(A)$.

(2) If $A \subseteq B$, then $\beta I_\sigma(A) \subseteq \beta I_\sigma(B)$ and $\beta C_\sigma(A) \subseteq \beta C_\sigma(B)$.

(3)A is σ-β-open iff $\beta I_\sigma(A) = A$.

(4)F is σ-β-closed iff $\beta C_\sigma(F) = F$.

(5) $\beta I_\sigma(\beta I_\sigma(A)) = \beta I_\sigma(A)$ and $\beta C_\sigma(\beta C_\sigma(A)) = \beta C_\sigma(A)$.

(6) $\beta C_\sigma(X - A) = X - \beta I_\sigma(A)$ and $\beta I_\sigma(X - A) = X - \beta C_\sigma(A)$.

Proof. It is easily obtained from Theorem 2.2 and Theorem 3.3. □

Theorem 3.9. *Let σ be a σ-structure on a nonempty set X and $A \subseteq X$. Then*

(1) $x \in \beta C_\sigma(A)$ if and only if $A \cap V \neq \emptyset$ for every σ-β-open set V containing x.

(2) $x \in \beta I_\sigma(A)$ if and only if there exists an σ-β-open set U such that $U \subseteq A$.

Proof. (1) Suppose that there is a σ-β-open set V containing x such that $A \cap V = \emptyset$. Then $X - V$ is a σ-β-closed set satisfying $A \subseteq X - V$ and $x \notin X - V$. Finally, $x \notin \beta C_\sigma(A)$.

The converse is obvious.

(2) Obvious. □

Definition 3.10. Let σ, σ' be σ-structures on X and Y, respectively. Then a function $f : (X, \sigma) \to (Y, \sigma')$ is said to be *σ-β-continuous* if for each x and each σ-open set V containing $f(x)$, there exists a σ-β-open set U containing x such that $f(U) \subseteq V$.

Let σ, σ' be σ-structures on X and Y, respectively. Then a function $f : X \to Y$ is said to be

(1) *σ-continuous* [2] if $f^{-1}(G) \in \sigma$ for every non empty set $G \in \sigma'$;

(2) *σ-semicontinuous* [3] if for each x and each σ-open set V containing $f(x)$, there exists a σ-semiopen set U containing x such that $f(U) \subseteq V$;

(3) *σ-precontinuous* [4] if for each x and each σ-open set V containing $f(x)$, there exists a σ-preopen set U containing x such that $f(U) \subseteq V$.

Then by Definition 3.10 and Remark 3.12 of [4], we have the diagram below:

σ-semicontinuity
↗ ↘
σ-continuity σ-β-continuity
↘ ↗
σ-precontinuity

Example 3.11. As in Example 3.2, we consider $X = \{a, b, c\}$, $\sigma_1 = \{\{a\}, \{b\}, \{a, b\}, X\}$ and $\sigma_2 = \{\{b, c\}, \{a, b\}, X\}$.

(1) Let us define a function $f : (X, \sigma_1) \to (X, \sigma_2)$ as $f(x) = x$ for $x \in X$. Then f is σ-β-continuous but not σ-precontinuous (See (1) of Example 3.2).

(2) Let us define a function $g : (X, \sigma_2) \to (X, \sigma_2)$ as $g(a) = a$; $g(b) = c$; $g(c) = b$. Then g is σ-β-continuous but not σ-semicontinuous (See (2) of Example 3.2).

Theorem 3.12. *Let $f : (X, \sigma_X) \to (Y, \sigma_Y)$ be a function between σ-structures σ_X and σ_Y. Then the following statements are equivalent:*

(1) f is σ-β-continuous.

(2) For each σ-open subset V in Y, $f^{-1}(V)$ is σ-β-open.

(3) For each σ-closed subset B in Y, $f^{-1}(B)$ is σ-β-closed.

(4) $f(\beta C_\sigma(A)) \subseteq C_\sigma(f(A))$ for $A \subseteq X$.

(5) $\beta C_\sigma(f^{-1}(B)) \subseteq f^{-1}(C_\sigma(B))$ for $B \subseteq Y$.

(6) $f^{-1}(I_\sigma(B)) \subseteq \beta I_\sigma(f^{-1}(B))$ for $B \subseteq Y$.

Proof. (1) $\Rightarrow$ (2) Let V be a σ-open set in Y. For each $x \in f^{-1}(V)$, there exists a σ-β-open set U_x containing x such that $x \in U_x \subseteq f^{-1}(V)$. It implies $f^{-1}(V) = \cup_{x \in f^{-1}(V)} U_x$, and so $f^{-1}(V)$ is σ-β-open.

(2) $\Leftrightarrow$ (3) Obvious.

(3) ⇒ (4) For $A \subseteq X$,

$$\begin{aligned} & f^{-1}(C_\sigma(f(A))) \\ &\quad = f^{-1}(\cap\{F \subseteq Y : f(A) \subseteq F \text{ and } F \text{ is } \sigma\text{-closed}\}) \\ &\quad = \cap\{f^{-1}(F) \subseteq X : A \subseteq f^{-1}(F) \text{ and } f^{-1}(F) \text{ is } \sigma\text{-}\beta\text{-closed}\} \\ &\quad \supseteq \cap\{K \subseteq X : A \subseteq K \text{ and } K \text{ is } \sigma\text{-}\beta\text{-closed}\} \\ &\quad = \beta C_\sigma(A) \end{aligned}$$

So we have $f(\beta C_\sigma(A)) \subseteq C_\sigma(f(A))$.

(4) ⇒ (5) For any $B \subseteq Y$, by (4), $f(\beta C_\sigma(f^{-1}(B))) \subseteq C_\sigma(f(f^{-1}(B))) \subseteq C_\sigma(B)$. This implies $\beta C_\sigma(f^{-1}(B)) \subseteq f^{-1}(C_\sigma(B))$.

(5) ⇒ (6) For $B \subseteq Y$,

$$\begin{aligned} f^{-1}(I_\sigma(B)) &= f^{-1}(Y - C_\sigma(Y - B)) \\ &= X - (f^{-1}(C_\sigma(Y - B))) \\ &\subseteq X - \beta C_\sigma(f^{-1}(Y - B)) \\ &= \beta I_\sigma(f^{-1}(B)). \end{aligned}$$

So we have (6).

(6) ⇒ (1) Let V be any σ-open set containing $f(x)$. Then $x \in f^{-1}(V) = f^{-1}(I_\sigma(V)) \subseteq \beta I_\sigma(f^{-1}(V))$. By Theorem 3.9, there exists a σ-β-open set U containing x such that $x \in U \subseteq f^{-1}(V)$. This implies that f is σ-β-continuous. □

Definition 3.13. Let σ, σ' be σ-structures on X and Y, respectively. Then a function $f : (X, \sigma) \to (Y, \sigma')$ is said to be *σ-β-open* (resp., *σ-β-closed*) if for each σ-open (resp., σ-closed) set V, $f(V)$ is σ-β-open (resp., σ-β-closed).

Theorem 3.14. *Let $f : (X, \sigma_X) \to (Y, \sigma_Y)$ be a function between σ-structures σ_X and σ_Y. Then the following statements are equivalent:*
(1) f is σ-β-open.
(2) $f(V) \subseteq C_\sigma(I_\sigma(C_\sigma(f(V))))$ for each σ-open set V in Y.
(3) $f(I_\sigma(B)) \subseteq C_\sigma(I_\sigma(C_\sigma(f(B))))$ for $B \subseteq Y$.

Proof. (1) ⇔ (2) For each σ-open set V in Y, since $f(V)$ is σ-β-open if and only if $f(V) \subseteq C_\sigma(I_\sigma(C_\sigma(f^{-1}(V))))$, the statement is obtained.

(1) ⇔ (3) Obvious. □

Theorem 3.15. *Let $f : (X, \sigma_X) \to (Y, \sigma_Y)$ be a function between σ-structures σ_X and σ_Y. Then the following statements are equivalent:*

(1) f is σ-β-closed.

(2) $I_\sigma(C_\sigma(I_\sigma(f(V)))) \subseteq f(V)$ for each σ-closed set V in Y.

(3) $I_\sigma(C_\sigma(I_\sigma(f(B)))) \subseteq f(C_\sigma(B))$ for $B \subseteq Y$.

Proof. It is easily obtained from Lemma 3.6 and Theorem 3.8.

□

Lemma 3.16. *Let σ be a σ-structure on a nonempty set X and $A \subseteq X$. Then we have the following:*

(1) $I_\sigma(C_\sigma(I_\sigma(A))) \subseteq I_\sigma(C_\sigma(I_\sigma(\beta C_\sigma(A)))) \subseteq \beta C_\sigma(A)$.

(2) $\beta I_\sigma(A) \subseteq C_\sigma(I_\sigma(C_\sigma(\beta I_\sigma(A)))) \subseteq C_\sigma(I_\sigma(C_\sigma(A)))$.

Proof. Obvious. □

Theorem 3.17. *Let $f : (X, \sigma_X) \to (Y, \sigma_Y)$ be a function between σ-structures σ_X and σ_Y. Then the following statements are equivalent:*

(1) f is σ-β-open.

(2) $f(V) \subseteq \beta I_\sigma(f(V))$ for each σ-open set V in Y.

(3) $f(I_\sigma(B)) \subseteq \beta I_\sigma(f(B))$ for $B \subseteq Y$.

Proof. (1) $\Leftrightarrow$ (2) By (3) of Theorem 3.8, $f(V)$ is σ-β-open if and only if $f(V) = \beta I_\sigma(f(V))$. So it is obtained.

(2) $\Rightarrow$ (3) Let $B \subseteq Y$. Then $I_\sigma(B)$ is σ-β-open and by (2), $f(I_\sigma(B)) \subseteq \beta I_\sigma(f(I_\sigma(B))) \subseteq \beta I_\sigma(f(B))$. So (3) is obtained.

(3) $\Rightarrow$ (1) For $B \subseteq Y$, by (3) and Lemma 3.16, $f(I_\sigma(B)) \subseteq \beta I_\sigma(f(B)) \subseteq C_\sigma(I_\sigma(C_\sigma(f(B))))$. So by (3) of Theorem 3.14, f is σ-β-open.

□

Theorem 3.18. *Let $f : (X, \sigma_X) \to (Y, \sigma_Y)$ be a function between σ-structures σ_X and σ_Y. Then the following statements are equivalent:*

(1) f is σ-β-closed.

(2) $\beta C_\sigma(f(V)) \subseteq f(V)$ for each σ-closed set V in Y.

(3) $\beta C_\sigma(f(B)) \subseteq f(C_\sigma(B)) \subseteq$ for $B \subseteq Y$.

Proof. It is similar to the proof of Theorem 3.17. □

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