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WALLED SIGNED BRAUER ALGEBRAS AS CENTRALIZER ALGEBRAS OF THE MIXED TENSOR REPRESENTATION OF LIE SUPERALGEBRAS

B. Kethesan
Ramanujan Institute for Advanced study in Mathematics
University of Madras
Chepauk, Chennai, 600 005, Tamilnadu, INDIA

Abstract: In this paper, we prove that the walled signed Brauer algebra $\overrightarrow{D}_{r,s}(x)$ is embedded in a canonical fashion as a subalgebra of a walled Brauer algebra $B_{2r,2s}(x)$ and the walled signed Brauer algebra $\overrightarrow{D}_{r,s}(x)$ is the centralizer algebra of $\sigma = (1\ 2)(3\ 4)\dots(2r{-}1\ 2r)(2r{+}1\ 2r{+}2)\dots(2r{+}2s{-}1\ 2r{+}2s)$ in the walled Brauer algebra $B_{2r,2s}(x)$. Finally we prove that, if $2r{+}2s < (m{+}1)(n{+}1)$, the walled signed Brauer algebra $\overrightarrow{D}_{r,s}(\delta)$ is isomorphic to the centralizer algebra $End_{\mathfrak{g}^{2r+2s}}(V^{\otimes r} \otimes (V^*)^{\otimes s})^{op}$ of the $\mathfrak{g}^{2r+2s}$- action on the mixed tensor space $(V^{\otimes r} \otimes (V^*)^{\otimes s})$; where $\delta = m - n$, $\mathfrak{g} = gl(m,n)$ is the complex general linear Lie superalgebra, $V = W \otimes W$, $W = \mathbb{C}^{m|n}$ is the natural representation of $\mathfrak{g}$ and V^* is the dual of V.

AMS Subject Classification: 17B10
Key Words: walled Brauer algebra, signed Brauer algebra, Lie superalgebra

1. Introduction

The general linear group $GL_n(\mathbb{C})$ of $n \times n$ invertible complex matrices acts on $V = \mathbb{C}^n$ by matrix multiplication. It follows that the diagonal action of $GL_n(\mathbb{C})$ is well defined on $V^{\otimes r}$. The symmetric group S_r acts on $V^{\otimes r}$ by place

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permutation. Clearly these actions commute with each other. In [10] and [11], Schur proved that, when $n \geq r$, these two group actions generate the full centralizer of each other. This fundamental result, reffered to as Schur-Weyl duality, connects the representation theories of $GL_n(\mathbb{C})$ and S_r. There are several generalizations of the Schur-Weyl duality.

Let V^* be the dual of V and $V^{\otimes r} \otimes (V^*)^{\otimes s}$ be the mixed tensor representation of $GL_n(\mathbb{C})$. The vector spaces $(V^*)^{\otimes s}$ and $V^{\otimes r} \otimes (V^*)^{\otimes s}$ inherit $GL_n(\mathbb{C})$-module structure in a natural way. The mixed tensor representation and the centralizer algebra $End_{GL_n(\mathbb{C})}\ (V^{\otimes r} \otimes (V^*)^{\otimes s})$ were studied in [1], [3], [6], [7] and [14].

For complex general linear Lie superalgebra $\mathfrak{g} = gl(m,n)$, Shader and Moon [13] studied the mixed tensor representation $(V^*)^{\otimes r} \otimes V^{\otimes s}$ of $gl(m,n)$, where $V = \mathbb{C}^{m+n}$ is the natural representation of $gl(m,n)$ and V^* is the dual of V and they proved that centralizer algebra $End_{\mathfrak{g}}((V^*)^{\otimes r} \otimes V^{\otimes s})$ is isomorphic to the walled Brauer algebra $B_{r,s}(m-n)$ for $r+s \leq m-n$. Moreover, Brundan and Stroppel [2], also studied the mixed tensor representation $V^{\otimes r} \otimes (V^*)^{\otimes s}$ of $\mathfrak{g} = gl(m,n)$ and proved that the centralizer algebra $End_{\mathfrak{g}}(V^{\otimes r} \otimes (V^*)^{\otimes s})^{op}$ is isomorphic to the walled Brauer algebra $B_{r,s}(m-n)$ when $r+s < (m+1)(n+1)$.

The walled signed Brauer algebra denoted by $\overrightarrow{D}_{r,s}(x)$, where $r, s \in \mathbb{N}$ and x is an indeterminant, is introduced in [5]. In this paper we proved that the walled signed Brauer algebra $\overrightarrow{D}_{r,s}(x)$ is embedded in a canonical fashion as a subalgebra of a walled Brauer algebra $B_{2r,2s}(x)$. Further we obtained the walled signed Brauer algebra $\overrightarrow{D}_{r,s}(x)$ is the centralizer algebra of $\sigma = (1\ 2)(3\ 4)\dots(2r-1\ 2r)(2r+1\ 2r+2)\dots(2r+2s-1\ 2r+2s)$ in the walled Brauer algebra $B_{2r,2s}(x)$.

These works motivated us to define a Lie superalgebra

$$\mathfrak{g}^{2r+2s} = \triangle^{2r+2s-1}(\mathfrak{g}) \oplus \mathbb{C}(\sigma \otimes \sigma \otimes \cdots \otimes \sigma),$$

where

$$\triangle^{2r+2s-1} : \mathfrak{g} \longrightarrow \mathfrak{g} \otimes \mathfrak{g} \otimes \cdots \otimes \mathfrak{g}$$

((2r+2s) -fold) is defined by $\triangle^{2r+2s-1}(g) = \sum_{i=1}^{2r+2s} g^i_{2r+2s}$; $g^i_{2r+2s} = 1 \otimes 1 \otimes \cdots \otimes 1 \otimes g \otimes 1 \otimes \cdots \otimes 1$, g in the i^{th} position and a map,

$$\Psi^{m,n}_{2r,2s} : \overrightarrow{D}_{r,s}(\delta) \longrightarrow End_{\mathfrak{g}^{2r+2s}}(V^{\otimes r} \otimes (V^*)^{\otimes s})^{op},$$

where $\delta = m-n$. Finally we show that this map $\Psi^{m,n}_{2r,2s}$ is an isomorphism whenever $2r+2s < (m+1)(n+1)$.

2. Preliminaries

In this section, we collect some preliminary results from [2] and [13] that we needed for the development of the paper.

Let $\mathfrak{g} = gl(m,n) = gl(m,n)_0 \oplus gl(m,n)_1$ denote the complex general linear Lie superalgebra. Let $W = \mathbb{C}^{m|n}$ be the natural representation of $\mathfrak{g}$. As a $\mathbb{C}-$ vector space $W = W_0 \oplus W_1$ with $dim(W_0) = m$ and $dim(W_1) = n$. Let $\{w_1, w_2 \ldots, w_m\}$ and $\{w_{m+1}, w_{m+2} \ldots, w_{m+n}\}$ be the bases of W_0 and W_1 respectively. For a homogeneous vector w in W, we will denote $\mid w \mid$ for its parity. $\mid w_i \mid := \mid i \mid \in \mathbb{Z}_2$, where $\mid i \mid = 0$ if $1 \leq i \leq m$ and $\mid i \mid = 1$ if $m+1 \leq i \leq m+n$.

Let W^* be the dual space of W with the standard dual basis $\{w_1^*, \ldots, w_m^*, w_{m+1}^*, \ldots, w_{m+n}^*\}$, so $w_i^*(w_j) = \delta_{ij}$ and $W^* = (W^*)_0 \oplus (W^*)_1$, where $(W^*)_a = (W_a)^*$ for $a \in \mathbb{Z}_2$. Then W^* is a $\mathfrak{g}$- supermodule via

$(gf)(w) := -(-1)^{|f||g|} f(gw)$ for homogeneous $g \in \mathfrak{g}, f \in W^*$ and $w \in W$.

The standard basis elements of $\mathfrak{g}$ consisting of matrix units $e_{i,j}$ acts on the bases of W and W^* by the following formulae,

$e_{i,j}(w_k) = \delta_{j,k} w_i$ and $e_{i,j}(w_k^*) = -\delta_{i,k}(-1)^{(|i|+|j|)|i|} w_j^*$ for all $1 \leq i,j,k \leq m+n$.

A $\mathfrak{g}$- action on a mixed tensor space $W^{\otimes 2r} \otimes (W^*)^{\otimes 2s}$ is given as follows: For

$$w = w_1, w_2, \ldots, w_{2r}, w_{2r+1}, \ldots, w_{2r+2s} \in W^{\otimes 2r} \otimes (W^*)^{\otimes 2s}$$

and $g \in \mathfrak{g}$,

$$g.w = \sum_{i=1}^{2r+2s} (-1)^{|g|(|w_1|+\cdots+|w_{i-1}|)} w_1 \otimes \cdots \otimes w_{i-1} \otimes gw_i \otimes \cdots \otimes w_{2r+2s}.$$

If M and N are $\mathfrak{g}$- supermodules, we write $Hom_{\mathfrak{g}}(M,N)$ for the vector superspace of all $\mathfrak{g}$- supermodule homomorphism from M to N. By definition, this is the set of all (not necessarily homogeneous) linear maps $f \in Hom(M,N)$ annihilated by all $g \in \mathfrak{g}$, where the linear action of $\mathfrak{g}$ on $Hom(M,N)$ is by

$$(gf)(m) = g(f(m)) - (-1)^{|f||g|} f(gm)$$

for homogeneous $g \in \mathfrak{g}, f \in Hom(M,N)$ and $m \in M$.

For $1 \leq a < b \leq 2r+2s$, let

$$\Omega_{a,b} := \sum_{i,j=1}^{m+n} (-1)^{|j|} 1^{\otimes(a-1)} \otimes e_{i,j} \otimes 1^{\otimes(b-a-1)} \otimes e_{j,i} \otimes 1^{(2r+2s-b)}$$

in $\mathfrak{g}^{\otimes(2r+2s)}$, then $\Omega_{a,b}$ is a $\mathfrak{g}$- supermodule endomorphism of $M_1 \otimes M_2 \otimes \cdots \otimes M_{(2r+2s)}$ given any $\mathfrak{g}$- supermodules $M_1, M_2 \ldots M_{2r+2s}$.

Let $I = \{1, 2, \ldots, m, m+1, \ldots, m+n\}$.

For $k \geq 0$, the symmetric group S_k acts naturally on the right on the set I^k by

$$\boldsymbol{i}.\sigma = (i_{\sigma(1)}, i_{\sigma(2)}, \ldots, i_{\sigma(k)})$$

for

$$\boldsymbol{i} = (i_1, i_2, \ldots, i_k) \in I^k$$

and $\sigma \in S_k$.

For $\boldsymbol{i} \in I^k$, let $\mid \boldsymbol{i} \mid := \mid i_1 \mid + \mid i_2 \mid + \cdots + \mid i_k \mid$ and $p(\boldsymbol{i}) = \sum_{1 \leq a < b \leq k} \mid i_a \mid\mid i_b \mid$.

Given $\boldsymbol{i} = (i_1, i_2, \ldots, i_{2r+2s}) \in I^{2r+2s}$, let $\boldsymbol{i}^L = (i_1, i_2, \ldots, i_{2r}) \in I^{2r}$ and $\boldsymbol{i}^R = (i_{2r+1}, i_{2r+2}, \ldots, i_{2r+2s}) \in I^{2r+2s}$ so that $\boldsymbol{i} = \boldsymbol{i}^L.\boldsymbol{i}^R$, where the product on the right is by concatenation of tuples.

Then the $\mathfrak{g}$- supermodules $W^{\otimes 2r}$ and $(W^*)^{\otimes 2s}$ have bases $\{w_{\boldsymbol{i}^L} \mid \boldsymbol{i}^L \in I^{2r}\}$ and $\{w^*_{\boldsymbol{i}^R} \mid \boldsymbol{i}^R \in I^{2s}\}$ respectively, where $w_{\boldsymbol{i}^L} = w_{i_1} \otimes w_{i_2} \otimes \cdots \otimes w_{i_{2r}}$ and $w^*_{\boldsymbol{i}^R} = w^*_{i_{2r+1}} \otimes w^*_{i_{2r+2}} \otimes \cdots \otimes w^*_{i_{2r+2s}}$.

Similarly, $W^{\otimes(2r+2s)}$ has basis $\{w_{\boldsymbol{i}^L} \otimes w_{\boldsymbol{i}^R} \mid \boldsymbol{i}^L \in I^{2r}, \boldsymbol{i}^R \in I^{2s}\}$ and $W^{\otimes 2r} \otimes (W^*)^{\otimes 2s}$ has basis $\{w_{\boldsymbol{i}^L} \otimes w^*_{\boldsymbol{i}^R} \mid \boldsymbol{i}^L \in I^{2r}, \boldsymbol{i}^R \in I^{2s}\}$.

For $\boldsymbol{i}, \boldsymbol{j} \in I^{2r+2s}$ and a diagram σ which is either in S_{2r+2s} or in a walled Brauer algebra $B_{2r,2s}(\delta)$.

$\boldsymbol{i}\sigma\boldsymbol{j}$ denote the labelled diagram obtained by colouring the vertices at the bottom of σ by $i_1, i_2, \ldots, i_{2r+2s}$ and the vertices at the top of σ by $j_1, j_2, \ldots, j_{2r+2s}$ (in order from left to right) $\boldsymbol{i}\sigma\boldsymbol{j}$ is consistently coloured if the vertices at the ends of each strand are coloured in the same way. In a consistently coloured permutation diagram $\boldsymbol{i}\sigma\boldsymbol{j}$, we have that, $\boldsymbol{j} = \boldsymbol{i}.\sigma$.

The weight of $\boldsymbol{i}\sigma\boldsymbol{j}$ is

$$Wt(\boldsymbol{i}\sigma\boldsymbol{j}) = \begin{cases} \prod_c (-1)^{|c|} \prod_h (-1)^{|h|} & \text{if } \boldsymbol{i}\sigma\boldsymbol{j} \text{ is consistently coloured,} \\ 0 & \text{other wise,} \end{cases}$$

where the notations are explained below:

1. The first product is taken over all proper coloured crossings c in $\boldsymbol{i}\sigma\boldsymbol{j}$, that is, the crossings that involve two different strands rather than self-intersections. The parity $\mid c \mid$ of a coloured crossing c of two different strands c_1 and c_2 is $\mid c \mid = \mid c_1 \mid\mid c_2 \mid$, where $\mid c_i \mid = \mid i_k \mid$, if the strand c_i is a vertical strand with label i_k at the bottom vertex or c_i is a horizontal strand with label i_k at the right vertex.

2. The second product is taken over all coloured horizontal strands h in $\boldsymbol{i\sigma j}$ whose end points are on the bottom row (if σ is a permutation diagram this product is empty so can be omitted). The parity $\mid h \mid$ of a coloured horizontal strand h with label i_k at the right vertex is $\mid h \mid = \mid i_k \mid$.

The following two lemmas from [2] give the actions of $\mathbb{C}S_{2r+2s}$ and $B_{2r,2s}(\delta)$ on $W^{\otimes(2r+2s)}$ and $W^{\otimes 2r} \otimes (W^*)^{\otimes 2s}$ respectively.

Lemma 2.1. (see [2], Lemma 7.3) *There is a well defined right $\mathbb{C}S_{2r+2s}$-module structure on $W^{\otimes(2r+2s)}$ such that*

$$w_{\mathbf{i}}.\sigma = \sum_{\boldsymbol{j} \in I^{2r+2s}} Wt(\boldsymbol{i\sigma j}) w_{\boldsymbol{j}},$$

for all $\boldsymbol{i} \in I^{2r+2s}$ and $\sigma \in S_{2r+2s}$.

Moreover for $1 \le a < b \le 2r + 2s$, the transposition $(a\ b)$ acts in the same way as the operator $\Omega_{a,b}$, defined in Section 2. Hence the action of $\mathbb{C}S_{2r+2s}$ commutes with the action of $\mathfrak{g}$.

Lemma 2.2. (see [2], Lemma 7.4) *There is a well defined right $B_{2r,2s}(\delta)$-module structure on $W^{\otimes 2r} \otimes (W^*)^{\otimes 2s}$ such that*

$$(w_{\mathbf{i}^L} \otimes w^*_{\mathbf{i}^R}).\sigma = \sum_{\boldsymbol{j} \in I^{2r+2s}} Wt(\boldsymbol{i\sigma j})\ w_{\mathbf{i}^L} \otimes w^*_{\mathbf{i}^R},$$

for all $\boldsymbol{i} \in I^{2r+2s}$ and $\sigma \in B_{2r,2s}(\delta)$.

Moreover for $1 \le a < b \le 2r + 2s$, the transposition $\overline{(a\ b)}$ in $B_{2r,2s}(\delta)$ acts in the same way as the operator $\Omega_{a,b}$. Hence the action of $B_{2r,2s}(\delta)$ commutes with the action of $\mathfrak{g}$.

3. Mixed Tensor Representation

3.1. Walled Signed Brauer Algebra

We recall the following definition from [5].

Definition 3.1. (see [5]) For $r, s \in \mathbb{N}$, the walled signed Brauer algebra $\overrightarrow{D}_{r,s}(x)$ is defined as a subalgebra of the signed Brauer algebra $\overrightarrow{D}_{r+s}(x)$ over the field $k(x)$, where k is any arbitrary field and x an indeterminate.

Recall that the signed Brauer algebra $\overrightarrow{D}_{r+s}(x)$ [8] is generated by the signed diagrams where a signed diagram is a diagram on $(2r+2s)$ vertices with $(r+s)$

signed edges, vertices being arranged in two rows, each row consisting of $(r+s)$ vertices.

An edge is labeled by a plus sign or a minus sign will be called a signed edge. An edge labeled by a plus (resp., minus) sign will be called a positive (resp., negative) edge. A positive vertical (resp., horizontal) edge will be denoted by $\downarrow$ (resp., $\rightarrow$) and a negative vertical (resp., horizontal) edge will be denoted by $\uparrow$ (resp., $\leftarrow$).

The Walled Signed Brauer Algebra $\overrightarrow{D}_{r,s}(x)$ is generated by the walled signed Brauer diagrams where a walled signed Brauer diagram is a signed diagram which has a vertical wall between the r^{th} and $(r+1)^{th}$ vertices of the upper and lower rows.

The multiplication of two walled signed Brauer diagrams $\vec{a}$ and $\vec{b}$ is obtained by placing $\vec{a}$ above $\vec{b}$ and identifying the vertices in the bottom row of $\vec{a}$ with the corresponding vertices in the top row of $\vec{b}$. Then $a.b = x^d c$, where a and b are underlying walled Brauer diagrams, d is the number of loops in $a.b$, and c is the resulting walled Brauer diagram. A new edge obtained in the product $\vec{a}.\vec{b}$ is labeled by a plus sign or a minus sign according to the number of negative edges involved from $\vec{a}$ and $\vec{b}$ to from this edge is even or odd.

A loop β in $\vec{a}.\vec{b}$ is said to be positive (resp., negative) if the number of negative edges obtained from $\vec{a}$ and $\vec{b}$ to form this loop is even(resp., odd). A positive (resp., negative) loop β in $\vec{a}.\vec{b}$ is replaced by the variable x^2 (resp., x) in $\vec{a}.\vec{b}$. Then $\vec{a}\cdot\vec{b} = x^{(2d_1+d_2)}\vec{c}$, where d_1(resp., d_2) is the number of positive (resp., negative) loops in $\vec{a}\cdot\vec{b}$.

3.2. Generators, Relations of the Walled Signed Brauer Algebra

Theorem 3.2. (see [5]) *The walled signed Brauer algebra* $\overrightarrow{D}_{r,s}(x)$ *is generated by the elements* $\vec{h}_1, \vec{h}_{r+1},\ g_1,\ g_2, ..., g_{r-1},\ e_r,\ g_{r+1}, ...,\ g_{r+s-1}$ *and satisfying the following relations:*

1. $g_i^2 = 1;\quad i = 1, 2, ..., (r-1), (r+1), ..., (r+s-1)$
2. $g_i g_j = g_j g_i$ *if* $\mid i-j \mid > 1;$
3. $g_{i+1} g_i g_{i+1} = g_i g_{i+1} g_i;$
4. $e_r g_i = g_i e_r \quad 1 \leq i \leq r-2$ *or* $r+2 \leq i \leq r+s-1;$
5. $e_r^2 = x^2 e_r;$
6. $e_r g_{r-1} e_r = e_r;$

7. $e_r g_{r+1} e_r = e_r$;

8. $g_{r-1} g_{r+1} e_r g_{r-1} g_{r+1} e_r = e_r g_{r-1} g_{r+1} e_r$;

9. $e_r g_{r-1} g_{r+1} e_r g_{r-1} g_{r+1} = e_r g_{r-1} g_{r+1} e_r$;

10. $\vec{h}_i^2 = 1; \quad i = 1, 2, \ldots\ldots, (r+s)$;

11. $\vec{h}_1 g_i = g_i \vec{h}_1; \quad i \neq 1$;

12. $\vec{h}_1 g_1 \vec{h}_1 g_1 = g_1 \vec{h}_1 g_1 \vec{h}_1$;

13. $\vec{h}_{r+1} g_i = g_i \vec{h}_{r+1}; \quad i \neq r, (r+1)$;

14. $\vec{h}_{r+1} g_{r+1} \vec{h}_{r+1} g_{r+1} = g_{r+1} \vec{h}_{r+1} g_{r+1} \vec{h}_{r+1}$;

15. $e_r \vec{h}_r e_r = x e_r$ and $e_r \vec{h}_{r+1} e_r = x e_r$;

16. $g_i \vec{h}_{i+1} = \vec{h}_i g_i; \quad i \neq r$;

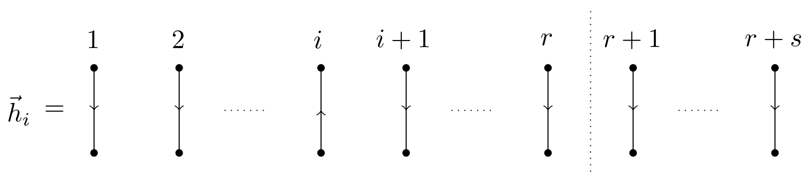
17. $e_r \vec{h}_{r+1} = e_r \vec{h}_r$;

18. $\vec{h}_{r+1} e_r = \vec{h}_r e_r$;

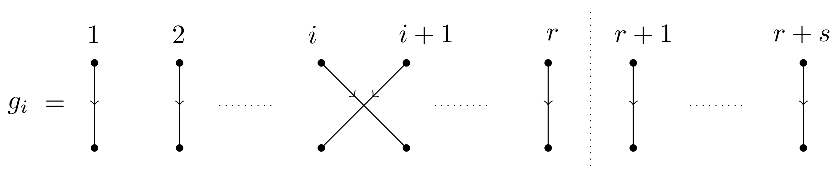
19. $e_r \vec{h}_i = \vec{h}_i e_r; \quad i \neq r, (r+1)$;

20. $e_r \vec{h}_r g_{r+1} e_r = e_r \vec{h}_{r+2}$ and $e_r \vec{h}_{r+1} g_{r+1} e_r = e_r \vec{h}_{r+2}$,

where

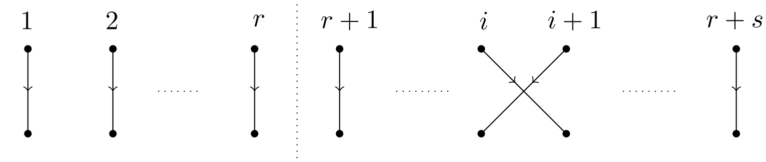


For $1 \leq i \leq r-1$,



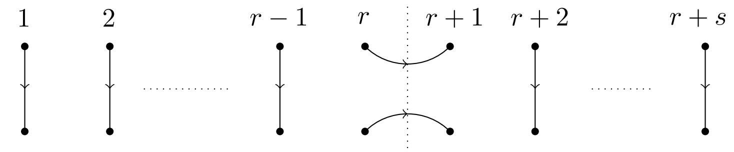
For $r+1 \leq i \leq r+s-1$,

$g_i =$



and

$e_r =$

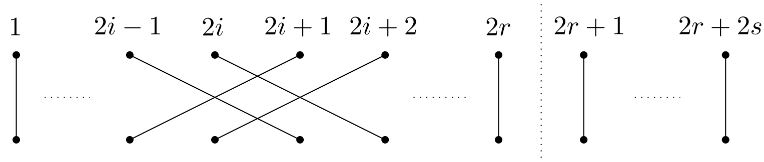


Now we shall show that the walled signed Brauer algebra $\overrightarrow{D}_{r,s}(x)$ is embedded in a canonical fashion as a subalgebra of a walled Brauer algebra $B_{2r,2s}(x)$.

Theorem 3.3. *There is a canonical embedding* $j : \overrightarrow{D}_{r,s}(x) \longrightarrow B_{2r,2s}(x)$, *where* j *is defined on the generators of walled signed Brauer algebra.*

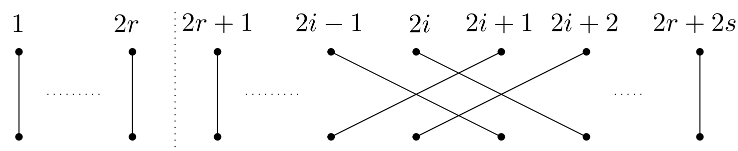
For $1 \leq 2i \leq 2r - 2$

$j(g_i) =$

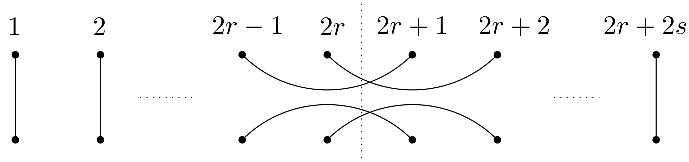


For $2r + 1 \leq 2i \leq 2r + 2s - 2$

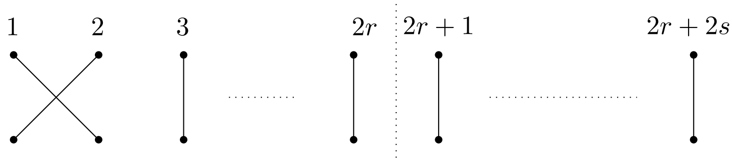
$j(g_i) =$



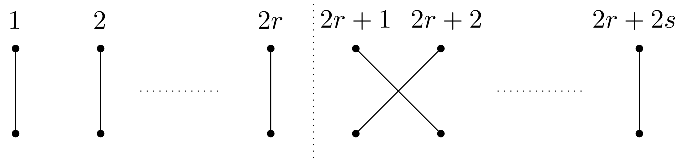
$j(e_r) =$



$j(\vec{h}_1) =$



and

$j(\vec{h}_{r+1}) =$ 

Proof. In [[9], *Theorem* 2.4], Parvathi and Selvaraj have proved that the signed Brauer algebra $\overrightarrow{D}_{r+s}(x)$ is embedded in a canonical fashion as a subalgebra of a Brauer algebra $B_{2r+2s}(x)$. Since $\overrightarrow{D}_{r,s}(x) \subseteq \overrightarrow{D}_{r+s}(x)$ and $B_{2r,2s}(x) \subseteq B_{2r+2s}(x)$. So we can use the same technique to prove our theorem.

It is easy to check that j preserves the relation given in the theorem 3.2.

Let $\vec{d}$ be a walled signed Brauer diagram in $\overrightarrow{D}_{r,s}(x)$ with upper vertices $1, 2, \ldots, r, r+1, \ldots, r+s$ and lower vertices $\overline{1}, \overline{2}, \ldots, \overline{r}, \overline{r+1}, \ldots, \overline{r+s}$.

1. Every vertical positive (resp., negative) edge $(i, \overline{k})_{i<\overline{k}}$ in $\vec{d}$ is replaced (under the mapping j) by two parallel (resp., intersecting) edges $(2i, \overline{2k})$ and $(2i-1, \overline{2k-1})$ (resp., $(2i, \overline{2k-1})$ and $(2i-1, \overline{2k})$).

2. Every positive (resp., negative) upper horizontal arc $(i, k)_{i<k}$ in $\vec{d}$ is replaced by two intersecting (resp., non-intersecting) upper horizontal arcs $(2i, 2k)$ and $(2i-1, 2k-1)$ (resp., $(2i, 2k-1)$ and $(2i-1, 2k)$).

3. Every positive (resp., negative) lower horizontal arc $(\overline{i}, \overline{k})_{\overline{i}<\overline{k}}$ in $\vec{d}$ is replaced by two intersecting (resp., non-intersecting) lower horizontal arcs $(\overline{2i}, \overline{2k})$ and $(\overline{2i-1}, \overline{2k-1})$ (resp., $(\overline{2i}, \overline{2k-1})$ and $(\overline{2i-1}, \overline{2k})$).

4. Every positive (resp., negative) loop in the product of two walled signed Brauer diagrams $\vec{d_1}$ and $\vec{d_2}$ in $\overrightarrow{D}_{r,s}(x)$ is replaced by two loops (resp., one loop) in the product of the walled Brauer diagrams $j(\vec{d_1})$ and $j(\vec{d_2})$.

Hence the mapping j preserves multiplication from $\overrightarrow{D}_{r,s}(x)$ into $B_{2r,2s}(x)$, proving that j is a canonical embedding. □

Lemma 3.4. *Let S_{2n} be a symmetric group of $2n$ symbols.*

Let $\sigma = (1\ 2)(3\ 4)\ldots(2n{-}1\ 2n)$ be a permutation in S_{2n} then the centralizer of σ in S_{2n} is $\mathbb{Z}_2 \wr S_n$.

That is, $\mathcal{C}_{S_{2n}}(\sigma) = \{\tau \in S_{2n} \mid \tau\sigma = \sigma\tau\} = \mathbb{Z}_2 \wr S_n$.

Proof. The proof follows from [4], Section 4.1.18. □

Now we can identify $\tau \in S_{2n}$ with its permutation diagram, the diagram with $2n$ vertices on its top and bottom rows and n vertical edges connecting the i^{th} vertex on the top row to the $\tau(i)^{th}$ vertex on the bottom row for each $1 \leq i \leq 2n$.

So the vertices of $\tau \in \mathcal{C}_{S_{2n}}(\sigma)$ are connected in the following ways:

Let $\tau \in \mathcal{C}_{S_{2n}}(\sigma)$ then $\tau\sigma = \sigma\tau$.

$\tau\sigma(2i-1) = \sigma\tau(2i-1)$, for $1 \leq i \leq n$, this implies $\tau(2i) = \sigma\tau(2i-1)$.

Case(1): If $\tau(2i-1) = 2i-1$ then $\tau(2i) = \sigma(2i-1) = 2i$, thus $\tau(2i-1) = 2i-1$ and $\tau(2i) = 2i$.

Case(2): If $\tau(2i-1) = 2i$ then $\tau(2i) = \sigma(2i) = 2i-1$, thus, $\tau(2i-1) = 2i$ and $\tau(2i) = 2i-1$.

Case(3): If $\tau(2i-1) = 2j-1,\ 1 \leq j \leq n$ then $\tau(2i) = \sigma(2j-1) = 2j$, thus, $\tau(2i-1) = 2j-1$ and $\tau(2i) = 2j$.

Case(4): If $\tau(2i-1) = 2j,\ 1 \leq j \leq n$ then $\tau(2i) = \sigma(2j) = 2j-1$, thus, $\tau(2i-1) = 2j$ and $\tau(2i) = 2j-1$.

So every consecutive pair of vertices $(2i-1, 2i);\ 1 \leq i \leq n$ in the top row of $\tau \in \mathcal{C}_{S_{2n}}(\sigma)$ is connected to the one of the pairs $(2i-1, 2i), (2i, 2i-1), (2j-1, 2j)$ or $(2j, 2j-1);\ 1 \leq j \leq n$ in the bottom row of τ.

Conversely if $\tau \in S_{2n}$ as in the above four cases then $\tau \in \mathcal{C}_{S_{2n}}(\sigma)$.

For, if $\tau(2i-1) = 2j-1$ and $\tau(2i) = 2j,\ 1 \leq i, j \leq n$, then $\sigma\tau(2i-1) = \sigma(2j-1) = 2j$ and $\tau\sigma(2i-1) = \tau(2i) = 2j$ and $\sigma\tau(2i) = \sigma(2j) = 2j-1$ and $\tau\sigma(2i) = \tau(2i-1) = 2j-1$. Thus,

$$\sigma\tau(2i-1) = \tau\sigma(2i-1) \text{ and } \sigma\tau(2i) = \tau\sigma(2i). \tag{3.1}$$

Similarly, equation (3.1) holds for other cases.

Thus, $\sigma\tau = \tau\sigma$ in S_{2n}, hence $\tau \in \mathcal{C}_{S_{2n}}(\sigma)$.

Let $\sigma = (1\ 2)(3\ 4)\ldots(2r-1\ 2r)\ldots(2r+2s-1\ 2r+2s)$ be a permutation in S_{2r+2s} then $\mathcal{C}_{S_{2r+2s}}(\sigma) = \{\tau \in S_{2r+2s} \mid \tau\sigma = \sigma\tau\} = \mathbb{Z}_2 \wr S_{r+s}$.

Lemma 3.5. *Let $\mathcal{C}_{B_{2r,2s}}(\sigma) = \{\tau' \in B_{2r,2s} \mid \tau'\sigma = \sigma\tau'\}$ then $\tau \in \mathcal{C}_{S_{2r+2s}}(\sigma)$ if and only if $flip_{2r,2s}(\tau) \in \mathcal{C}_{B_{2r,2s}}(\sigma)$, where $flip_{2r,2s} : \mathbb{C}S_{2r+2s} \longrightarrow B_{2r,2s}$ is a vector space isomorphism.*

Proof. Suppose that $\tau \in \mathcal{C}_{S_{2r+2s}}(\sigma)$ then $\tau\sigma = \sigma\tau$ in S_{2r+2s}. From the previous result, every pair of vertices $(2i-1, 2i)$ in τ is connected to the one of the pairs $(2i-1, 2i), (2i, 2i-1), (2j-1, 2j)$ *or* $(2j, 2j-1)$ *in* τ; $1 \leq i \leq r+s$ and $1 \leq j \leq r+s$.

Case(1): If $\tau(2i-1) = 2i-1$ and $\tau(2i) = 2i$ then For $1 \leq 2i-1 < 2r$ or $2r+1 \leq 2i-1 < 2r+2$, $flip_{2r,2s}(\tau)(2i-1) = 2i-1$ and $flip_{2r,2s}(\tau)(2i) =$

$2i$. Thus, $\sigma flip_{2r,2s}(\tau)(2i-1) = flip_{2r,2s}(\tau)\sigma(2i-1)$ and $\sigma flip_{2r,2s}(\tau)(2i) = flip_{2r,2s}(\tau)\sigma(2i)$.

Case(2): If $\tau(2i-1) = 2i$ and $\tau(2i) = 2i-1$ then For $1 \leq 2i-1 < 2r$ or $2r+1 \leq 2i-1 < 2r+2s$, $flip_{2r,2s}(\tau)(2i-1) = 2i$ and $flip_{2r,2s}(\tau)(2i) = 2i-1$. Thus, $\sigma flip_{2r,2s}(\tau)(2i-1) = flip_{2r,2s}(\tau)\sigma(2i-1)$ and $\sigma flip_{2r,2s}(\tau)(2i) = flip_{2r,2s}(\tau)\sigma(2i)$.

Case(3): If $\tau(2i-1) = 2j-1$ and $\tau(2i) = 2j$ then (i) if $1 \leq 2i-1 < 2r$ and $1 \leq 2j-1 < 2r$ then $flip_{2r,2s}(\tau)(2i-1) = 2j-1$ and $flip_{2r,2s}(\tau)(2i) = 2j$. Thus, $\sigma flip_{2r,2s}(\tau)(2i-1) = fip_{2r,2s}(\tau)\sigma(2i-1) = 2j$ and $\sigma flip_{2r,2s}(\tau)(2i) = flip_{2r,2s}(\tau)\sigma(2i) = 2j-1$. (ii) if $2r+1 \leq 2i-1 < 2r+2s$ and $2r+1 \leq 2j-1 < 2r+2s$ then $flip_{2r,2s}(\tau)(2j-1) = 2i-1$ and $flip_{2r,2s}(\tau)(2j) = 2i$. Thus, $\sigma flip_{2r,2s}(\tau)(2j-1) = flip_{2r,2s}(\tau)\sigma(2j-1)$ and $\sigma flip_{2r,2s}(\tau)(2j) = flip_{2r,2s}(\tau)\sigma(2j)$. (iii) if $1 \leq 2i-1 < 2r$ and $2r+1 \leq 2j-1 < 2r+2s$ then $flip_{2r,2s}(\tau)$ has a pair of upper horizontal arcs $(2i-1, 2j-1)$ and $(2i, 2j)$, that is, $flip_{2r,2s}(\tau)(2i-1) = 2j-1$ and $flip_{2r,2s}(\tau)(2i) = 2j$.

Thus, $\sigma flip_{2r,2s}(\tau)(2i-1) = flip_{2r,2s}(\tau)\sigma(2i-1)$ and $\sigma flip_{2r,2s}(\tau)(2i) = flip_{2r,2s}(\tau)\sigma(2i)$. (iv) if $2r+1 \leq 2i-1 < 2r+2s$ and $1 \leq 2j-1 < 2r$ then $flip_{2r,2s}(\tau)$ has a pair of lower horizontal arcs $(2i-1, 2j-1)$ and $(2i, 2j)$, that is, $flip_{2r,2s}(\tau)(2j-1) = 2i-1$ and $flip_{2r,2s}(\tau)(2j) = 2i$.

Thus, $\sigma flip_{2r,2s}(\tau)(2j-1) = flip_{2r,2s}(\tau)\sigma(2j-1) = 2i$ and $\sigma flip_{2r,2s}(\tau)(2j) = flip_{2r,2s}(\tau)\sigma(2j) = 2i-1$.

Case(4): If $\tau(2i-1) = 2j$ and $\tau(2i) = 2j-1$ then we have, (same as in case(3)), (i) if $1 \leq 2i-1 < 2r$ and $1 \leq 2j-1 < 2r$ then $\sigma flip_{2r,2s}(\tau)(2i-1) = flip_{2r,2s}(\tau)\sigma(2i-1)$ and $\sigma flip_{2r,2s}(\tau)(2i) = flip_{2r,2s}(\tau)\sigma(2i)$. (ii) if $2r+1 \leq 2i-1 < 2r+2s$ and $2r+1 \leq 2j-1 < 2r+2s$ then $\sigma flip_{2r,2s}(\tau)(2j-1) = flip_{2r,2s}(\tau)\sigma(2j-1) = 2i-1$ and $\sigma flip_{2r,2s}(\tau)(2j) = flip_{2r,2s}(\tau)\sigma(2j) = 2i$. (iii) if $1 \leq 2i-1 < 2r$ and $2r+1 \leq 2j-1 < 2r+2s$ then $\sigma flip_{2r,2s}(\tau)(2i-1) = flip_{2r,2s}(\tau)\sigma(2i-1) = 2j-1$ and $\sigma flip_{2r,2s}(\tau)(2i) = flip_{2r,2s}(\tau)\sigma(2i) = 2j$. (iv) if $2r+1 \leq 2i-1 < 2r+2s$ and $1 \leq 2j-1 < 2r$ then $\sigma flip_{2r,2s}(\tau)(2j-1) = flip_{2r,2s}(\tau)\sigma(2j-1) = 2i-1$ and $\sigma flip_{2r,2s}(\tau)(2j) = flip_{2r,2s}(\tau)\sigma(2j) = 2i$. From the above four cases, we have $\sigma flip_{2r,2s}(\tau) = flip_{2r,2s}(\tau)\sigma$ in $B_{2r,2s}(x)$, therefore, $flip_{2r,2s}(\tau) \in \mathcal{C}_{B_{2r,2s}}(\sigma)$.

Conversely suppose that $flip_{2r,2s}(\tau) \in \mathcal{C}_{B_{2r,2s}}(\sigma)$; for $\tau \in S_{2r+2s}$ then $\sigma flip_{2r,2s}(\tau) = flip_{2r,2s}(\tau)\sigma$ in $B_{2r,2s}(x)$. That is

$$flip_{2r,2s}(\tau)\sigma(2i-1) = \sigma flip_{2r,2s}(\tau)(2i-1)$$

and $flip_{2r,2s}(\tau)(2i) = \sigma flip_{2r,2s}(\tau)(2i-1), 1 \leq i \leq r+s$.

Case(1): If $flip_{2r,2s}(\tau)(2i-1) = 2i-1$ then $flip_{2r,2s}(\tau)(2i) = \sigma(2i-1) = 2i$. For $1 \leq 2i-1 < 2r$ or $2r+1 \leq 2i-1 < 2r+2s$, $\tau(2i-1) = 2i-1$ and $\tau(2i) = 2i$, therefore, $\sigma\tau(2i-1) = \tau\sigma(2i-1)$ and $\sigma\tau(2i) = \tau\sigma(2i)$.

Case(2): If $flip_{2r,2s}(\tau)(2i-1) = 2i$ then $flip_{2r,2s}(\tau)(2i) = \sigma(2i) = 2i-1$. For $1 \leq 2i-1 < 2r$ or $2r+1 \leq 2i-1 < 2r+2s$, $\tau(2i-1) = 2i$ and $\tau(2i) = 2i-1$, therefore, $\sigma\tau(2i-1) = \tau\sigma(2i-1)$ and $\sigma\tau(2i) = \tau\sigma(2i)$.

Case(3): If $flip_{2r,2s}(\tau)(2i-1) = 2j-1$ then $flip_{2r,2s}(\tau)(2i) = \sigma(2j-1) = 2j$. (i) if $1 \leq 2i-1 < 2r$ and $1 \leq 2j-1 < 2r$ then $\tau(2i-1) = 2j-1$ and $\tau(2i) = 2j$. Therefore, $\sigma\tau(2i-1) = \tau\sigma(2i-1) = 2j$ and $\sigma\tau(2i) = \tau\sigma(2i) = 2j-1$. (ii) if $2r+1 \leq 2i-1 < 2r+2s$ and $2r+1 \leq 2j-1 < 2r+2s$ then $\tau(2j-1) = 2i-1$ and $\tau(2j) = 2i$. Therefore, $\sigma\tau(2j-1) = \tau\sigma(2j-1) = 2i$ and $\sigma\tau(2j) = \tau\sigma(2j) = 2i-1$. (iii) if $1 \leq 2i-1 < 2r$ and $2r+1 \leq 2j-1 < 2r+2s$ then $flip_{2r,2s}(\tau)$ has either a pair of upper horizontal arcs $(2i-1, 2j-1)$ and $(2i, 2j)$ or a pair of lower horizontal arcs $(2i-1, 2j-1)$ and $(2i, 2j)$, that is, either $\tau(2i-1) = 2j-1$ and $\tau(2i) = 2j$ or $\tau(2j-1) = 2i-1$ and $\tau(2j) = 2i$. Therefore, either $\sigma\tau(2i-1) = \tau\sigma(2i-1) = 2j$ and $\sigma\tau(2i) = \tau\sigma(2i) = 2j-1$ or $\sigma\tau(2j-1) = \tau\sigma(2j-1) = 2i$ and $\sigma\tau(2j) = \tau\sigma(2j) = 2i-1$. (iv) if $1 \leq 2j-1 < 2r$ and $2r+1 \leq 2i-1 < 2r+2s$. This is impossible, because $flip_{2r,2s}(\tau)$ has no vertical edges that crosses the wall.

Case(4): if $flip_{2r,2s}(\tau)(2i-1) = 2j$ then $flip_{2r,2s}(\tau)(2i) = \sigma(2j) = 2j-1$. Same as in case(3), we have, (i) if $1 \leq 2i-1 < 2r$ and $1 \leq 2j-1 < 2r$ then $\sigma\tau(2i-1) = \tau\sigma(2i-1) = 2j-1$ and $\sigma\tau(2i) = \tau\sigma(2i) = 2j$. (ii) if $2r+1 \leq 2i-1 < 2r+2s$ and $2r+1 \leq 2j-1 < 2r+2s$ then $\sigma\tau(2j-1) = \tau\sigma(2j-1) = 2i-1$ and $\sigma\tau(2j) = \tau\sigma(2j) = 2i$. (iii) if $1 \leq 2i-1 < 2r$ and $2r+1 \leq 2j-1 < 2r+2s$ then $flip_{2r,2s}(\tau)$ has either a pair of upper horizontal arcs $(2i-1, 2j)$ and $(2i, 2j-1)$ or a pair of lower horizontal arcs $(2i-1, 2j)$ and $(2i, 2j-1)$. Therefore, either $\sigma\tau(2i-1) = \tau\sigma(2i-1) = 2j-1$ and $\sigma\tau(2i) = \tau\sigma(2i) = 2j$ or $\sigma\tau(2j-1) = \tau\sigma(2j-1) = 2i-1$ and $\sigma\tau(2j) = \tau\sigma(2j) = 2i$. (iv) $1 \leq 2j-1 < 2r$ and $2r+1 \leq 2i-1 < 2r+2s$. This is impossible. From the above four cases, we have , $\sigma\tau = \tau\sigma$ in S_{2r+2s}. Hence $\tau \in \mathcal{C}_{S_{2r+2s}}(\sigma)$. □

Let $\mathfrak{g} = gl(m, n)$ be the complex general linear Lie superalgebra with Lie super bracket, $[a, b] = a.b - (-1)^{|a||b|} b.a$, $a, b \in \mathfrak{g}$. Then we have an embedding, $\triangle^{2r+2s-1} : \mathfrak{g} \longrightarrow \mathfrak{g} \otimes \mathfrak{g} \otimes \cdots \otimes \mathfrak{g}$ ((2r+2s) -fold) by $\triangle^{2r+2s-1}(g) = \sum_{i=1}^{2r+2s} g^i_{2r+2s}$; $g^i_{2r+2s} = 1 \otimes 1 \otimes \cdots \otimes 1 \otimes g \otimes 1 \otimes \cdots \otimes 1$, g in the i^{th} position.

Let $\mathfrak{g}^{2r+2s} = \triangle^{2r+2s-1}(\mathfrak{g}) \oplus \mathbb{C}(\sigma \otimes \sigma \otimes \cdots \otimes \sigma)$ ((2r+2s) -fold).

Lemma 3.6. *$\mathfrak{g}^{2r+2s}$ is a Lie superalgebra.*

Proof. Since $\triangle^{2r+2s-1}(\mathfrak{g})$ is a Lie superalgebra, we have,

$$\triangle^{2r+2s-1}(\mathfrak{g}) = (\triangle^{2r+2s-1}(\mathfrak{g}))_0 \oplus (\triangle^{2r+2s-1}(\mathfrak{g}))_1$$

and

$$[(\triangle^{2r+2s-1}(\mathfrak{g}))_\alpha, (\triangle^{2r+2s-1}(\mathfrak{g}))_\beta] \subseteq (\triangle^{2r+2s-1}(\mathfrak{g}))_{\alpha+\beta},$$

for $\alpha, \beta \in \mathbb{Z}_2$. Therefore, $\mathfrak{g}^{2r+2s} = (\mathfrak{g}^{2r+2s})_0 \oplus (\mathfrak{g}^{2r+2s})_1$, where

$$(\mathfrak{g}^{2r+2s})_1 = (\triangle^{2r+2s-1}(\mathfrak{g}))_1$$

and

$$(\mathfrak{g}^{2r+2s})_0 = (\triangle^{2r+2s-1}(\mathfrak{g}))_0 \oplus \mathbb{C}(\sigma \otimes \sigma \otimes \cdots \otimes \sigma).$$

Let $a \in (\mathfrak{g}^{2r+2s})_0$ and $b \in (\mathfrak{g}^{2r+2s})_1$ then $a = x_1 + y_1,\ x_1 \in (\triangle^{2r+2s-1}(\mathfrak{g}))_0$ and $y_1 \in \mathbb{C}(\sigma \otimes \sigma \otimes \cdots \otimes \sigma)$.

Since $[a, b] = [x_1+y_1, b] = [x_1, b] \in (\mathfrak{g}^{2r+2s})_1$, we have, $[(\mathfrak{g}^{2r+2s})_0, (\mathfrak{g}^{2r+2s})_1] \subseteq (\mathfrak{g}^{2r+2s})_{0+1}$.

Similarly, we can show that, $[(\mathfrak{g}^{2r+2s})_\alpha, (\mathfrak{g}^{2r+2s})_\beta] \subseteq (\mathfrak{g}^{2r+2s})_{\alpha+\beta}$, for $\alpha, \beta \in \mathbb{Z}_2$.

Now let $a, b \in \mathfrak{g}^{2r+2s}$ then $a = x_1 + y_1$ and $b = x_2 + y_2$, where $x_1, x_2 \in \triangle^{2r+2s-1}(\mathfrak{g})$ and $y_1, y_2 \in \mathbb{C}(\sigma \otimes \sigma \otimes \cdots \otimes \sigma)$.

Let $x_1 = \sum_{i=1}^{2r+2s} g^i_{2r+2s}$, $x_2 = \sum_{i=1}^{2r+2s} \bar{g}^i_{2r+2s}$, $y_1 = \alpha(\sigma \otimes \sigma \otimes \cdots \otimes \sigma)$ and $y_2 = \beta(\sigma \otimes \sigma \otimes \cdots \otimes \sigma)$; where $\alpha, \beta \in \mathbb{C}$ and $g, \bar{g} \in \mathfrak{g}$.

Consider,

$$\begin{aligned} [a, b] &= [x_1 + y_1, x_2 + y_2] \\ &= [x_1, x_2];\ \text{since } [x_1, y_2] = 0 = [y_1,\ x_2] \text{ and } [y_1, y_2] = 0 \\ &= [\sum_{i=1}^{2r+2s} g^i_{2r+2s}, \sum_{i=1}^{2r+2s} \bar{g}^i_{2rr+2s}]. \end{aligned}$$

since $[g^i_{2r+2s}, \bar{g}^j_{2r+2s}] = 0$, for all $i \neq j$, $\mid a \mid=\mid x_1 \mid=\mid g \mid$, $\mid b \mid=\mid x_2 \mid=\mid \bar{g} \mid$ and $g^i_{2r+2s}.\bar{g}^i_{2r+2s} = 1 \otimes 1 \otimes \cdots \otimes 1 \otimes g.\bar{g} \otimes 1 \otimes \cdots \otimes 1 = (g\bar{g})^i_{2r+2s}$,

$$\begin{aligned} [a, b] &= \sum_{i=1}^{2r+2s} [g^i_{2r+2s}, \bar{g}^i_{2r+2s}] \\ &= \sum_{i=1}^{2r+2s} (g\bar{g})^i_{2r+2s} - (-1)^{|g||\bar{g}|}(\bar{g}g)^i_{2r+2s} \end{aligned}$$

$$= \quad -(-1)^{|g||\bar{g}|}[b,a] = -(-1)^{|a||b|}[b,a].$$

Let $a, b, c \in \mathfrak{g}^{2r+2s}$ then $a = x_1 + y_1,\ b = x_2 + y_2$ and $c = x_3 + y_3$, where $x_1,\ x_2,\ x_3 \in \triangle^{2r+2s-1}(\mathfrak{g})$ and $y_1,\ y_2,\ y_3 \in \mathbb{C}(\sigma \otimes \sigma \otimes \cdots \otimes \sigma)$.

Also we have, $[a,b] = [x_1, x_2],\ [b,c] = [x_2, x_3]$ and $[c,a] = [x_3, x_1]$.

Consider

$$\begin{aligned}[a,[b,c]] &= [x_1 + y_1, [x_2, x_3]] \\ &= [x_1,[x_2,x_3]] + [y_1,[x_2,x_3]] = [x_1,[x_2,x_3]]\end{aligned}$$

since

$$(-1)^{|x_1||x_3|}[x_1,[x_2,x_3]] + (-1)^{|x_1||x_2|}[x_2,[x_3,x_1]] + (-1)^{|x_2||x_3|}[x_3,[x_1,x_2]] = 0,$$

we have

$$\begin{aligned}(-1)^{|a||c|}[a,[b,c]] &= (-1)^{|x_1||x_3|}[x_1,[x_2,x_3]] \\ &= -(-1)^{|a||b|}[b,[c,a]] - (-1)^{|b||c|}[c,[a,b]].\end{aligned}$$

Hence $\mathfrak{g}^{2r+2s}$ is a Lie superalgebra. □

In [2], Brundan and Stroppel have proved that the map, $\Psi^{m,n}_{2r,2s} : B_{2r,2s}(\delta) \longrightarrow End_{\mathfrak{g}}(W^{\otimes 2r} \otimes (W^*)^{\otimes 2s})^{op}$ is an isomorphism when $2r + 2s < (m+1)(n+1)$, where $\delta = m - n$ and $W = \mathbb{C}^{m+n}$ is the natural representation of $\mathfrak{g}$.

Let $End_{\mathfrak{g}^{2r+2s}}(V^{\otimes r} \otimes (V^*)^{\otimes s})$ be the centralizer algebra of the $\mathfrak{g}^{2r+2s}$- action on $V^{\otimes r} \otimes (V^*)^{\otimes s}$, where $V = W \otimes W$ and $V^* = W^* \otimes W^*$. Then

$$End_{\mathfrak{g}^{2r+2s}}(V^{\otimes r} \otimes (V^*)^{\otimes s}) = \{T \in End(V^{\otimes r} \otimes (V^*)^{\otimes s}) \mid T.\triangle^{2r+2s-1}(g) = \triangle^{2r+2s-1}(g).T$$

and

$$T.(\sigma \otimes \sigma \otimes \cdots \otimes \sigma) = (\sigma \otimes \sigma \otimes \cdots \otimes \sigma).T, \quad \text{for all} \quad g \in \mathfrak{g}\}.$$

This is motivated us to define a map, $\Psi^{m,n}_{2r,2s} : \overrightarrow{D}_{r,s}(\delta) \longrightarrow End_{\mathfrak{g}^{2r+2s}}(V^{\otimes r} \otimes (V^*)^{\otimes s})^{op}$.

Next we will show that this map $\Psi^{m,n}_{2r,2s}$ is an isomorphism whenever $2r+2s < (m+1)(n+1)$. First we will prove the following lemma.

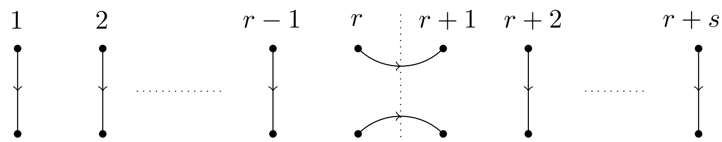
Lemma 3.7. *The walled signed Brauer algebra* $\overrightarrow{D}_{r,s}(x)$ *is the centralizer of* σ *in the walled Brauer algebra* $B_{2r,2s}(x)$*, where* $\sigma = (1\ 2)(3\ 4)\ldots(2r-1\ 2r)(2r+1\ 2r+2)\ldots(2r+2s-1\ 2r+2s)$ *is a permutation in* S_{2r+2s}*. That is*

$$\overrightarrow{D}_{r,s}(x) = \{d \in B_{2r,2s} \mid d\sigma = \sigma d\} = \mathcal{C}_{B_{2r,2s}}(\sigma).$$

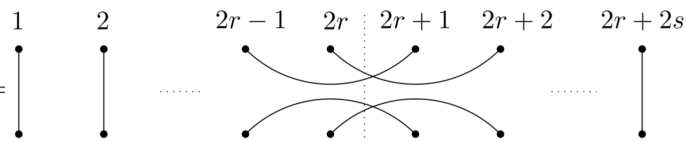
Proof. By the Theorem 3.2, the walled signed Brauer algebra $\overrightarrow{D}_{r,s}(x)$ is generated by the elements $\vec{h}_1, \vec{h}_{r+1},\ g_1,\ g_2, \ldots, g_{r-1},\ e_r, g_{r+1}, \ldots, g_{r+s-1}$. Since $\vec{h}_1, \vec{h}_{r+1},\ g_1,\ g_2, \ldots, g_{r-1},\ g_{r+1}, \ldots,\ g_{r+s-1} \in (\mathbb{Z}_2 \wr S_r \times \mathbb{Z}_2 \wr S_s) = \mathcal{C}_{S_{2r} \times S_{2s}}(\sigma) \subseteq \mathcal{C}_{S_{2r+2s}}(\sigma)$, $flip_{2r,2s}(g_i) = g_i$ for $i = 1, 2, \ldots, r-1, r+1, \ldots, r+s-1$ and $flip_{2r,2s}(\vec{h_j}) = h_j$ for $j = 1, r+1$, by Lemma 3.5, we have

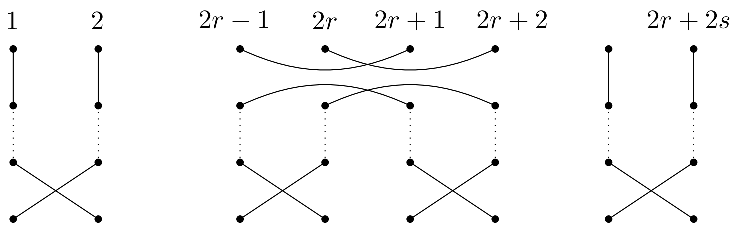
$$\vec{h}_1, \vec{h}_{r+1},\ g_1,\ g_2, \ldots, g_{r-1},\ g_{r+1}, \ldots,\ g_{r+s-1} \in \mathcal{C}_{B_{2r,2s}}(\sigma).$$

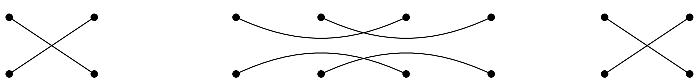
Since $j : \overrightarrow{D}_{r,s}(x) \longrightarrow B_{2r,2s}(x)$ is an embedding and

$e_r =$ 

Let $e'_r = j(e_r)$ then

$e'_r = j(e_r) =$ 

$e'_r.\sigma\ =$ 

$=$ 

$= \sigma.e'_r$

That is, $e_r \in \mathcal{C}_{B_{2r,2s}}(\sigma)$, therefore, $\overrightarrow{D}_{r,s}(x) \subseteq \mathcal{C}_{B_{2r,2s}}(\sigma)$.

Now let $d \in \mathcal{C}_{B_{2r,2s}}(\sigma)$ then $d\sigma = \sigma d$ in $B_{2r,2s}(x)$.

Since $flip_{2r+2s} : \mathbb{C}S_{2r+2s} \longrightarrow B_{2r,2s}(x)$, $flip_{r,s} : \mathcal{C}_{S_{2r+2s}}(\sigma) = \mathbb{Z}_2 \wr S_{r+s} \longrightarrow \overrightarrow{D}_{r,s}(x) \subseteq \mathcal{C}_{B_{2r,2s}}(\sigma)$ are vector space isomorphism and $\mathcal{C}_{B_{2r,2s}}(\sigma) \subseteq B_{2r,2s}(x)$.

By Lemma 3.5, $flip^{-1}_{2r,2s}(d).\sigma = \sigma.flip^{-1}_{2r,2s}(d)$ in S_{2r+2s}.

That is, $flip^{-1}_{2r,2s}(d) \in \mathcal{C}_{S_{2r+2s}}(\sigma) = \mathbb{Z}_2 \wr S_{r+s}$.

So $d \in \overrightarrow{D}_{r,s}(x)$, thus, $\mathcal{C}_{B_{2r,2s}}(\sigma) \subseteq \overrightarrow{D}_{r,s}(x)$.

Hence $\overrightarrow{D}_{r,s}(x) = \mathcal{C}_{B_{2r,2s}}(\sigma)$. □

From Lemma 7.7 and Theorem 7.8 in [2], we have, the following results: (i) the diagram

$$\begin{CD}
\mathbb{C}S_{2r+2s} @>{flip_{2r,2s}}>> B_{2r,2s}(\delta) \\
@V{\Phi^{m,n}_{2r,2s}}VV @VV{\Psi^{m,n}_{2r,2s}}V \\
End_{\mathfrak{g}}(V^{\otimes(r+s)}) @>{flip_{2r,2s}}>> End_{\mathfrak{g}}(V^{\otimes r} \otimes (V^*)^{\otimes s})
\end{CD}$$

commutes, and

(ii) when $2r+2s < (m+1)(n+1)$, $\Phi^{m,n}_{2r,2s}$ and $\Psi^{m,n}_{2r,2s}$ are algebra isomorphism, where $V = W \otimes W$ and $V^* = W^* \otimes W^*$.

Then the following diagram commutes:

$$\begin{CD}
\mathcal{C}_{S_{2r+2s}}(\sigma) \subset S_{2r+2s} @>{flip_{2r,2s}}>> \mathcal{C}_{B_{2r,2s}}(\sigma) \subset B_{2r,2s}(\delta) \\
@V{\Phi^{m,n}_{2r,2s}}VV @VV{\Psi^{m,n}_{2r,2s}}V \\
\mathcal{C}_{End_{\mathfrak{g}}(V^{\otimes(r+s)})}(\Phi^{m,n}_{2r,2s}(\sigma)) @>{flip_{2r,2s}}>> \mathcal{C}_{End_{\mathfrak{g}}(V^{\otimes r} \otimes (V^*)^{\otimes s})}(\Psi^{m,n}_{2r,2s}(\sigma))
\end{CD}$$

where,

$$\begin{aligned}
&\mathcal{C}_{End_{\mathfrak{g}}(V^{\otimes(r+s)})}(\Phi^{m,n}_{2r,2s}(\sigma)) \\
&= \{T' \in End_{\mathfrak{g}}(V^{\otimes(r+s)} \mid T'.\Phi^{m,n}_{2r,2s}(\sigma) = \Phi^{m,n}_{2r,2s}(\sigma).T'\} \\
&= \{T' \in End(V^{\otimes(r+s)}) \mid T'. \triangle^{2r+2s-1}(g) = \triangle^{2r+2s-1}(g).T'
\end{aligned}$$

$$\text{for all } g \in \mathfrak{g} \text{ and } T'.(\sigma \otimes \cdots \otimes \sigma) = (\sigma \otimes \cdots \otimes \sigma).T'\}$$
$$= End_{\mathfrak{g}^{2r+2s}}(V^{\otimes(r+s)})$$

and

$$\mathcal{C}_{End_{\mathfrak{g}}}(V^{\otimes r} \otimes (V^*)^{\otimes s})(\Psi^{m,n}_{2r,2s}(\sigma))$$
$$= \{T \in End(V^{\otimes r} \otimes (V^*)^{\otimes s}) \mid T. \triangle^{2r+2s-1} (g) = \triangle^{2r+2s-1}(g).T$$
$$\text{for all } g \in \mathfrak{g} \text{ and } T.(\sigma \otimes \cdots \otimes \sigma) = (\sigma \otimes \cdots \otimes \sigma).T\}$$
$$= End_{\mathfrak{g}^{2r+2s}}(V^{\otimes r} \otimes (V^*)^{\otimes s}).$$

Theorem 3.8. *When $2r+2s < (m+1)(n+1)$ the walled signed Brauer algebra $\overrightarrow{D}_{r,s}(\delta)$ is isomorphic to the centralizer algebra $End_{\mathfrak{g}^{2r+2s}}(V^{\otimes r}\otimes(V^*)^{\otimes s})^{op}$, where $\delta = m - n$.*

Proof. The proof follows from Lemma 3.6 and the above results. □

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