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TOPOLOGICAL PROPERTIES OF *L*-FUZZY UNIFORM SPACES

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Abstract: In this paper, we define L-neighborhood spaces and investigated the topological properties of L-fuzzy uniformity in complete residuated lattices. We obtain L-fuzzy topology and L-neighborhood spaces induced by L-fuzzy uniformity. Moreover, we study the relations among L-fuzzy topology, L-neighborhood system and L-fuzzy uniformity.

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1. Introduction

Lowen [9] introduced the notion of fuzzy uniformities as a viewpoint of the enourage approach. Many researchers [5-8,12] studied the different approach as powerset [6,12] or the uniform covering [5]. Kim [8] introduced the notion of fuzzy uniformities as an extension of Lowen in a stsc-quantale lattice L.

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On the other hand, Hájek [4] introduced a complete residuated lattice which is an algebraic structure for many valued logic. Bělohlávek [2] investigated information systems and decision rules in complete residuated lattices.

In this paper, we define L-neighborhood spaces and investigated the topological properties of L-fuzzy uniformity in complete residuated lattices. We obtain L-fuzzy topology and L-neighborhood spaces induced by L-fuzzy uniformity. Moreover, we study the relations among L-fuzzy topology, L-neighborhood system and L-fuzzy uniformity.

2. Preliminaries

Definition 2.1. (see [2], [4]) An algebra $(L, \wedge, \vee, \odot, \rightarrow, \perp, \top)$ is called a complete residuated lattice if it satisfies the following conditions:

(C1) $L = (L, \leq, \vee, \wedge, \perp, \top)$ is a complete lattice with the greatest element $\top$ and the least element $\perp$;

(C2) $(L, \odot, \top)$ is a commutative monoid;

(C3) $x \odot y \leq z$ iff $x \leq y \rightarrow z$ for $x, y, z \in L$.

For $\alpha \in L, \lambda \in L^X$, we denote $(\alpha \rightarrow \lambda), (\alpha \odot \lambda), \alpha_X \in L^X$ as $(\alpha \rightarrow \lambda)(x) = \alpha \rightarrow \lambda(x)$, $(\alpha \odot \lambda)(x) = \alpha \odot \lambda(x)$, $\alpha_X(x) = \alpha$.

In this paper, we assume that $(L, \vee, \wedge, \odot, \rightarrow, \perp, \top)$ is a complete residuated lattice.

Lemma 2.2. (see [2], [4]) *For each $x, y, z, w, x_i, y_i \in L$, the following properties hold:*

(1) *If $y \leq z$, then $x \odot y \leq x \odot z$.*

(2) *If $y \leq z$, then $x \rightarrow y \leq x \rightarrow z$ and $z \rightarrow x \leq y \rightarrow x$.*

(3) *$x \rightarrow y = \top$ iff $x \leq y$.*

(4) *$x \rightarrow \top = \top$ and $\top \rightarrow x = x$.*

(5) *$x \odot y \leq x \wedge y$.*

(6) *$x \odot (\bigvee_{i\in\Gamma} y_i) = \bigvee_{i\in\Gamma}(x \odot y_i)$ and $(\bigvee_{i\in\Gamma} x_i) \odot y = \bigvee_{i\in\Gamma}(x_i \odot y)$.*

(7) *$x \rightarrow (\bigwedge_{i\in\Gamma} y_i) = \bigwedge_{i\in\Gamma}(x \rightarrow y_i)$ and $(\bigvee_{i\in\Gamma} x_i) \rightarrow y = \bigwedge_{i\in\Gamma}(x_i \rightarrow y)$.*

(8) *$\bigvee_{i\in\Gamma} x_i \rightarrow \bigvee_{i\in\Gamma} y_i \geq \bigwedge_{i\in\Gamma}(x_i \rightarrow y_i)$ and $\bigwedge_{i\in\Gamma} x_i \rightarrow \bigwedge_{i\in\Gamma} y_i \geq \bigwedge_{i\in\Gamma}(x_i \rightarrow y_i)$.*

(9) $(x \to y) \odot x \leq y$ *and* $(x \to y) \odot (y \to z) \leq (x \to z)$.

(10) $x \to y \leq (y \to z) \to (x \to z)$ *and* $x \to y \leq (z \to x) \to (z \to y)$.

(11) $(x \odot y) \to z = x \to (y \to z) = y \to (x \to z)$.

(12) $y \to z \leq x \odot y \to x \odot z$ *and* $(x \to z) \odot (y \to w) \leq x \odot y \to z \odot w$.

Definition 2.3. (see [3], [10]) Let X be a set. A function $R : X \times X \to L$ is called an *L-partial order* if it satisfies the following conditions:

(E1) reflexive if $R(x, x) = \top$ for all $x \in X$,

(E2) transitive if $R(x, y) \odot R(y, z) \leq R(x, z)$, for all $x, y, z \in X$,

(E3) antisymmetric if $R(x, y) = R(y, x) = \top$, then $x = y$.

Lemma 2.4. (see [3], [10]) *For a given set X, define a binary mapping $S : L^X \times L^X \to L$ by*

$$S(\lambda, \mu) = \bigwedge_{x \in X} (\lambda(x) \to \mu(x)).$$

Then, for each $\lambda, \mu, \rho, \nu \in L^X$, and $\alpha \in L$, the following properties hold:

(1) *S is an L-partial order on L^X.*

(2) *$\lambda \leq \mu$ iff $S(\lambda, \mu) \geq \top$,*

(3) *If $\lambda \leq \mu$, then $S(\rho, \lambda) \leq S(\rho, \mu)$ and $S(\lambda, \rho) \geq S(\mu, \rho)$ for each $\rho \in L^X$,*

(4) *$S(\lambda, \mu) \odot S(\nu, \rho) \leq S(\lambda \odot \nu, \mu \odot \rho)$.*

Lemma 2.5. (see [3]) *Let $\phi : X \to Y$ be an ordinary mapping. Define $\phi^{\to} : L^X \to L^Y$ and $\phi^{\leftarrow} : L^Y \to L^X$ by*

$$\phi^{\to}(\lambda)(y) = \bigvee_{\phi(x)=y} \lambda(x), \quad \forall \lambda \in L^X,\ y \in Y,$$

$$\phi^{\leftarrow}(\mu)(x) = \mu(\phi(x)) = \mu \circ \phi(x), \ \ \forall \mu \in L^Y.$$

Then for $\lambda, \mu \in L^X$ and $\rho, \nu \in L^Y$,

$$S(\lambda, \mu) \leq S(\phi^{\to}(\lambda), \phi^{\to}(\mu)), \ \ S(\rho, \nu) \leq S(\phi^{\leftarrow}(\rho), \phi^{\leftarrow}(\nu)),$$

and the equalities hold if ϕ is bijective.

Definition 2.6. (see [5]) A map $\mathcal{T} : L^X \to L$ is called an L-fuzzy topology on X if it satisfies the following conditions:

(LO1) $\mathcal{T}(\bot_X) = \mathcal{T}(\top_X) = \top$,

(LO2) $\mathcal{T}(\lambda \odot \mu) \geq \mathcal{T}(\lambda) \odot \mathcal{T}(\mu), \quad \forall \quad \lambda, \mu \in L^X$, (LO3) $\mathcal{T}(\bigvee_i \lambda_i) \geq \bigwedge_i \mathcal{T}(\lambda_i), \quad \forall \quad \{\lambda_i\}_{i\in\Gamma} \subseteq L^X$.

An L-fuzzy topology is called enriched if

(R) $\mathcal{T}(\alpha \odot \lambda) \geq \mathcal{T}(\lambda)$ for all $\lambda \in L^X$ and $\alpha \in L$.

The pair $(X, \mathcal{T})$ is called an L-fuzzy topological space.

Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be two L-fuzzy topological spaces. A mapping $\phi : X \to Y$ is said to be L-continuous iff for each $\lambda \in L^Y$,

$$\mathcal{T}_2(\lambda) \leq \mathcal{T}_1(\phi^{\leftarrow}(\lambda)).$$

Definition 2.7. (see [8]) A map $\mathcal{U} : L^{X\times X} \to L$ is called an L-fuzzy uniformity on X iff the following conditions hold:

(LU1) There exists $u \in L^{X\times X}$ such that $\mathcal{U}(u) = \top$.

(LU2) If $v \leq u$, then $\mathcal{U}(v) \leq \mathcal{U}(u)$.

(LU3) For every $u, v \in L^{X\times X}$,$\mathcal{U}(u \odot v) \geq \mathcal{U}(u) \odot \mathcal{U}(v)$.

(LU4) If $\mathcal{U}(u) \neq \bot$ then $\top_\triangle \leq u$ where

$$\top_\triangle(x, y) = \begin{cases} \top, & \text{if x = y} \\ \bot, & \text{if } x \neq y, \end{cases}$$

(LU5) $\mathcal{U}(u) \leq \mathcal{U}(u^{-1})$, where $u^{-1}(x, y) = u(y, x)$.

(LU6) $\mathcal{U}(u) \leq \bigvee\{\mathcal{U}(v) \mid v \circ v \leq u\}, \quad \forall \, u \in L^{X\times X}$, where

$$v \circ v(x, y) = \bigvee_{z\in X} v(x, z) \odot v(z, y), \quad \forall \, x, y \in X.$$

An L-fuzzy uniformity $\mathcal{U}$ on X is said to be stratified if:

(R) $\mathcal{U}(\alpha \odot u) \geq \alpha \odot \mathcal{U}(u), \quad \forall \, u \in L^{X\times X}$.

The pair $(X,\mathcal{U})$ is called an L-fuzzy uniform space.

Let $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ be L-fuzzy uniform spaces, and $\phi : X \to Y$ be a mapping. Then ϕ is said to be L-fuzzy uniformly continuous if $\mathcal{V}(v) \leq \mathcal{U}((\phi \times \phi)^{\leftarrow}(v))$, for every $v \in L^{Y \times Y}$.

Remark 2.8. Let $(X,\mathcal{U})$ be an L-fuzzy uniform space.

(1) By (LU1) and (LU2), we have $\mathcal{U}(\top_{X\times X}) = \top$ because $u \leq \top_{X\times X}$ for all $u \in L^{X\times X}$.

(2) Since $\mathcal{U}(u) \leq \mathcal{U}(u^{-1}) \leq \mathcal{U}((u^{-1})^{-1}) = \mathcal{U}(u)$, then $\mathcal{U}(u) = \mathcal{U}(u^{-1})$.

3. Topological Properties of L-Fuzzy Uniform Spaces

Definition 3.1. A map $N : X \to L^{L^X}$ is called an L-neighborhood system on X if $N = \{N_x \mid x \in X\}$ satisfies the following conditions

(LN1) $N_x(\top_X) = \top$ and $N_x(\bot_X) = \bot$,

(LN2) $N_x(\lambda \odot \mu) \geq N_x(\lambda) \odot N_x(\mu)$ for each $\lambda, \mu \in L^X$,

(LN3) If $\lambda \leq \mu$, then $N_x(\lambda) \leq N_x(\mu)$,

(LN4) $N_x(\lambda) \leq \lambda(x)$ for all $\lambda \in L^X$.

An L-neighborhood system is called stratified if

(R) $N_x(\alpha \odot \lambda) \geq \alpha \odot N_x(\lambda)$ for all $\lambda \in L^X$ and $\alpha \in L$.

The pair (X, N) is called an L-neighborhood space.

Let (X, N) and (Y, M) be two L-neighborhood spaces. A mapping $\phi : X \to Y$ is said to be L-continuous at $x \in X$ iff $M_{\phi(x)}(\lambda) \leq N_x(\phi^{\leftarrow}(\lambda))$ for each $\lambda \in L^Y$, ϕ is L-continuous if it is L-continuous at every $x \in X$.

Theorem 3.2. *Let $(X,\mathcal{U})$ be an L-fuzzy uniform space. Define a map $N^{\mathcal{U}} : X \to L^{L^X}$ by*

$$N^{\mathcal{U}}_x(\lambda) = \bigvee_u \mathcal{U}(u) \odot S(u[x], \lambda), \quad \forall\, \lambda \in L^X,\ x \in X,$$

where $u[x](y) = u(y, x)$. Then the following properties hold:

(1) *$(X, N^{\mathcal{U}})$ is an L-neighborhood space.*

(2) *If $\mathcal{U}$ is stratified, then N is also stratified.*

Proof. (1) (LN1) For $\mathcal{U}(u) \neq \bot, \ \top_\triangle \leq u$. Then

$$\begin{aligned} N_x^{\mathcal{U}}(\bot_X) &= \bigvee_u \mathcal{U}(u) \odot S(u[x], \bot_X) \\ &\leq \bigvee_u \big(\mathcal{U}(u) \odot (u(x,x) \to \bot)\big) \\ &= \bigvee_u \big(\mathcal{U}(u) \odot (\top_\triangle(x,x) \to \bot)\big) = \bot. \end{aligned}$$

Hence $N_x^{\mathcal{U}}(\bot_X) = \bot$. Also, $N_x^{\mathcal{U}}(\top_X) = \top$, because

$$N_x^{\mathcal{U}}(\top_X) \geq \mathcal{U}(\top_{X\times X}) \odot \bigwedge_{y\in X} (\top_{X\times X}(x,y) \to \top_X(y)) = \top.$$

(LN2) By Lemma 2.4 (4) , we have

$$\begin{aligned} & N_x^{\mathcal{U}}(\lambda) \odot N_x^{\mathcal{U}}(\mu) \\ &= \big(\bigvee_u \mathcal{U}(u) \odot S(u[x], \lambda)\big) \odot \big(\bigvee_u \mathcal{U}(v) \odot S(v[x], \mu)\big) \\ &= \bigvee_{u,v} \mathcal{U}(u) \odot \mathcal{U}(v) \odot S(u[x], \lambda) \odot S(v[x], \mu) \\ &\leq \bigvee_{u,v} \mathcal{U}(u \odot v) \odot S((u \odot v)[x], \lambda \odot \mu) \\ &\leq \bigvee_w \mathcal{U}(w) \odot S(w[x], \lambda \odot \mu) = N_x^{\mathcal{U}}(\lambda \odot \mu). \end{aligned}$$

(LN3) By Lemma 2.4 (3), we have

$$\begin{aligned} N_x^{\mathcal{U}}(\lambda) &= \bigvee_u \mathcal{U}(u) \odot S(u[x], \lambda) \\ &\leq \bigvee_u \mathcal{U}(u) \odot S(u[x], \mu) = N_x^{\mathcal{U}}(\mu). \end{aligned}$$

(LN4) For $\mathcal{U}(u) \neq \bot, \ \top_\triangle \leq u$.

$$N_x^{\mathcal{U}}(\lambda) = \bigvee_u \mathcal{U}(u) \odot \bigwedge_{y\in X} (u(y,x) \to \lambda(y))$$

$$\leq \bigvee_u \{\mathcal{U}(u) \odot (u(x,x) \to \lambda(x))\} \leq \lambda(x).$$

This implies that $(X, N_x^{\mathcal{U}})$ is an L-neighborhood space.

(2)

$$\begin{aligned}\alpha \odot N_x^{\mathcal{U}}(\lambda) &= \alpha \odot \bigvee_u \mathcal{U}(u) \odot S(u[x], \lambda)\\ &= \bigvee_u \alpha \odot \mathcal{U}(u) \odot S(\alpha, \alpha) \odot S(u[x], \lambda)\\ &\leq \bigvee_u \mathcal{U}(\alpha \odot u) \odot S(\alpha \odot u[x], \alpha \odot \lambda)\\ &\leq N_x^{\mathcal{U}}(\alpha \odot \lambda).\end{aligned}$$

Theorem 3.3. *If $\phi : (X, \mathcal{U}) \to (Y, \mathcal{V})$ is L-fuzzy uniformly continuous, then $\phi : (X, N^{\mathcal{U}}) \to (Y, N^{\mathcal{V}})$ is L-continuous.*

Proof. First we show that $\phi^{\leftarrow}(v[\phi(x)]) = (\phi \times \phi)^{\leftarrow}(v)[x]$ from

$$\begin{aligned}\phi^{\leftarrow}(v[\phi(x)])(z) &= v[\phi(x)](\phi(z)) = v(\phi(z), \phi(x))\\ &= (\phi \times \phi)^{\leftarrow}(v)(z,x) = (\phi \times \phi)^{\leftarrow}(v)[x](z).\end{aligned}$$

Thus, by Lemma 2.5, we have

$$\begin{aligned}S(v[\phi(x)], \lambda) &\leq S(\phi^{\leftarrow}(v[\phi(x)]), \phi^{\leftarrow}(\lambda))\\ &= S((\phi \times \phi)^{\leftarrow}(v)[x], \phi^{\leftarrow}(\lambda)).\end{aligned}$$

$$\begin{aligned}N_{\phi(x)}^{\mathcal{V}}(\lambda) &= \bigvee_v \mathcal{V}(v) \odot S(v[\phi(x)], \lambda)\\ &\leq \bigvee_v \mathcal{V}(v) \odot S((\phi \times \phi)^{\leftarrow}(v)[x], \phi^{\leftarrow}(\lambda))\\ &\leq \bigvee_u \mathcal{U}((\phi \times \phi)^{\leftarrow}(v)) \odot S((\phi \times \phi)^{\leftarrow}(v)[x], \phi^{\leftarrow}(\lambda))\\ &\leq N_x^{\mathcal{U}}(\phi^{\leftarrow}(\lambda)).\end{aligned}$$

Theorem 3.4. *(1) The L-neighborhood system* $N^{\mathcal{U}} = \{N_x^{\mathcal{U}} \mid x \in X\}$ *can be constructed from the cuts* $\{u \in L^{X\times X} \mid \mathcal{U} \geq \alpha\}$ *as follows:*

$$N_x^{\mathcal{U}}(\lambda) = \bigvee_{\alpha\in L} \alpha \odot N_x^{\mathcal{U}}(\lambda, \alpha),$$

where $N_x^{\mathcal{U}}(\lambda, \alpha) = \bigvee_v \{S(u[x], \lambda) \mid \mathcal{U}(u) \geq \alpha\}$.

(2)

$$N_x^{\mathcal{U}}(\lambda) \leq \bigvee_v \{N_x^{\mathcal{U}}(\rho) \mid \rho(z) \leq N_z^{\mathcal{U}}(\lambda, \mathcal{U}(v))\}.$$

Proof. (1) If for some $y \in X$ we have $A(y) \geq \alpha$, then we can write $A(y) \odot B(y) \geq \alpha \odot B(y)$ and

$$\bigvee \{A(x) \odot B(x) \mid A(x) \geq \alpha\} \geq \bigvee \{\alpha \odot B(y) \mid A(x) \geq \alpha\}.$$

Suppose

$$\bigvee \{A(x) \odot B(x) \mid x \in X\} \not\leq \bigvee_{\alpha\in L} \bigvee \{\alpha \odot B(x) \mid A(x) \geq \alpha\}.$$

There exists $x_0 \in X$ such that

$$A(x_0) \odot B(x_0) \not\leq \bigvee_{\alpha\in L} \bigvee \{\alpha \odot B(x) \mid A(x) \geq \alpha\}.$$

It is a contradiction. Hence

$$\bigvee \{A(x) \odot B(x) \mid x \in X\} = \bigvee_{\alpha\in L} \bigvee \{\alpha \odot B(x) \mid A(x) \geq \alpha\}.$$

Applying this equality to the formula giving $N_x^{\mathcal{U}}(\lambda)$, we obtain

$$\begin{aligned} N_x^{\mathcal{U}}(\lambda) &= \bigvee_{\alpha\in L} \{\bigvee \alpha \odot S(u[x], \lambda) \mid \mathcal{U}(u) \geq \alpha\} \\ &= \bigvee_{\alpha\in L} \{\alpha \odot \bigvee \{S(u[x], \lambda) \mid \mathcal{U}(u) \geq \alpha\}\} \\ &= \bigvee_{\alpha\in L} \{\alpha \odot N_x^{\mathcal{U}}(\lambda, \alpha)\}. \end{aligned}$$

(2) For $u \in L^{X\times X}$ and $\lambda \in L^X$, we have

$$N_x^{\mathcal{U}}(\lambda) = \bigvee_u \mathcal{U}(u) \odot S(u[x],\lambda)$$
$$= \bigvee_u \{\mathcal{U}(u) \odot \bigwedge_{y\in X} (u(y,x) \to \lambda(y))\} \text{ (by LU(6))}$$
$$\leq \bigvee_v \{\mathcal{U}(v) \odot \bigwedge_{y\in X} ((v \circ v)(y,x) \to \lambda(y))\}$$
$$= \bigvee_v \{\mathcal{U}(v) \odot \bigwedge_{y\in X} ((\bigvee_{z\in X} v(z,x) \odot v(y,z)) \to \lambda(y))\}$$
$$= \bigvee_v \{\mathcal{U}(v) \odot \bigwedge_{y\in X} \bigwedge_{z\in X} ((v(z,x) \odot v(y,z)) \to \lambda(y))\}$$
$$= \bigvee_v \{\mathcal{U}(v) \odot \bigwedge_{y\in X} \bigwedge_{z\in X} (v(z,x) \to (v(y,z) \to \lambda(y))\} \text{ (by Lemma 2.2 (12))}$$
$$= \bigvee_v \{\mathcal{U}(v) \odot \bigwedge_{z\in X} (v(z,x) \to \bigwedge_{y\in X} (v(y,z) \to \lambda(y))\}.$$

Put $\rho(z) = \bigwedge_{y\in X}(v(y,z) \to \lambda(y))$. Then $\rho(z) \leq N_z^{\mathcal{U}}(\lambda,\mathcal{U}(v))$ for all $z \in X$. Thus,

$$N_x^{\mathcal{U}}(\lambda) \leq \bigvee_v \{\mathcal{U}(v) \odot \bigwedge_{z\in X} (v(z,x) \to \rho(z)) \mid \rho(z) \leq N_z^{\mathcal{U}}(\lambda,\mathcal{U}(v))\}$$
$$\leq \bigvee_v \{N_x^{\mathcal{U}}(\rho) \mid \rho(z) \leq N_z^{\mathcal{U}}(\lambda,\mathcal{U}(v))\}.$$

Theorem 3.5. Let X be any nonempty set and (Y, M) be an L-neighborhood space. If $\phi : X \to Y$ is a mapping, then (X,N) is an L-neighborhood space, where

$$N_x(\lambda) = \bigvee\{M_{\phi(x)}(\mu) \mid \phi^{\leftarrow}(\mu) \leq \lambda,\ \mu \in L^Y\}.$$

Moreover, if ϕ is surjective, then ϕ is L-continuous and if M is stratified, then N is also stratified.

Proof. (LN1) and (LN3) are clearly true.

(LN2) Let $\lambda_1, \lambda_2 \in L^X$, $\mu_1, \mu_2 \in L^Y$ and $x \in X$, then we have

$N_x(\lambda_1) \odot N_x(\lambda_2)$

$$
\begin{aligned}
&= \bigvee\{M_{\phi(x)}(\mu_1) \mid \phi^{\leftarrow}(\mu_1) \le \lambda_1\} \odot \bigvee\{M_{\phi(x)}(\mu_2) \mid \phi^{\leftarrow}(\mu_2) \le \lambda_2\} \\
&= \bigvee\{M_{\phi(x)}(\mu_1) \odot M_{\phi(x)}(\mu_2) \mid \phi^{\leftarrow}(\mu_1) \le \lambda_1,\ \phi^{\leftarrow}(\mu_2) \le \lambda_2\} \\
&\le \bigvee\{M_{\phi(x)}(\mu_1 \odot \mu_2) \mid \phi^{\leftarrow}(\mu_1) \odot \phi^{\leftarrow}(\mu_2) \le \lambda_1 \odot \lambda_2\} \\
&= \bigvee\{M_{\phi(x)}(\mu) \mid \phi^{\leftarrow}(\mu) \le \lambda_1 \odot \lambda_2\} \\
&= N_x(\lambda_1 \odot \lambda_2).
\end{aligned}
$$

(LN4) For any $\lambda \in L^X$, $\mu \in L^Y$ and $x \in X$, we have

$$
\begin{aligned}
&N_x(\lambda) = \bigvee\{M_{\phi(x)}(\mu) \mid \phi^{\leftarrow}(\mu) \le \lambda\} \le \bigvee\{\mu(\phi^{\rightarrow}(x)) \mid \phi^{\leftarrow}(\mu) \le \lambda\} \\
&= \bigvee\{\phi^{\leftarrow}(\mu)(x) \mid \phi^{\leftarrow}(\mu) \le \lambda\} \le \lambda(x).
\end{aligned}
$$

Now, since the mapping $\phi : (X, N) \to (Y, M)$ is L-continuous and if $x \in X, \mu \in L^Y$, it follows that

$$
\begin{aligned}
&N_x(\phi^{\leftarrow}(\mu)) = \bigvee\{M_{\phi(x)}(\lambda) \mid \phi^{\leftarrow}(\lambda) \le \phi^{\leftarrow}(\mu)\} \\
&\ge \bigvee\{M_{\phi(x)}(\lambda) \mid \lambda \le \mu\} \ge M_{\phi(x)}(\mu).
\end{aligned}
$$

Finally, if M is stratified, then N is also stratified. In fact, for any $\alpha \in L$ and $\lambda \in L^X$, we have

$$
\begin{aligned}
&\alpha \odot N_x(\lambda) = \alpha \odot \bigvee\{M_{\phi(x)}(\mu) \mid \phi^{\leftarrow}(\mu) \le \lambda\} \\
&\le \bigvee\{\alpha \odot M_{\phi(x)}(\mu) \mid \alpha \odot \phi^{\leftarrow}(\mu) \le \alpha \odot \lambda\} \\
&\le \bigvee\{M_{\phi(x)}(\alpha \odot \mu) \mid \phi^{\leftarrow}(\alpha \odot \mu) \le \alpha \odot \lambda\} \\
&= N_x(\alpha \odot \lambda).
\end{aligned}
$$

Theorem 3.6. *Let (X, N) be a L-neighborhood space. Define a map $\mathcal{T}_N : L^X \to L$ by*

$$\mathcal{T}_N(\lambda) = \bigwedge_{x \in X} (\lambda(x) \to N_x(\lambda)).$$

Then (1) $\mathcal{T}_N$ *is an L-fuzzy topology on* X*,*

(2) If N_x *is stratified, then* $\mathcal{T}_N$ *is an enriched L-fuzzy topology.*

Proof. (1) (LO1)

$$\mathcal{T}_N(\top_X) = \bigwedge_{x\in X} (\top_X(x) \to N_x(\top_X)) = \top \to \top = \top,$$

$$\mathcal{T}_N(\bot_X) = \bigwedge_{x\in X} (\bot_X(x) \to N_x(\bot_X)) = \bot \to \bot = \bot.$$

(LO2)

$$\begin{aligned}
&\mathcal{T}_N(\lambda \odot \mu) = \bigwedge_{x\in X} ((\lambda \odot \mu)(x) \to N_x(\lambda \odot \mu)) \\
&\geq \bigwedge_{x\in X} ((\lambda \odot \mu)(x) \to (N_x(\lambda) \odot N_x(\mu))) \\
&\qquad \text{(by Lemma 2.2 (12))} \\
&\geq \bigwedge_{x\in X} (\lambda(x) \to N_x(\lambda)) \odot \bigwedge_{x\in X} (\mu(x) \to N_x(\mu)) \\
&= \mathcal{T}_N(\lambda) \odot \mathcal{T}_N(\mu).
\end{aligned}$$

(LO3)

$$\begin{aligned}
\mathcal{T}_N(\bigvee_i \lambda_i) &= \bigwedge_{x\in X} ((\bigvee_i \lambda_i)(x) \to N_x(\bigvee_i \lambda_i)) \\
&\geq \bigwedge_{x\in X} ((\bigvee_i \lambda_i)(x) \to \bigvee_i N_x(\lambda_i)) \\
&\qquad \text{(by Lemma 2.2 (8))} \\
&\geq \bigwedge_i \bigwedge_{x\in X} (\lambda_i(x) \to N_x(\lambda_i)) = \bigwedge_i \mathcal{T}_N(\lambda_i).
\end{aligned}$$

(2) By Lemma 2.2 (12), we have

$$\begin{aligned}
&\mathcal{T}_N(\alpha \odot \lambda) = \bigwedge_{x\in X} ((\alpha \odot \lambda)(x) \to N_x(\alpha \odot \lambda)) \\
&\geq \bigwedge_{x\in X} ((\alpha \odot \lambda)(x) \to (\alpha \odot N_x(\lambda)))
\end{aligned}$$

$$\geq \bigwedge_{x\in X} (\lambda(x) \to N_x(\lambda)) = \mathcal{T}_N(\lambda).$$

Corollary 3.7. *Let Let $(X,\mathcal{U})$ be an L-fuzzy uniform space and $\{N_x^{\mathcal{U}} \mid x \in X\}$ be an L-neighorhood system on X. Define a map $\mathcal{T}_{\mathcal{U}} : L^X \to L$ by*

$$\mathcal{T}_{\mathcal{U}}(\lambda) = \bigwedge_{x\in X} (\lambda(x) \to N_x^{\mathcal{U}}(\lambda)).$$

Then:

(1) $\mathcal{T}_{\mathcal{U}}$ is an L-fuzzy topology on X,

(2) If $N_x^{\mathcal{U}}$ is stratified, then $\mathcal{T}_{\mathcal{U}}$ is an enriched L-fuzzy topology.

Theorem 3.8. *Let $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ be two L-fuzzy uniform spaces. If a map $\phi : (X,\mathcal{U}) \to (Y,\mathcal{V})$ is L-fuzzy uniformly continuous, then a map $\phi : (X,\mathcal{T}_{\mathcal{U}}) \to (Y,\mathcal{T}_{\mathcal{V}})$ is L- continuous.*

Proof.

$$\begin{aligned}
&\mathcal{T}_{\mathcal{V}}(\lambda) \to \mathcal{T}_{\mathcal{U}}(\phi^{\leftarrow}(\lambda))\\
&= \bigwedge_{y\in Y} (\lambda(y) \to N_y^{\mathcal{V}}(\lambda)) \to \bigwedge_{x\in X} (\phi^{\leftarrow}(\lambda)(x) \to N_x^{\mathcal{U}}(\phi^{\leftarrow}(\lambda)))\\
&\geq \bigwedge_{x\in X} (\phi^{\leftarrow}(\lambda)(x) \to N_{\phi^{\leftarrow}(x)}^{\mathcal{V}}(\lambda)) \to \bigwedge_{x\in X} (\phi^{\leftarrow}(\lambda)(x) \to N_x^{\mathcal{U}}(\phi^{\leftarrow}(\lambda)))\\
&\geq \bigwedge_{x\in X} \Big((\phi^{\leftarrow}(\lambda) \to N_{\phi^{\leftarrow}(x)}^{\mathcal{V}}(\lambda)) \to (\phi^{\leftarrow}(\lambda) \to N_x^{\mathcal{U}}(\phi^{\leftarrow}(\lambda)))\Big)\\
&\text{(by Lemma 2.2 (8))}\\
&\geq \bigwedge_{x\in X} \Big(N_{\phi^{\leftarrow}(x)}^{\mathcal{V}}(\lambda) \to N_x^{\mathcal{U}}(\phi^{\leftarrow}(\lambda))\Big) \text{ (by Lemma 2.2 (10)).}
\end{aligned}$$

Thus, if $N_{\phi^{\leftarrow}(x)}^{\mathcal{V}}(\lambda) \leq N_x^{\mathcal{U}}(\phi^{\leftarrow}(\lambda))$, then $\mathcal{T}_{\mathcal{V}}(\lambda) \leq \mathcal{T}_{\mathcal{U}}(\phi^{\leftarrow}(\lambda))$. So, ϕ is L-continuous.

Example 3.9. Let $(L = [0,1], \odot, \to)$ be a complete residuated lattice defined by

$$x \odot y = (x + y - 1) \vee 0, \quad x \to y = (1 - x + y) \wedge 1.$$

Let $X = \{x, y, z\}$ be a set and $w, w \odot w \in L^{X \times X}$ such that

$$w = \begin{pmatrix} 1 & 0.7 & 0.6 \\ 0.7 & 1 & 0.3 \\ 0.6 & 0.3 & 1 \end{pmatrix}, w \odot w = \begin{pmatrix} 1 & 0.4 & 0.2 \\ 0.4 & 1 & 0 \\ 0.2 & 0 & 1 \end{pmatrix}.$$

Define $\mathcal{U} : L^X \to L$ as follows

$$\mathcal{U}(u) = \begin{cases} 1, & \text{if } u = 1_{X \times X}, \\ 0.6, & \text{if } w \leq u \neq 1_{X \times X}, \\ 0.3, & \text{if } \ w \odot w \leq u \not\geq w, \\ 0, & \text{otherwise.} \end{cases}$$

Since $0.3 = \mathcal{U}(w \odot w) \geq \mathcal{U}(w) \odot \mathcal{U}(w) = 0.2$ and $w \circ w = w, (w \odot w) \circ (w \odot w) = (w \odot w)$, $w = w^{-1}$, $\mathcal{U}$ is an L-fuzzy uniformity on X.

$$N_x^{\mathcal{U}}(\lambda) = \bigvee_u \mathcal{U}(u) \odot S(u[x], \lambda), \quad \forall\, \lambda \in L^X, \ x \in X,$$

$$\begin{aligned} N_x^{\mathcal{U}}(\lambda) &= \textstyle\bigvee_u \mathcal{U}(u) \odot S(u[x], \lambda) \\ &= (\lambda(x) \wedge \lambda(y) \wedge \lambda(z)) \vee \Big(0.6 \odot (\lambda(x) \wedge (0.3 + \lambda(y)) \wedge (0.4 + \lambda(z))\Big) \\ &\quad \vee \Big(0.3 \odot (\lambda(x) \wedge (0.6 + \lambda(y)) \wedge (0.8 + \lambda(z))\Big) \end{aligned}$$

$$\begin{aligned} N_y^{\mathcal{U}}(\lambda) &= (\lambda(x) \wedge \lambda(y) \wedge \lambda(z)) \vee \Big(0.6 \odot ((0.3 + \lambda(x)) \wedge \lambda(y) \wedge (0.7 + \lambda(z))\Big) \\ &\quad \vee \Big(0.3 \odot ((0.6 + \lambda(x)) \wedge \lambda(y))\Big) \end{aligned}$$

$$\begin{aligned} N_z^{\mathcal{U}}(\lambda) &= (\lambda(x) \wedge \lambda(y) \wedge \lambda(z)) \vee \Big(0.6 \odot ((0.4 + \lambda(x)) \wedge (0.7 + \lambda(y)) \wedge \lambda(z))\Big) \\ &\quad \vee \Big(0.3 \odot ((0.8 + \lambda(x)) \wedge \lambda(z))\Big) \end{aligned}$$

For $\lambda = (0.4, 0.6, 0.8)$,

$$N_x^{\mathcal{U}}(\lambda) = 0.4, N_y^{\mathcal{U}}(\lambda) = 0.4, N_z^{\mathcal{U}}(\lambda) = 0.4$$

$$\mathcal{T}_N(\lambda) = \bigwedge_{x \in X} (\lambda(x) \to N_x(\lambda)) = 0.7.$$

For $\rho = (0.8, 0.2, 0.7)$,

$$N_x^{\mathcal{U}}(\lambda) = 0.2, N_y^{\mathcal{U}}(\lambda) = 0.2, N_z^{\mathcal{U}}(\lambda) = 0.3$$

$$\mathcal{T}_N(\lambda) = \bigwedge_{x \in X} (\lambda(x) \to N_x(\lambda)) = 0.4.$$

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