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ON THE DIOPHANTINE EQUATION $323^x + 323^{2s}n^y = z^{2t}$ WHERE s, t, n ARE NON-NEGATIVE INTEGERS AND $n \equiv 5 \pmod{20}$

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Abstract: Let s, t, n be non negative integers such that $n \equiv 5 \pmod{20}$. In this paper, we found that all non-negative integer solutions (x, y, z) of the Diophantine equation $323^x + 323^{2s}n^y = z^{2t}$ are in the following form:

$$(x, y, z) = \begin{cases} (1+2s, 0, 18(323)^s) & ; \quad t = 1, \\ \text{no solution} & ; \quad \text{otherwise.} \end{cases}$$

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1. Introduction

There are many mathematicians solved all non-negative integer solutions (x, y, z) of the Diophantine equation $a^x + b^y = z^2$ where a, b are positive integers and x, y, z are non-negative integers. For example, in 2013, Chotchaisthit [1] showed that $(p, x, y, z) = (7, 0, 1, 3)$ and $(p, x, y, z) = (3, 2, 2, 5)$ are the only solutions of the Diophantine equation $p^x + (p+1)^y = z^2$ where x, y, z are non-negative integers and p is the Mersenne prime.

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In 2013-2014, there are numbers of paper of Sroysang, for examples, in [4, 5] solved all non-negative integer solutions (x, y, z) of the Diophantine equations $3^x + 5^y = z^2$, $3^x + 17^y = z^2$ and he found that both Diophantine equations have exactly one non-negative integer solution, namely, $(x, y, z) = (1, 0, 2)$.

In recently 2014, Sroysang [6] showed that there is a unique non-negative integer solution $(x, y, z) = (1, 0, 18)$ for the Diophantine equation $323^x + 325^y = z^2$. Now we find all possible non-negative integer solutions (x, y, z) of the Diophantine equation $323^x + 323^{2s} n^y = z^{2t}$ where $n \equiv 5 \pmod{20}$, which is a generalization of the Diophantine equation $323^x + 325^y = z^2$ where $s = 0$ and $n = 325$.

2. Preliminaries

Throughout this paper, we assume that n is a non-negative integer such that $n \equiv 5 \pmod{20}$. It is clear that $n \equiv 1 \pmod 4$ and $n \equiv 0 \pmod 5$.

Corollary 2.1. *[3] Let a and b be nonzero integers and c be an integer. If $a|c$ and $b|c$, with* $\gcd(a, b) = 1$*, then $ab|c$.*

Theorem 2.2. *[3] Let a, b, p be integers. If p is a prime and $p|ab$, then $p|a$ or $p|b$.*

Lemma 2.3. *[6]* $(1, 18)$ *is a unique non-negative integer solution* (x, z) *for the Diophantine equation* $323^x + 1 = z^2$.

Let p be an odd prime and a be a positive integer where $gcd(a, p) = 1$. If the quadratic congruence $x^2 \equiv a \pmod p$ has a solution, then a is said to be *a quadratic residue of* p. Otherwise, a is called *a quadratic non-residue of* p. In 1798 Adrien-Marie Legendre introduced *the Legendre symbol* $\left(\frac{a}{p}\right)$ which is defined by

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & ; \text{if } a \text{ is a quadratic residue of } p, \\ -1 & ; \text{if } a \text{ is a quadratic non-residue of } p. \end{cases}$$

In the present paper, we need the following well-known facts about the Legendre symbols.

Theorem 2.4. *[3] If p is an odd prime, then*

$$\left(\frac{2}{p}\right) = \begin{cases} 1 & ;\textit{if } p \equiv 1 \pmod 8 \textit{ or } p \equiv 7 \pmod 8, \\ -1 & ;\textit{if } p \equiv 3 \pmod 8 \textit{ or } p \equiv 5 \pmod 8. \end{cases}$$

Theorem 2.5. *[3] If $p \neq 3$ is an odd prime, then*

$$\left(\frac{3}{p}\right) = \begin{cases} 1 & ;\textit{if } p \equiv \pm 1 \pmod{12}, \\ -1 & ;\textit{if } p \equiv \pm 5 \pmod{12}. \end{cases}$$

3. Main Results

First, we find all non-negative integer solutions (x, y, z) of the Diophantine equation $323^x + n^y = z^2$ where x, y, z are non-negative integers.

Theorem 3.1. $(1, 0, 18)$ *is a unique non-negative integer solution* (x, y, z) *for the Diophantine equation* $323^x + n^y = z^2$.

Proof. Note that z is an even integer. Then $z^2 \equiv 0 \pmod 4$.
Case 1. $y = 0$. By Lemma 2.3, we have $(x, y, z) = (1, 0, 18)$ is a unique non-negative integer solution of the Diophantine equation $323^x + n^y = z^2$.
Case 2. $y \geq 1$. Suppose that there exists a non-negative integer solution (x, y, z) for the Diophantine equation $323^x + n^y = z^2$. Since $n \equiv 5 \pmod{20}$, $n \equiv 1 \pmod 4$ and $n \equiv 0 \pmod 5$. Then we have $0 \equiv z^2 = 323^x + n^y \equiv 3^x + 1 \pmod 4$. Thus $3^x \equiv -1 \pmod 4$. This implies that x is odd. Using x is odd, we have $3^x \equiv 2 \pmod 5$ or $3^x \equiv 3 \pmod 5$. Since $n^y \equiv 0 \pmod 5$, we obtain $z^2 = 323^x + n^y = 3^x + 0 \equiv 2$ or $3 \pmod 5$. That is $\left(\frac{2}{5}\right) = 1$ and $\left(\frac{3}{5}\right) = 1$. This is a contradiction to Theorem 2.4 and Theorem 2.5, respectively. In this case, there is no non-negative integer solution.

It can be seen that $(1, 0, 18)$ is a unique non-negative integer solution (x, y, z) for the Diophantine equation $323^x + n^y = z^2$. This completes the proof. □

We know that $n = 325 \equiv 5 \pmod{20}$. The following example is the main theorem of Sroysang [6] which is a special case of Theorem 3.1.

Example 3.2. $(1, 0, 18)$ is a unique non-negative integer solution (x, y, z) for the Diophantine equation $323^x + 325^y = z^2$.

Lemma 3.3. *[2] Let m be a positive integer with $m \equiv 1 \pmod 4$. The Diophantine equation $1 + mn^y = z^2$ has no non-negative integer solution where y, z are non-negative integers.*

Lemma 3.4. *Let m be an integer greater than 1. The Diophantine equation $323 + 323^m n^y = z^2$ has no non-negative integer solution (y, z).*

Proof. Suppose that there exist non-negative integers y, z such that $323 + 323^m n^y = z^2$. This implies $323|z^2$. Since $17|z^2$, $19|z^2$ and $\gcd(17, 19) = 1$, we conclude $323|z$ by Corollary 2.1 and Theorem 2.2. We write $z = 323r$ for some positive integers r. Substituting $z = 323r$ in the Diophantine equation $323+323^m n^y = z^2$, then $323+323^m n^y = 323^2 r^2$ and thus $1+323^{m-1}n^y = 323r^2$. We get $1 = 323(r^2 - 323^{m-2}n^y)$. So that $323|1$. This is a contradiction. Hence $323 + 323^m n^y = z^2$ has no non-negative integer solution. □

The next theorem is the main result of this paper.

Theorem 3.5. *Let s be a non-negative integer. Then $(1+2s, 0, 18(323)^s)$ is a unique non-negative integer solution (x, y, z) for the Diophantine equation $323^x + 323^{2s} n^y = z^2$.*

Proof. We prove by induction on s.
Let $P(s)$: The Diophantine equation $323^x + 323^{2s} n^y = z^2$ has a unique non-negative integer solution $(x, y, z) = (1 + 2s, 0, 18(323)^s)$.
By Theorem 3.1, $P(0)$ is true.

Suppose that $P(k)$ is true, that is the Diophantine equation $323^x+323^{2k} n^y = z^2$ has a unique non-negative integer solution $(x, y, z) = (1 + 2k, 0, 18(323)^k)$.

Consider the Diophantine equation $323^x + 323^{2(k+1)} n^y = z^2$ into the following cases:

Case $x = 0$. Since $323^{2(k+1)} \equiv 1 \pmod 4$ and by Lemma 3.3, we obtain that $1 + 323^{2(k+1)} n^y = z^2$ has no non-negative integer solution.

Case $x = 1$. By Lemma 3.4, the Diophantine equation $323+323^{2(k+1)} n^y = z^2$ has no non-negative integer solution.

Case $x \geq 2$. Note that the Diophantine equation $323^x + 323^{2(k+1)} n^y = z^2$ can be written as $323^{x-2} + 323^{2k} n^y = \left(\frac{z}{323}\right)^2$ and $x - 2, \frac{z}{323}$ are non-negative integers. Let $u = x - 2$ and $v = \frac{z}{323}$. By assumption $P(k)$ is true, we obtain that $323^u + 323^{2k} n^y = v^2$ has a unique non-negative integer solution $(u, y, v) = (1 + 2k, 0, 18(323)^k)$. That is $u = 1 + 2k$ and $v = 18(323)^k$. Thus

$(x, y, z) = (1+2(k+1), 0, 18(323)^{k+1})$ is a unique non-negative integer solution of the Diophantine equation $323^x + 323^{2(k+1)}n^y = z^2$. Therefore $P(k+1)$ is true.

By mathematical induction, the Diophantine equation $323^x + 323^{2s}n^y = z^2$ has a unique non-negative integer solution $(x, y, z) = (1 + 2s, 0, 18(323)^s)$. □

As a consequence of Theorem 3.5, we obtain:

Corollary 3.6. *Let s, t be non-negative integers such that $t \geq 2$. The Diophantine equation $323^x + 323^{2s}n^y = z^{2t}$ has no non-negative integer solution (x, y, z).*

Proof. Suppose that (x, y, z) is a non-negative integer solution of the Diophantine equation $323^x + 323^{2s}n^y = z^{2t}$. Thus (x, y, z^t) is a non-negative integer solution of the Diophantine equation $323^x + 323^{2s}n^y = z^2$. By Theorem 3.5, we have $(x, y, z^t) = (1 + 2s, 0, 18(323)^s)$. Thus $z^t = 18(323)^s$ which is a contradiction. Therefore the Diophantine equation $323^x + 323^{2s}n^y = z^{2t}$ has no non-negative integer solution (x, y, z). □

The Diophantine equation $323^x + 323^{2s}n^y = z^{2t}$, when s, t are non-negative integers, is a generalization of the Diophantine equation $323^x + 325^y = z^2$. It easy to verify that the Diophantine equation $323^x + 323^{2s}n^y = z^{2t}$ has no non-negative integer solution when $t = 0$. Theorem 3.5 and Corollary 3.6 give the following result.

Theorem 3.7. *Let n, s, t be any non-negative integers where $n \equiv 5 \pmod{20}$. All non-negative integer solutions (x, y, z) of the Diophantine equation $323^x + 323^{2s}n^y = z^{2t}$ are the following:*

$$(x, y, z) = \begin{cases} (1 + 2s, 0, 18(323)^s) & ; \quad t = 1, \\ \text{no solution} & ; \quad \text{otherwise.} \end{cases}$$

We can apply Theorem 3.7 to solve non-negative solutions of many Diophantine equations, and we show as the following examples:

Example 3.8. $(x, y, z) = (5, 0, 208658)$ is a unique non-negative integer solution (x, y, z) for the Diophantine equation $323^x + 10884540241(145)^y = 81z^2$.

Proof. Note that the Diophantine equation $323^x + 10884540241(145)^y = 81z^2$ can be written as $323^x + (323)^4(145)^y = (9z)^2$. By Theorem 3.7, we have $(x, y, 9z) = (5, 0, 18(323)^2)$. Thus $9z = 18(323)^2$ or $z = 208658$. Therefore, $(x, y, z) = (5, 0, 208658)$ is a unique non-negative integer solution. □

Example 3.9. $(x, y, z) = (4, 0, 323)$ is a unique non-negative integer solution (x, y, z) for the Diophantine equation $323^x + 33698267(65)^y = 104652z^2$.

Proof. Note that the Diophantine equation $323^x+33698267(65)^y = 104652z^2$ can be written as $323^x + (323)^3(65)^y = 323(18z)^2$. If $x = 0$, then $323|1$. This is a contradiction. Suppose $x \geq 1$. So we have $323^{x-1} + 323^2(65)^y = (18z)^2$. By Theorem 3.7, we have $(x - 1, y, 18z) = (3, 0, 18(323))$. Thus $x - 1 = 3$ and $18z = 18(323)$ or $x = 4$ and $z = 323$. Therefore, $(x, y, z) = (4, 0, 323)$ is a unique non-negative integer solution. □

Example 3.10. The Diophantine equation $323^x+33698267(45)^y = 5168z^4$ has no non-negative integer solution (x, y, z).

Proof. Note that the Diophantine equation $323^x + 33698267(45)^y = 5168z^4$ can be written as $323^x + (323)^3(45)^y = 323(2z)^4$. If $x = 0$, then $323|1$. This is a contradiction. Suppose $x \geq 1$. So we have $323^{x-1} + 323^2(45)^y = (2z)^4$. By Theorem 3.7, this equation has no non-negative integer solution (x, y, z). □

Example 3.11. The Diophantine equation $323^x + 323(25)^y = 8075z^2$ has no non-negative integer solution (x, y, z).

Proof. Note that the Diophantine equation $323^x + 323(25)^y = 8075z^2$ can be written as $323^x + 323(25)^y = 323(5z)^2$. If $x = 0$, then $323|1$. This is a contradiction. Suppose $x \geq 1$. So we have $323^{x-1} + 25^y = (5z)^2$. By Theorem 3.7, we have $(x - 1, y, 5z) = (1, 0, 18)$. This implies that $5z = 18$, i.e., $z = \frac{18}{5}$. This is a contradiction since z is integer. Therefore, this equation has no non-negative integer solution (x, y, z). □

Using Theorem 3.7, it is easy to verify the following examples.

Example 3.12. For $n = 5, s = 0, 1, 2, 3, 4, ...$

s	$323^x + 323^{2s}5^y = z^2$	solution (x, y, z)
$s = 0$	$323^x + 5^y = z^2$	$(1, 0, 18)$
$s = 1$	$323^x + (104329)5^y = z^2$	$(3, 0, 5814)$
$s = 2$	$323^x + (10884540241)5^y = z^2$	$(5, 0, 1877922)$
$s = 3$	$323^x + (1135573198803289)5^y = z^2$	$(7, 0, 606568806)$
$s = 4$	$323^x + (118473216257948338081)5^y = z^2$	$(9, 0, 195921724338)$
$\vdots$	$\vdots$	$\vdots$

For $n = 25, s = 0, 1, 2, 3, 4, ...$

s	$323^x + 323^{2s}25^y = z^2$	solution (x, y, z)
$s = 0$	$323^x + 25^y = z^2$	$(1, 0, 18)$
$s = 1$	$323^x + (104329)25^y = z^2$	$(3, 0, 5814)$
$s = 2$	$323^x + (10884540241)25^y = z^2$	$(5, 0, 1877922)$
$s = 3$	$323^x + (1135573198803289)25^y = z^2$	$(7, 0, 606568806)$
$s = 4$	$323^x + (118473216257948338081)25^y = z^2$	$(9, 0, 195921724338)$
$\vdots$	$\vdots$	$\vdots$

For $n = 45, s = 0, 1, 2, 3, 4, ...$

s	$323^x + 323^{2s}45^y = z^2$	solution (x, y, z)
$s = 0$	$323^x + 45^y = z^2$	$(1, 0, 18)$
$s = 1$	$323^x + 104329(45)^y = z^2$	$(3, 0, 5814)$
$s = 2$	$323^x + (10884540241)45^y = z^2$	$(5, 0, 1877922)$
$s = 3$	$323^x + (1135573198803289)45^y = z^2$	$(7, 0, 606568806)$
$s = 4$	$323^x + (118473216257948338081)45^y = z^2$	$(9, 0, 195921724338)$
$\vdots$	$\vdots$	$\vdots$

Moreover, for any $n \equiv 5 \pmod{20}$, we obtain the non-negative integer solutions (x, y, z) of the Diophantine equation of the form $323^x + 323^{2s}n^y = z^{2t}$ in a similar way.

4. Open Problem

It is important to note that all solutions (x, y, z) of the Diophantine equation $323^x + n^y = z^2$, where n is any positive integer and $n \not\equiv 5 \pmod{20}$, are still an open problem. For example, it is not known how to find all non-negative integer solutions (x, y, z) of the Diophantine equation $323^x + n^y = z^2$ where $n = 2, 3, 6, 7, ...$ etc.

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