ON THE DIOPHANTINE EQUATION $323^x + 323^{2s}n^y = z^{2t}$
WHERE $s, t, n$ ARE NON-NEGATIVE INTEGERS
AND $n \equiv 5 \pmod{20}$

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Abstract: Let $s, t, n$ be non-negative integers such that $n \equiv 5 \pmod{20}$. In this paper, we found that all non-negative integer solutions $(x, y, z)$ of the Diophantine equation $323^x + 323^{2s}n^y = z^{2t}$ are in the following form:

$$(x, y, z) = \begin{cases} (1 + 2s, 0, 18(323)^s) & : t = 1, \\ \text{no solution} & : \text{otherwise}. \end{cases}$$

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1. Introduction

There are many mathematicians solved all non-negative integer solutions $(x, y, z)$ of the Diophantine equation $a^x + b^y = z^2$ where $a, b$ are positive integers and $x, y, z$ are non-negative integers. For example, in 2013, Chotchaisthit [1] showed that $(p, x, y, z) = (7, 0, 1, 3)$ and $(p, x, y, z) = (3, 2, 2, 5)$ are the only solutions of the Diophantine equation $p^x + (p + 1)^y = z^2$ where $x, y, z$ are non-negative integers and $p$ is the Mersenne prime.
In 2013-2014, there are numbers of paper of Sroysang, for examples, in [4, 5] solved all non-negative integer solutions \((x, y, z)\) of the Diophantine equations
\[
3^x + 5^y = z^2, \quad 3^x + 17^y = z^2
\]
and he found that both Diophantine equations have exactly one non-negative integer solution, namely, \((x, y, z) = (1, 0, 2)\).

In recently 2014, Sroysang [6] showed that there is a unique non-negative integer solution \((x, y, z) = (1, 0, 18)\) for the Diophantine equation
\[
323^x + 325^y = z^2
\]
Now we find all possible non-negative integer solutions \((x, y, z)\) of the Diophantine equation
\[
323^x + 323^2sn^y = z^2
\]
where \(n \equiv 5 \pmod{20}\), which is a generalization of the Diophantine equation \(323^x + 325^y = z^2\) where \(s = 0\) and \(n = 325\).

2. Preliminaries

Throughout this paper, we assume that \(n\) is a non-negative integer such that \(n \equiv 5 \pmod{20}\). It is clear that \(n \equiv 1 \pmod{4}\) and \(n \equiv 0 \pmod{5}\).

**Corollary 2.1.** [3] Let \(a\) and \(b\) be nonzero integers and \(c\) be an integer. If \(a|c\) and \(b|c\), with \(\gcd(a, b) = 1\), then \(ab|c\).

**Theorem 2.2.** [3] Let \(a, b, p\) be integers. If \(p\) is a prime and \(p|ab\), then \(p|a\) or \(p|b\).

**Lemma 2.3.** [6] \((1, 18)\) is a unique non-negative integer solution \((x, z)\) for the Diophantine equation \(323^x + 1 = z^2\).

Let \(p\) be an odd prime and \(a\) be a positive integer where \(\gcd(a, p) = 1\). If the quadratic congruence \(x^2 \equiv a \pmod{p}\) has a solution, then \(a\) is said to be a **quadratic residue of** \(p\). Otherwise, \(a\) is called a **quadratic non-residue of** \(p\). In 1798 Adrien-Marie Legendre introduced the **Legendre symbol** \((\frac{a}{p})\) which is defined by
\[
(\frac{a}{p}) = \begin{cases} 
1 & \text{if } a \text{ is a quadratic residue of } p, \\
-1 & \text{if } a \text{ is a quadratic non-residue of } p.
\end{cases}
\]

In the present paper, we need the following well-known facts about the Legendre symbols.
\textbf{Theorem 2.4.} [3] If $p$ is an odd prime, then
\[
\left(\frac{2}{p}\right) = \begin{cases} 
1 & \text{if } p \equiv 1 \pmod{8} \text{ or } p \equiv 7 \pmod{8}, \\
-1 & \text{if } p \equiv 3 \pmod{8} \text{ or } p \equiv 5 \pmod{8}.
\end{cases}
\]

\textbf{Theorem 2.5.} [3] If $p \neq 3$ is an odd prime, then
\[
\left(\frac{3}{p}\right) = \begin{cases} 
1 & \text{if } p \equiv \pm 1 \pmod{12}, \\
-1 & \text{if } p \equiv \pm 5 \pmod{12}.
\end{cases}
\]

\section{3. Main Results}

First, we find all non-negative integer solutions $(x, y, z)$ of the Diophantine equation $323^x + 325^y = z^2$ where $x, y, z$ are non-negative integers.

\textbf{Theorem 3.1.} $(1, 0, 18)$ is a unique non-negative integer solution $(x, y, z)$ for the Diophantine equation $323^x + n^y = z^2$.

\text{Proof.} Note that $z$ is an even integer. Then $z^2 \equiv 0 \pmod{4}$.

\textbf{Case 1.} $y = 0$. By Lemma 2.3, we have $(x, y, z) = (1, 0, 18)$ is a unique non-negative integer solution of the Diophantine equation $323^x + n^y = z^2$.

\textbf{Case 2.} $y \geq 1$. Suppose that there exists a non-negative integer solution $(x, y, z)$ for the Diophantine equation $323^x + n^y = z^2$. Since $n \equiv 5 \pmod{20}$, $n \equiv 1 \pmod{4}$ and $n \equiv 0 \pmod{5}$. Then we have $0 \equiv z^2 = 323^x + n^y \equiv 3^x + 1 \pmod{4}$. Thus $3^x \equiv -1 \pmod{4}$. This implies that $x$ is odd. Using $x$ is odd, we have $3^x \equiv 2 \pmod{5}$ or $3^x \equiv 3 \pmod{5}$. Since $n^y \equiv 0 \pmod{5}$, we obtain $z^2 = 323^x + n^y = 3^x + 0 \equiv 2$ or $3 \pmod{5}$. That is $\left(\frac{2}{5}\right) = 1$ and $\left(\frac{3}{5}\right) = 1$. This is a contradiction to Theorem 2.4 and Theorem 2.5, respectively. In this case, there is no non-negative integer solution.

It can be seen that $(1, 0, 18)$ is a unique non-negative integer solution $(x, y, z)$ for the Diophantine equation $323^x + 325^y = z^2$. This completes the proof. \hfill $\square$

We know that $n = 325 \equiv 5 \pmod{20}$. The following example is the main theorem of Sroysang [6] which is a special case of Theorem 3.1.

\textbf{Example 3.2.} $(1, 0, 18)$ is a unique non-negative integer solution $(x, y, z)$ for the Diophantine equation $323^x + 325^y = z^2$. 

Lemma 3.3. [2] Let \( m \) be a positive integer with \( m \equiv 1 \pmod{4} \). The Diophantine equation \( 1 + mn^y = z^2 \) has no non-negative integer solution where \( y, z \) are non-negative integers.

Lemma 3.4. Let \( m \) be an integer greater than 1. The Diophantine equation \( 323 + 323^m n^y = z^2 \) has no non-negative integer solution \( (y, z) \).

Proof. Suppose that there exist non-negative integers \( y, z \) such that \( 323 + 323^m n^y = z^2 \). This implies \( 323 | z^2 \). Since \( 17 | z^2, 19 | z^2 \) and \( \gcd(17, 19) = 1 \), we conclude \( 323 | z \) by Corollary 2.1 and Theorem 2.2. We write \( z = 323r \) for some positive integer \( r \). Substituting \( z = 323r \) in the Diophantine equation \( 323 + 323^m n^y = z^2 \), then \( 323 + 323^m n^y = 323^2 r^2 \) and thus \( 1 + 323^{m-1} n^y = 323r^2 \). We get \( 1 = 323(r^2 - 323^{m-2} n^y) \). So that \( 323 | 1 \). This is a contradiction. Hence \( 323 + 323^m n^y = z^2 \) has no non-negative integer solution.

The next theorem is the main result of this paper.

Theorem 3.5. Let \( s \) be a non-negative integer. Then \((1 + 2s, 0, 18(323)^s)\) is a unique non-negative integer solution \((x, y, z)\) for the Diophantine equation \( 323^x + 323^{2s} n^y = z^2 \).

Proof. We prove by induction on \( s \).

Let \( P(s) : \) The Diophantine equation \( 323^x + 323^{2s} n^y = z^2 \) has a unique non-negative integer solution \((x, y, z) = (1 + 2s, 0, 18(323)^s)\).

By Theorem 3.1, \( P(0) \) is true.

Suppose that \( P(k) \) is true, that is the Diophantine equation \( 323^x + 323^{2k} n^y = z^2 \) has a unique non-negative integer solution \((x, y, z) = (1 + 2k, 0, 18(323)^k)\).

Consider the Diophantine equation \( 323^x + 323^{2(k+1)} n^y = z^2 \) into the following cases:

Case \( x = 0 \). Since \( 323^{2(k+1)} \equiv 1 \pmod{4} \) and by Lemma 3.3, we obtain that \( 1 + 323^{2(k+1)} n^y = z^2 \) has no non-negative integer solution.

Case \( x = 1 \). By Lemma 3.4, the Diophantine equation \( 323 + 323^{2(k+1)} n^y = z^2 \) has no non-negative integer solution.

Case \( x \geq 2 \). Note that the Diophantine equation \( 323^x + 323^{2(k+1)} n^y = z^2 \) can be written as \( 323^{x-2} + 323^{2k} n^y = \left(\frac{z}{323}\right)^2 \) and \( x - 2, \frac{z}{323} \) are non-negative integers. Let \( u = x - 2 \) and \( v = \frac{z}{323} \). By assumption \( P(k) \) is true, we obtain that \( 323^u + 323^{2k} n^y = v^2 \) has a unique non-negative integer solution \((u, y, v) = (1 + 2k, 0, 18(323)^k)\). That is \( u = 1 + 2k \) and \( v = 18(323)^k \). Thus
ON THE DIOPHANTINE EQUATION $323^x + 323^{2s}ny = z^{2t}$...

$(x, y, z) = (1 + 2(k + 1), 0, 18(323)^{k+1})$ is a unique non-negative integer solution of the Diophantine equation $323^x + 323^{2(k+1)}ny = z^2$. Therefore $P(k+1)$ is true.

By mathematical induction, the Diophantine equation $323^x + 323^{2s}ny = z^2$ has a unique non-negative integer solution $(x, y, z) = (1 + 2s, 0, 18(323)^s)$. □

As a consequence of Theorem 3.5, we obtain:

**Corollary 3.6.** Let $s, t$ be non-negative integers such that $t \geq 2$. The Diophantine equation $323^x + 323^{2s}ny = z^{2t}$ has no non-negative integer solution $(x, y, z)$.

**Proof.** Suppose that $(x, y, z)$ is a non-negative integer solution of the Diophantine equation $323^x + 323^{2s}ny = z^{2t}$. Thus $(x, y, z^t)$ is a non-negative integer solution of the Diophantine equation $323^x + 323^{2s}ny = z^2$. By Theorem 3.5, we have $(x, y, z^t) = (1 + 2s, 0, 18(323)^s)$. Thus $z^t = 18(323)^s$ which is a contradiction. Therefore the Diophantine equation $323^x + 323^{2s}ny = z^{2t}$ has no non-negative integer solution $(x, y, z)$. □

The Diophantine equation $323^x + 323^{2s}ny = z^{2t}$, when $s, t$ are non-negative integers, is a generalization of the Diophantine equation $323^x + 325^y = z^2$. It easy to verify that the Diophantine equation $323^x + 323^{2s}ny = z^{2t}$ has no non-negative integer solution when $t = 0$. Theorem 3.5 and Corollary 3.6 give the following result.

**Theorem 3.7.** Let $n, s, t$ be any non-negative integers where $n \equiv 5 \pmod{20}$. All non-negative integer solutions $(x, y, z)$ of the Diophantine equation $323^x + 323^{2s}ny = z^{2t}$ are the following:

$$(x, y, z) = \begin{cases} (1 + 2s, 0, 18(323)^s) ; & t = 1, \\ \text{no solution} ; & \text{otherwise.} \end{cases}$$

We can apply Theorem 3.7 to solve non-negative solutions of many Diophantine equations, and we show as the following examples:

**Example 3.8.** $(x, y, z) = (5, 0, 208658)$ is a unique non-negative integer solution $(x, y, z)$ for the Diophantine equation $323^x + 10884540241(145)^y = 81z^2$. 
Proof. Note that the Diophantine equation $323^x + 10884540241(145)^y = 81z^2$ can be written as $323^x + (323)^4(145)^y = (9z)^2$. By Theorem 3.7, we have $(x, y, 9z) = (5, 0, 18(323)^2)$. Thus $9z = 18(323)^2$ or $z = 208658$. Therefore, $(x, y, z) = (5, 0, 208658)$ is a unique non-negative integer solution.

Example 3.9. $(x, y, z) = (4, 0, 323)$ is a unique non-negative integer solution $(x, y, z)$ for the Diophantine equation $323^x + 33698267(65)^y = 104652z^2$.

Proof. Note that the Diophantine equation $323^x + 33698267(65)^y = 104652z^2$ can be written as $323^x + (323)^3(65)^y = 323(18z)^2$. If $x = 0$, then $323 | 1$. This is a contradiction. Suppose $x \geq 1$. So we have $323^{x-1} + 323^2(65)^y = (18z)^2$. By Theorem 3.7, we have $(x-1, y, 18z) = (3, 0, 18(323))$. Thus $x-1 = 3$ and $18z = 18(323)$ or $x = 4$ and $z = 323$. Therefore, $(x, y, z) = (4, 0, 323)$ is a unique non-negative integer solution.

Example 3.10. The Diophantine equation $323^x + 33698267(45)^y = 5168z^4$ has no non-negative integer solution $(x, y, z)$.

Proof. Note that the Diophantine equation $323^x + 33698267(45)^y = 5168z^4$ can be written as $323^x + (323)^3(45)^y = 323(2z)^4$. If $x = 0$, then $323 | 1$. This is a contradiction. Suppose $x \geq 1$. So we have $323^{x-1} + 323^2(45)^y = (2z)^4$. By Theorem 3.7, this equation has no non-negative integer solution $(x, y, z)$.

Example 3.11. The Diophantine equation $323^x + 323(25)^y = 8075z^2$ has no non-negative integer solution $(x, y, z)$.

Proof. Note that the Diophantine equation $323^x + 323(25)^y = 8075z^2$ can be written as $323^x + 323(25)^y = 323(5z)^2$. If $x = 0$, then $323 | 1$. This is a contradiction. Suppose $x \geq 1$. So we have $323^{x-1} + 25^y = (5z)^2$. By Theorem 3.7, we have $(x-1, y, 5z) = (1, 0, 18)$. This implies that $5z = 18$, i.e., $z = \frac{18}{5}$. This is a contradiction since $z$ is integer. Therefore, this equation has no non-negative integer solution $(x, y, z)$.

Using Theorem 3.7, it is easy to verify the following examples.

Example 3.12. For $n = 5, s = 0, 1, 2, 3, 4, ...$
ON THE DIOPHANTINE EQUATION $323^x + 323^{2s}n^y = z^{2t}$

<table>
<thead>
<tr>
<th>s</th>
<th>$323^x + 323^{2s}n^y = z^{2t}$</th>
<th>solution $(x, y, z)$</th>
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<tbody>
<tr>
<td>$s = 0$</td>
<td>$323^x + 5^y = z^{2t}$</td>
<td>$(1, 0, 18)$</td>
</tr>
<tr>
<td>$s = 1$</td>
<td>$323^x + (104329)5^y = z^{2t}$</td>
<td>$(3, 0, 5814)$</td>
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<tr>
<td>$s = 2$</td>
<td>$323^x + (10884540241)5^y = z^{2t}$</td>
<td>$(5, 0, 1877922)$</td>
</tr>
<tr>
<td>$s = 3$</td>
<td>$323^x + (1135573198803289)5^y = z^{2t}$</td>
<td>$(7, 0, 606568806)$</td>
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<tr>
<td>$s = 4$</td>
<td>$323^x + (118473216257948338081)5^y = z^{2t}$</td>
<td>$(9, 0, 195921724338)$</td>
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<tr>
<td>For $n = 25$, $s = 0, 1, 2, 3, 4, ...$</td>
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</tbody>
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<tr>
<th>s</th>
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<tr>
<td>For $n = 45$, $s = 0, 1, 2, 3, 4, ...$</td>
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<tr>
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<th>solution $(x, y, z)$</th>
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Moreover, for any $n \equiv 5 \pmod{20}$, we obtain the non-negative integer solutions $(x, y, z)$ of the Diophantine equation of the form $323^x + 323^{2s}n^y = z^{2t}$ in a similar way.

### 4. Open Problem

It is important to note that all solutions $(x, y, z)$ of the Diophantine equation $323^x + n^y = z^2$, where $n$ is any positive integer and $n \not\equiv 5 \pmod{20}$, are still an open problem. For example, it is not known how to find all non-negative integer solutions $(x, y, z)$ of the Diophantine equation $323^x + n^y = z^2$ where $n = 2, 3, 6, 7, ...$ etc.
Acknowledgments

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References


[2] N. Sarasit, S. Chotchaisthit, On the Diophantine equation $3^x + 3^{2s} n^y = z^{2t}$ where $n \equiv 0 \pmod{5}$, *Int. J. Pure Appl. Math.*, 97, No. 2 (2014), 211-218, doi: http://dx.doi.org/10.12732/ijpam.v97i2.9


