

**NEW MIXED FINITE VOLUME SPACES
FOR ELLIPTIC PROBLEMS**

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Abstract: We introduce a new family of mixed finite volume spaces of higher order for second order elliptic problems on rectangular grids. We also show that these volume methods are equivalent to the nonconforming finite element methods used to define them.

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1. Introduction

Mixed finite element methods for elliptic problems have been the subject of extensive research(see [1], [2], [3], [4], [5], [6]). The main idea of mixed finite element methods is to introduce the Darcy velocity as a new variable and write the equation into a system of differential equations. By discretizing this system, we can compute velocity and pressure together. However, the resulting mixed system has many more variables and gives rise to a saddle point problem. Thus there were some restrictions to use in industry. For the lowest order mixed methods, it is well known that there exist equivalent nonconforming finite element methods[7], [8]. For higher order cases, there exist some equivalent nonconforming finite element spaces. But this methods involve projections into

the vector part of the mixed finite element spaces, which are not easy to handle. To overcome this difficulty, a finite volume type of mixed method and its variants were introduced (see [9], [10], [11], [12]). The idea is to integrate the system over each element with weight functions. Assuming the velocity variable in the Raviart-Thomas space, and the scalar variables lies in Q_1 nonconforming finite element space, they show the equivalence between the mixed finite element method and the nonconforming finite element method. Thus the solution of the mixed finite volume methods can be obtained through these nonconforming finite element method.

In this paper, we define new finite volume type of mixed methods with judiciously chosen test spaces and show that they are equivalent to some nonconforming finite element methods used to define them.

The organization of this paper is as follows: In the next section, we start with a problem description and presentation of the basic materials necessary to study mixed finite volume methods. In section 3, we define our new family of finite volume mixed method. Under a series of hypotheses, we show these methods are equivalent to some nonconforming finite element methods.

2. Mixed Finite Volume Methods

In this section, we introduce mixed finite volume methods and related nonconforming finite element methods. Let Ω be a boundary polygonal domain in \mathbb{R}^2 with the boundary $\partial\Omega$. We consider the following second order elliptic boundary value problem:

$$\begin{cases} -\operatorname{div}(\kappa\nabla p) = f, & \text{in } \Omega, \\ p = 0, & \text{on } \partial\Omega, \end{cases} \quad (1)$$

where $\kappa = \kappa(\mathbf{x})$ is a symmetric and uniformly positive definite matrix, i.e., there exist two positive constants c_1 and c_2 such that

$$c_1\xi^T\xi \leq \xi^T\kappa(\mathbf{x})\xi \leq c_2\xi^T\xi, \quad \forall\xi \in \mathbb{R}^2, \mathbf{x} \in \bar{\Omega}.$$

For the discussion of higher order methods, we shall require $f \in H^k(\Omega)$ for some integer $k \geq 0$. Let us introduce a vector variable $\mathbf{u} = -\kappa\nabla p$ and reformulate the problem (1) in the following mixed form:

$$\begin{cases} \mathbf{u} + \kappa\nabla p = 0, & \text{in } \Omega, \\ \operatorname{div} \mathbf{u} = f, & \text{in } \Omega, \\ p = 0, & \text{on } \partial\Omega. \end{cases} \quad (2)$$

First, we introduce some function spaces. For any domain Ω , we let $L^2(\Omega)$ be the set of all square integrable functions on Ω equipped with the usual inner product $(\cdot, \cdot)_\Omega$. Let $H^k(\Omega) = W^{k,2}(\Omega)$ be the Sobolev spaces of order $k \geq 0$ with obvious norms and $\mathbf{H}^k(\Omega)$ be the space of vectors each of whose component lies in $H^k(\Omega)$. For both of the spaces $H^k(\Omega)$ and $\mathbf{H}^k(\Omega)$, we denote the norms(semi-norms) by $\|\cdot\|_{k,\Omega}$ ($|\cdot|_{k,\Omega}$) or $\|\cdot\|_k$ ($|\cdot|_k$) when no confusion arises. Throughout this paper, we assume that the solution (\mathbf{u}, p) of (2) satisfies $\mathbf{u} \in \mathbf{H}^{k+1}(\Omega)$, $p \in H^{k+2}(\Omega)$, and there exists some constant $C > 0$ such that

$$\|\mathbf{u}\|_{k+1} + \|p\|_{k+2} \leq C \|f\|_k .$$

Also, let $\mathbf{V} = H(\text{div}, \Omega) = \{\mathbf{v} \in (L^2(\Omega))^2 \mid \text{div } \mathbf{v} \in L^2(\Omega)\}$ and $W = L^2(\Omega)$. Then we have the following variational form for (2):

$$\begin{cases} (\kappa^{-1}\mathbf{u}, \mathbf{v}) - (p, \text{div } \mathbf{v}) = 0, & \forall \mathbf{v} \in \mathbf{V}, \\ (\text{div } \mathbf{u}, q) = (f, q), & \forall q \in W. \end{cases} \tag{3}$$

This problem has a unique solution pair $(\mathbf{u}, p) \in \mathbf{V} \times W$. Now we consider the finite element method. From now on, we assume the domain can be partitioned into rectangles. Assume that we have some finite element subspaces $\mathbf{V}_h \subset \mathbf{V}$ and $W_h \subset W$ based on these grids. Then the corresponding finite dimensional approximation $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times W_h$ is defined as the solution of the following equations:

$$\begin{cases} (\kappa^{-1}\mathbf{u}_h, \mathbf{v}_h) - (p_h, \text{div } \mathbf{v}_h) = 0, & \forall \mathbf{v}_h \in \mathbf{V}_h, \\ (\text{div } \mathbf{u}_h, q_h) = (f, q_h), & \forall q_h \in W_h. \end{cases} \tag{4}$$

To explain a higher order mixed finite volume method, we need some hypotheses[12]. Let \mathbf{V}_h be the vector part of any mixed finite element space and let Q be a rectangular element.

(H1) There is a nonconforming finite element space N_h of order k associated with the same triangulation such that for $\chi \in N_h$ the moments up to k

$$\int_e \chi q ds, \quad q \in P_k(e), \quad \text{for any edge } e \text{ of } Q, \tag{5}$$

are continuous across interior edges,

$$\int_e \chi q ds, \quad q \in P_k(e), \quad \text{if edge } e \text{ is part of } \partial\Omega, \tag{6}$$

and the interior degrees of freedom are given by

$$\int_Q \chi \phi dx, \quad \phi \in N_h^i(Q), \tag{7}$$

for some space $N_h^i(Q)$ of polynomials.

(H2) The space $N_h(Q)$ has the approximation property

$$\inf_{v_h \in N_h(Q)} (\|v - v_h\|_0 + h |v - v_h|_1) \leq Ch^{k+2} \|v\|_{k+2}, \tag{8}$$

whenever $v \in H^{k+2}(Q)$.

(H3) Let $U_h(Q) = \text{div } \mathbf{V}_h(Q)$ and $\Psi_h(Q)$ be the interior degrees of freedom of $\mathbf{V}_h(Q)$. Then the space $U_h(Q)$ satisfies

$$\nabla U_h(Q) \subset \Psi_h(Q),$$

and

$$U_h(Q) \subset N_h(Q).$$

(H4) There is a space of polynomial $\mathbf{V}_h^s(Q)$ such that

$$\mathbf{V}_h^s(Q) \oplus \nabla U_h(Q) \supset \Psi_h(Q), \tag{9}$$

and

$$\dim \mathbf{V}_h^s(Q) = \dim \Psi_h(Q) - \dim U_h(Q) + 1. \tag{10}$$

Now, we present a higher order mixed finite volume method. Find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h(Q) \times N_h(Q)$ which satisfies on every element $Q \in \mathcal{T}_h$

$$\int_Q (\mathbf{u}_h + \kappa \nabla p_h) \cdot \nabla \chi dx = 0, \quad \forall \chi \in N_h(Q), \tag{11}$$

$$\int_Q (\mathbf{u}_h + \kappa \nabla p_h) \cdot \mathbf{v} dx = 0, \quad \forall \mathbf{v} \in \mathbf{V}_h^s(Q), \tag{12}$$

$$\int_Q \text{div } \mathbf{u}_h \phi dx = \int_Q f \phi dx, \quad \forall \phi \in U_h(Q). \tag{13}$$

From (13), we know that

$$\text{div } \mathbf{u}_h = P_{U_h} f, \tag{14}$$

where P_{U_h} is an L^2 -projection onto U_h . If $\chi \in N_h(Q)$, we have

$$\begin{aligned} \sum_Q \int_Q \kappa \nabla p_h \cdot \nabla \chi dx &= - \sum_Q \int_Q \mathbf{u}_h \nabla \chi dx \\ &= - \sum_Q \int_{\partial Q} \mathbf{u}_h \cdot \mathbf{n} ds + \sum_Q \int_Q \text{div } \mathbf{u}_h \chi dx. \end{aligned}$$

Since χ has continuous moments up to degree k across internal edges and has vanishing moments on $\partial\Omega$,

$$\sum_Q \int_Q \kappa \nabla p_h \cdot \nabla \chi dx = \int_\Omega (P_{U_h} f) \chi dx.$$

Since

$$\int_\Omega (P_{U_h} f) \chi dx = \int_\Omega f(P_{U_h} \chi) dx,$$

we can obtain the following equation:

$$\sum_Q \int_Q \kappa \nabla p_h \cdot \nabla \chi dx = \int_\Omega f(P_{U_h} \chi) dx, \text{ for } \chi \in N_h(Q). \tag{15}$$

From (15), we know that the mixed finite volume method is equivalent to a standard nonconforming finite element method, except that L^2 -projections are used on the lower order terms. Thus to implement (11)-(13), it suffices to solve the equivalent system (15).

3. New mixed finite volume spaces for rectangular element

For simplicity, we let $Q = [-1, 1] \times [-1, 1]$. Let $\mathbf{V}_h(Q)$ be the following element of order k :

$$\mathbf{V}_h(Q) = P_{k+1, k} \times P_{k, k+1},$$

where the two elements $(x^{k+1}y^k, 0)^T$ and $(0, x^k y^{k+1})^T$ are replaced by the single element $(x^{k+1}y^k, -x^k y^{k+1})^T$. To define the degrees of freedom, we first let

$$\mathbf{\Psi}_h(Q) = P_{k-1, k} \times P_{k, k-1},$$

where $(x^{k-1}y^k, 0)^T$ and $(0, x^k y^{k-1})^T$ are replaced by the element $(x^{k-1}y^k, -x^k y^{k-1})^T$. Then, for any $\mathbf{u} \in \mathbf{V}_h(Q)$, we have the following degrees of freedom:

$$\int_e \mathbf{u}_h \times \mathbf{n} \, q \, ds, \quad q \in P_k(e), \quad \text{for each edge } e \text{ of } Q, \tag{16}$$

$$\int_Q \mathbf{u}_h \cdot \phi \, d\mathbf{x}, \quad \phi \in \mathbf{\Psi}_h(Q). \tag{17}$$

Now consider a new nonconforming finite element space:

$$N_h(Q) = R_h(Q) \oplus X^k(Q) \oplus Y^k(Q), \tag{18}$$

where $R_h(Q)$ be the space of all polynomials in each variable up to degree k except a constant multiple of the term $x^k y^k$. And

$$X^k(Q) = \left\{ \sum_{i=0}^k [a_{i,0}l_{k+1}(x) + a_{i,1}l_{k+2}(x)]l_i(y) \right\}, \tag{19}$$

$$Y^k(Q) = \left\{ \sum_{i=0}^k [b_{i,0}l_{k+1}(y) + b_{i,1}l_{k+2}(y)]l_i(x) \right\}, \tag{20}$$

where l_i is the Legendre polynomial on $[-1, 1] \times [-1, 1]$. Then the function in $N_h(Q)$ is completely determined by (5) and (7) with the choice of $N_h^i(Q) = R_h(Q)$. Hence **(H1)** holds. Since $N_h(Q) \supset P_{k+1}(Q)$, **(H2)** holds. Also, **(H3)** is trivial, since $U_h(Q) = R_h(Q)$. We choose

$$\mathbf{V}_h^s(Q) = P_{k-1, k} \times P_{k, k-1} \cap \nabla P_{k,k}^\perp.$$

Since $\nabla P_{k,k} \subset P_{k-1, k} \times P_{k, k-1}$, the dimension of $\mathbf{V}_h^s(Q)$ is

$$\dim \mathbf{V}_h^s(Q) = \dim (P_{k-1, k} \times P_{k, k-1}) - \dim (\nabla P_{k,k}) = 2k(k+1) - (k+1)^2 + 1 = k^2.$$

To check **(H4)**, we first see that

$$\mathbf{V}_h^s(Q) \oplus \nabla U_h(Q) \supset \Psi_h(Q)$$

and

$$\dim \mathbf{V}_h^s(Q) = k^2, \dim \Psi_h = 2k(k+1) - 1, \dim U_h(Q) = (k+1)^2 - 1.$$

So (10) is satisfied. This completes the verification of **(H4)**.

Therefore we can define a new mixed finite volume scheme using the spaces $\mathbf{V}_h(Q)$ and $N_h(Q)$ for the vector part and the corresponding pressure part, respectively. To obtain the solution (\mathbf{u}_h, p_h) of higher order mixed volume methods, it suffices to solve the equivalent system (15) using the nonconforming space $N_h(Q)$.

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