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A CHARACTERIZATION OF PRIMITIVE ALGEBRAS VIA B^*- SEMINORMS

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Abstract: Primitive algebras are characterized in terms of sub-seminorms. We prove a necessary and a sufficient condition for a B^*- seminorm to be minimal. The results of this note are also valid for M^*- seminorms introduced by Bhatt and others in [2].

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1. Introduction

Let A be a complex B^*-algebra. For this algebra A, the norm is given by a B^*- seminorm. A B^*- seminorm on A is a seminorm p with the following conditions: For all $a, b \in A, p(ab) \leq p(a) \cdot p(b)$ and $p(a^*a) = (p(a))^2$. A seminorm p on A is an M^*- seminorm if for all $a, b \in A$, $p(ab) \leq p(a) \cdot p(b)$ and $p(a^*a) = p(a)$. Clearly, every B^*- seminorm on A is an M^*- seminorm on A [2].

Let $N(A)$ be the set of all B^*- seminorms on A. A partial ordering on $N(A)$ is defined as follows. For $p, q \in N(A), p \leq q$ if and only if $p(a) \leq q(a)$ for all $a \in A$. We say that a B^*- seminorm p on A is a subseminorm if $p \in Q$, whenever Q is a subset of $N(A)$ such that $p(a) = \max\{q(a) : q \in Q\}$. The set of all sub-seminorms on A is denoted by $NS(A)$.

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Let A be a complex $Banach^*-$algebras where norm is a B^*- seminorm on A. We call a B^*- seminorm p on A is minimal if for a non-zero p, $p = q$, whenever p is a nonzero B^*- seminorm on A dominated by p. Let $MN(A)$ denote the set of all minimal B^*- seminorms on A. Then $MN(A) \subseteq NS(A)$.

Remark 1. We note that the norm on A coincides with the largest B^*- seminorm on A. To see this, let $g(a) = \sup\{p(a) : p \in N(A), a \in A\}$. Then for all a in A, $\| a \| \leq g(a)$. Also, $g(a) \leq \| a^*a \|^{1/2} = \| a \|$. Hence, $g(a) = \| a \|, \forall a \in A$. This idea is also true in case of M^*- seminorms, see [2].

We denote by $I_d(A)$ the set of all ideals of A. The set of primitive ideals and maximal ideals of A that are written as $PR(A)$ and $M(A)$ respectively.

2. Minimal Norms and Maximal Ideals of A

In the following proposition, we present a characterization of minimal norms on A in terms of maximal ideal space of A.

Proposition 1. *A B^*- seminorm p, is minimal if and only if M_p is a maximal ideal.*

Proof. For each B^*- seminorm p on A, let $M_p = \{a \in A : p(a) = 0\}$. Then the quotient star algebra A/M_p with the norm $\| \cdot \|_p$ is a B^*- algebra, say B_p. Let $a \to \lambda_a$ be the canonical map from A onto A/M_p. Then for all $\lambda_a \in B_p$, $\| \lambda_a \|_p = \inf\{\| a + x \| : x \in M_p\}$. We write $\overline{p}(\lambda_a) = p(a)$. Since, $\overline{p}$ is a B^*- seminorm on B_p, we have by Remark 1, $\overline{p}(\lambda_a) \leq \| \lambda_a \|_p$.

Further, by (1.8.1) in [5] $\| \lambda_a \|_p \leq p(\lambda_a)$. Thus $p(a) = \inf\{\| a + x \| : x \in M_p\}$, $a \in A$.

Suppose that I is an ideal in A and A/I is the quotient B^*-algebra with the quotient norm $\| \cdot \|_Q$. Then a functional p on A can be defined by $p(a) = \| \lambda_a \|_Q, a \in A$. This functional p is a B^*- seminorm p on A. Also, $I = I_d(A)$. Hence the mapping $p \to I_p$ maps $N(A)$ onto $I_d(A)$. Furthermore, if p and q are in N(A) with $p \leq q$, we have $I_q \leq I_p$.

Next, we will show that if $p, q \in N(A)$ with $I_q \subseteq I_p$ then $p(a) \leq q(a)$, $a \in A$. Since $p(a) \leq \inf\{\| a + x \| : x \in I_q\}$, we have $p(a) \leq q(a)$. Thus we have shown that the mapping $p \to I_p$ is one to one and p is minimal if and only if M_p is a maximal ideal. In fact, we have established that $p \to I_p$ maps $MN(A)$ one to one onto $M(A)$. □

Definition 1. **a** A nonzero B^*- seminorm p is external if I_p is a prime ideal of A.

b With the null kernel topology, $PR(A)$ is called the structure space of A. The structure space is a locally compact T_o space whose points are the T_1 elements of $M(A)$. Thus it is a $\pi-$ space if and only if $M(A) = PR(A)$.

Proposition 2. *Let $p \in NS(A), p \neq 0$. The map $p \to I_p$ is a continuous one to one mapping from the set of all such B^*- seminorms under the pointwise convergence topology, into the structure space of A.*

Proof. Let $\widehat{A}$ be the set if equivalence classes of non-zero topologically irreducible star-representations of A. If $p \in NS(A)$, then there exists a $\pi \in \widehat{A}$ such that $p(a) = |\pi(a)|$. With this, I_p is a primitive ideal because $\ker \pi = I_p$. Thus $NS(A) \setminus \{0\} \to PR(A)$ is a one to one map.

Let $F : NS(A) \setminus \{0\} \to PR(A)$ be the mapping defined by $p \to I_p$. Suppose that S is a subset of $PR(A)$, which is closed in the null-kernel topology. Let $\{p_\lambda\}$ be a net in $F^{-1}(S)$ such that $p_\lambda \to p$. Since $I_{p_\lambda} \in S$, for all λ, we have $I_p \supseteq \bigcap\{I_{p_\lambda}\}$. Since S is closed, it follows that $I_p \in S$. Thus $p \in F^{-1}(S)$ and $F^{-1}(S)$ is closed in the topology of pointwise convergence. □

Definition 2. An algebra A is said to be primitive if the mapping $p \to I_p$ defines an onto map.

The following theorem establishes a necessary and sufficient condition for A to be Primitive.

Theorem 1. *The algebra A is primitive if and only if for each $\pi \in \widehat{A}$, the B^*- seminorm p_π is a sub nonzero seminorm on A.*

Proof. For each $\pi \in \widehat{A}$, define the B^*- seminorm $p_\pi(a) = |\pi(a)|, a \in A$. Suppose that A is primitive, then $\ker(\pi) = I_{p_\pi}$. Hence, p_π is a nonzero seminorm because $I_{p_\pi} \in PR(A)$.

To prove the converse, let p_π be in $NS(A)$ and $I_p \in PR(A)$. Then there exists $\pi \in \widehat{A}$ with $\ker = I_p$. This means that $I_{p_\pi} = I_p, p = p_\pi$, and $p \in NS(A)$. □

Corollary 1. *If the structure space of A is T_1, then A is a primitive algebra.*

Proof. Let I_p be a primitive ideal in A for some p in $N(A)$. Since $PR(A)$ is a T_1-space, we have $I_p \in M(A)$. Hence by Theorem 1, p is minimal and therefore sub and nonzero. Thus, A is primitive. □

Remark 2. If A has a T_1- structure space then the nonzero sub seminorms are minimal B^*- seminorms. Further, if A is primitive then the bijection map F (which is continuous) is not a homeomorphism.

The following theorem gives a necessary and a sufficient condition for the map $p \to I_p$ to be a homeomorphism.

Theorem 2. *The map $p \to I_p$ is a homeomorphism if and only if the structure space of A is a T_2- space.*

Proof. Let the structure space, $PR(A)$ be a T_2- space. The $PR(A)$ is also a T_1- space and the map $F : NS(A) \setminus \{0\} \to PR(A)$ is one to one and onto. Denote the set of all closed subsets of $PR(A)$ by $Cl(A)$. The space $Cl(A)$ is a compact T_2- space. Define the map $\overline{F} : N(A) \to Cl(A)$ by $\overline{F}(p) = \overline{I}_p$. Then by Theorem 2.2 in [6] $\overline{F}$ is a homeomorphism.

Next, we define the map $\phi : PR(A) \to Cl(A)$ by $\phi(I_p) = \{I_p\}$. Then by [1], the map $\phi : PR(A) \to \phi(PR(A))$ is a homeomorphism. Our main objective is to show that the map F is open. Suppose $G = \phi \circ F$. Then G is well defined. Let V be an open subset of $NS(A) \setminus \{0\}$. Then for an open set, U, in $N(A)$, V can be written as $V = U \cap (NS(A) \setminus \{0\})$. Therefore, $G(V) = \overline{F}(U) \cap \phi(PR(A))$ is open in $\phi(PR(A))$ because $\overline{F}(U)$ is open in $\overline{PR(A)}$. We also note that $\phi^{-1}(G(V))$ is open in $PR(A)$ since ϕ is a homeomorphism onto its range. In fact, $F(V) = \phi^{-1}(G(V))$ and $F(V)$ is open. Accordingly, F is a homeomorphism. □

Remark 3. In 2006, [2] the authors have used a similar seminorm, M^*-defined on a $*$- subalgebra of a given $*$- algebra. They have shown that an M^*-seminorm in unbounded if an $*$- representation of A exists. More on this to [3] [4]. The results of this paper hold for M^*-seminorms.

Remark 4. A B^*- seminorm p on A is extremal if, whenever q, r are B^*-seminorms on A with $p(a) = \max\{r(a), q(a)\}, a \in A$ we have $p = q$ or $p = r$. Let $EN(A)$ denote the set of all extremal B^*- seminorms on A. Then we have the following

$$MN(A) \subseteq NS(A) \subseteq EN(A)$$

It is easy to show that a nonzero B^*- seminorm p is extremal if and only if I_p is a prime ideal. The mapping $p \to I_p$ maps $EN(A) \setminus \{0\}$ one to one onto PR(A).

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