A CHARACTERIZATION OF
PRIMITIVE ALGEBRAS VIA B*–SEMINORMS

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Abstract: Primitive algebras are characterized in terms of sub-seminorms. We prove a necessary and a sufficient condition for a B*–seminorm to be minimal. The results of this note are also valid for M*–seminorms introduced by Bhatt and others in [2].

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1. Introduction

Let A be a complex B*–algebra. For this algebra A, the norm is given by a B*–seminorm. A B*–seminorm on A is a seminorm p with the following conditions: For all \( a, b \in A, p(ab) \leq p(a) \cdot p(b) \) and \( p(a^*a) = (p(a))^2 \). A seminorm p on A is an M*–seminorm if for all \( a, b \in A, p(ab) \leq p(a) \cdot p(b) \) and \( p(a^*a) = p(a) \). Clearly, every B*–seminorm on A is an M*–seminorm on A [2].

Let \( N(A) \) be the set of all B*–seminorms on A. A partial ordering on \( N(A) \) is defined as follows. For \( p, q \in N(A), p \leq q \) if and only if \( p(a) \leq q(a) \) for all \( a \in A \). We say that a B*–seminorm \( p \) on A is a subseminorm if \( p \in Q \), whenever Q is a subset of \( N(A) \) such that \( p(a) = \max\{q(a) : q \in Q\} \). The set of all sub-seminorms on A is denoted by \( NS(A) \).
Let $A$ be a complex Banach*—algebras where norm is a $B^*$—seminorm on $A$. We call a $B^*$—seminorm $p$ on $A$ is minimal if for a non-zero $p$, $p = q$, whenever $p$ is a nonzero $B^*$—seminorm on $A$ dominated by $p$. Let $MN(A)$ denote the set of all minimal $B^*$—seminorms on $A$. Then $MN(A) \subseteq NS(A)$.

**Remark 1.** We note that the norm on $A$ coincides with the largest $B^*$—seminorm on $A$. To see this, let $g(a) = \sup\{p(a) : p \in N(A), a \in A\}$. Then for all $a$ in $A$, $\|a\| \leq g(a)$. Also, $g(a) \leq \|a^*a\|^{1/2} = \|a\|$. Hence, $g(a) = \|a\|, \forall a \in A$. This idea is also true in case of $M^*$—seminorms, see [2].

We denote by $I_d(A)$ the set of all ideals of $A$. The set of primitive ideals and maximal ideals of $A$ that are written as $PR(A)$ and $M(A)$ respectively.

**2. Minimal Norms and Maximal Ideals of $A$**

In the following proposition, we present a characterization of minimal norms on $A$ in terms of maximal ideal space of $A$.

**Proposition 1.** A $B^*$—seminorm $p$, is minimal if and only if $M_p$ is a maximal ideal.

**Proof.** For each $B^*$—seminorm $p$ on $A$, let $M_p = \{a \in A : p(a) = 0\}$. Then the quotient star algebra $A/M_p$ with the norm $\| \cdot \|_p$ is a $B^*$—algebra, say $B_p$. Let $a \to \lambda_a$ be the canonical map from $A$ onto $A/M_p$. Then for all $\lambda_a \in B_p$, $\|\lambda_a\|_p = \inf\{\|a + x\| : x \in M_p\}$. We write $\overline{p}(\lambda_a) = p(a)$. Since, $\overline{p}$ is a $B^*$—seminorm on $B_p$, we have by Remark 1, $\overline{p}(\lambda_a) \leq \|\lambda_a\|_p$.

Further, by (1.8.1) in [5] $\|\lambda_a\|_p \leq p(\lambda_a)$. Thus $p(a) = \inf\{\|a + x\| : x \in M_p\}, a \in A$.

Suppose that $I$ is an ideal in $A$ and $A/I$ is the quotient $B^*$—algebra with the quotient norm $\| \cdot \|_Q$. Then a functional $p$ on $A$ can be defined by $p(a) = \|\lambda_a\|_Q, a \in A$. This functional $p$ is a $B^*$—seminorm $p$ on $A$. Also, $I = I_d(A)$. Hence the mapping $p \to I_p$ maps $N(A)$ onto $I_d(A)$. Furthermore, if $p$ and $q$ are in $N(A)$ with $p \leq q$, we have $I_q \leq I_p$.

Next, we will show that if $p, q \in N(A)$ with $I_q \subseteq I_p$ then $p(a) \leq q(a), a \in A$. Since $p(a) \leq \inf\{\|a + x\| : x \in I_q\}$, we have $p(a) \leq q(a)$. Thus we have shown that the mapping $p \to I_p$ is one to one and $p$ is minimal if and only if $M_p$ is a maximal ideal. In fact, we have established that $p \to I_p$ maps $MN(A)$ one to one onto $M(A)$. \qed
**Definition 1.** a A nonzero $B^*$–seminorm $p$ is external if $I_p$ is a prime ideal of $A$.

b With the null kernel topology, $PR(A)$ is called the structure space of $A$. The structure space is a locally compact $T_0$ space whose points are the $T_1$ elements of $M(A)$. Thus it is a $\pi$– space if and only if $M(A) = PR(A)$.

**Proposition 2.** Let $p \in NS(A), p \neq 0$. The map $p \to I_p$ is a continuous one to one mapping from the set of all such $B^*$–seminorms under the pointwise convergence topology, into the structure space of $A$.

**Proof.** Let $\hat{A}$ be the set if equivalence classes of non-zero topologically irreducible star-representations of $A$. If $p \in NS(A)$, then there exists a $\pi \in \hat{A}$ such that $p(a) = |\pi(a)|$. With this, $I_p$ is a primitive ideal because ker $\pi = I_p$. Thus $NS(A) \setminus \{0\} \to PR(A)$ is a one to one map.

Let $F : NS(A) \setminus \{0\} \to PR(A)$ be the mapping defined by $p \to I_p$. Suppose that $S$ is a subset of $PR(A)$, which is closed in the null-kernel topology. Let $\{p_\lambda\}$ be a net in $F^{-1}(S)$ such that $p_\lambda \to p$. Since $I_{p_\lambda} \in S$, for all $\lambda$, we have $I_p \supseteq \bigcap\{I_{p_\lambda}\}$. Since $S$ is closed, it follows that $I_p \in S$. Thus $p \in F^{-1}(S)$ and $F^{-1}(S)$ is closed in the topology of pointwise convergence.

**Definition 2.** An algebra $A$ is said to be primitive if the mapping $p \to I_p$ defines an onto map.

The following theorem establishes a necessary and sufficient condition for $A$ to be Primitive.

**Theorem 1.** The algebra $A$ is primitive if and only if for each $\pi \in \hat{A}$, the $B^*$–seminorm $p_\pi$ is a sub nonzero seminorm on $A$.

**Proof.** For each $\pi \in \hat{A}$, define the $B^*$–seminorm $p_\pi(a) = |\pi(a)|, a \in A$. Suppose that $A$ is primitive, then ker$(\pi) = I_{p_\pi}$. Hence, $p_\pi$ is a nonzero seminorm because $I_{p_\pi} \in PR(A)$.

To prove the converse, let $p_\pi$ be in $NS(A)$ and $I_p \in PR(A)$. Then there exists $\pi \in \hat{A}$ with ker $= I_p$. This means that $I_{p_\pi} = I_p, p = p_\pi$, and $p \in NS(A)$.

**Corollary 1.** If the structure space of $A$ is $T_1$, then $A$ is a primitive algebra.
Proof. Let $I_p$ be a primitive ideal in $A$ for some $p$ in $N(A)$. Since $PR(A)$ is a $T_1$-space, we have $I_p \in M(A)$. Hence by Theorem 1, $p$ is minimal and therefore sub and nonzero. Thus, $A$ is primitive.

Remark 2. If $A$ has a $T_1$-structure space then the nonzero sub seminorms are minimal $B^*$-seminorms. Further, if $A$ is primitive then the bijection map $F$ (which is continuous) is not a homeomorphism.

The following theorem gives a necessary and a sufficient condition for the map $p \mapsto I_p$ to be a homeomorphism.

Theorem 2. The map $p \mapsto I_p$ is a homeomorphism if and only if the structure space of $A$ is a $T_2$-space.

Proof. Let the structure space, $PR(A)$ be a $T_2$-space. The $PR(A)$ is also a $T_1$-space and the map $F : NS(A) \setminus \{0\} \rightarrow PR(A)$ is one to one and onto. Denote the set of all closed subsets of $PR(A)$ by $Cl(A)$. The space $Cl(A)$ is a compact $T_2$-space. Define the map $\overline{F} : N(A) \rightarrow Cl(A)$ by $\overline{F}(p) = \overline{I_p}$. Then by Theorem 2.2 in [6] $\overline{F}$ is a homeomorphism.

Next, we define the map $\phi : PR(A) \rightarrow Cl(A)$ by $\phi(I_p) = \{I_p\}$. Then by [1], the map $\phi : PR(A) \rightarrow \phi(PR(A))$ is a homeomorphism. Our main objective is to show that the map $F$ is open. Suppose $G = \phi \circ F$. Then $G$ is well defined. Let $V$ be an open subset of $NS(A) \setminus \{0\}$. Then for an open set, $U$, in $N(A)$, $V$ can be written as $V = U \cap (NS(A) \setminus \{0\})$. Therefore, $G(V) = \overline{F(U)} \cap \phi(PR(A))$ is open in $\phi(PR(A))$ because $\overline{F(U)}$ is open in $PR(A)$. We also note that $\phi^{-1}(G(V))$ is open in $PR(A)$ since $\phi$ is a homeomorphism onto its range. In fact, $F(V) = \phi^{-1}(G(V))$ and $F(V)$ is open. Accordingly, $F$ is a homeomorphism.

Remark 3. In 2006, [2] the authors have used a similar seminorm, $M^*$-defined on a $*$-subalgebra of a given $*$-algebra. They have shown that an $M^*$-seminorm in unbounded if an $*$-representation of $A$ exists. More on this to [3] [4]. The results of this paper hold for $M^*$-seminorms.

Remark 4. A $B^*$-seminorm $p$ on $A$ is extremal if, whenever $q, r$ are $B^*$-seminorms on $A$ with $p(a) = \max\{r(a), q(a)\}, a \in A$ we have $p = q$ or $p = r$. Let $EN(A)$ denote the set of all extremal $B^*$-seminorms on $A$. Then we have
the following

\[ MN(A) \subseteq NS(A) \subseteq EN(A) \]

It is easy to show that a nonzero \( B^* \)-seminorm \( p \) is extremal if and only if \( I_p \) is a prime ideal. The mapping \( p \to I_p \) maps \( EN(A) \setminus \{0\} \) one to one onto \( PR(A) \).

References


