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STABILITY OF LIE HOMOMORPHISMS ON LIE BANACH ALGEBRAS

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Abstract: In this paper we prove the Hyers-Ulam stability of n-ary Lie homomorphisms on n-ary Lie Banach algebras associated to a generalized functional equation.

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1. Introduction

A classical question in the theory of functional equations is: When is it true that a function which approximately satisfies a functional equation must be close to an exact solution of the equation? If the problem accepts a solution, we say that the equation is *stable*. For some sources see [1]-[3].

Definition 1.1. An n-ary algebra A is a complex linear space, endowed with an n-array product $(x_1, \dots, x_n) \to [x_1, \dots, x_n]_A$ from A^n into A such that

$$[[x_1, \dots, x_n]_A y_1, \dots, y_{n-1}]_A = [x_1[x_2, \dots, x_n y_1]_A y_2, \dots, y_{n-1}]_A = \cdots$$
$$= [x_1, \dots, x_{n-1}[x_n y_1, \dots, y_{n-1}]_A]_A,$$

for all $x_1, \dots, x_n, y_1, \dots, y_{n-1}$ in A.

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Definition 1.2. A Banach algebra A, endowed with the Lie product $[x, y] := (xy - yx)/2$ on A, is called a Lie Banach algebra.

Definition 1.3. Assume that the algbras A and B are complex n-ary algebras. A linear mapping $H : A \to B$ is said to be an n-ary Lie homomorphism if

$$H([x_1 \ , \dots , \ x_n]_A) = [H(x_1), \dots , H(x_n)]_B,$$

$$H[[x_1 \ , \dots , \ x_n]_A, [y_1 \ , \dots , \ y_n]_A] = [H[x_1 \ , \dots , \ x_n]_A, H[y_1 \ , \dots , \ y_n]_A]$$

hold for all $x_1, \dots , x_n$ in A.

We want to investigate the Hyers-Ulam stability of n-ary Lie homomorphisms.

2. Hyers-Ulam Stability of n-ary Lie Homomorphisms

In this section, we want to investigate the Hyers-Ulam stability of n-ary Lie homomorphisms acting on n-ary Lie Banach algebras associated with the following generalized functional equation:

$$\left(\frac{\sum_{i=1}^{n} x_i}{n+1}\right) + \mathrm{f}\left(\frac{nx_1 - \sum_{i=2}^{n-1} x_i - (n+1)x_n}{n+1}\right) + \mathrm{f}\left(\frac{(n+1)x_1 + nx_n}{n+1}\right) - 2\mathrm{f}(x_1) = 0.$$

Let A be an n-ary Banach algebra. For simplicity, for $x_1, \dots , x_n \in A$ and $\alpha \in \mathbb{C}$, we will denote the element $[x_1 \ , \dots , \ x_n]_A$, and n-tuples $(x_1, \dots , x_n)$ and $(\alpha x_1, \dots , \alpha x_n)$, respectively by $[X]_A$, X and αX. The same notations can be defined for $Y, Z, W, \alpha Y, \alpha Z, \alpha W, [Y]_A, [Z]_A, [W]_A$. Also, O denotes the zero n-tuple $(0, \dots , 0)$ in A^n.

For a given mapping $g : A \to B$, we define

$$\begin{aligned} C_\mu g(X, Y, Z, W) =& \frac{1}{\mu} g\left(\frac{\mu x_2 + \sum_{i=1, i\neq 2}^{n} x_i}{n+1}\right) \\ &+ g\left(\frac{nx_1 - \sum_{i=2}^{n-1} x_i - (n+1)x_n}{n+1}\right) \\ &+ g\left(\frac{(n+1)x_1 + nx_n}{n+1}\right) - 2g(x_1) + g\left([Y]_A\right) \\ &- [g(y_1) \ , \dots , \ g(y_n)]_B + g\big[[Z]_A, [W]_a\big] - \big[g[Z]_A, g[W]_A\big]. \end{aligned}$$

Theorem 2.1. *Assume that $(A, []_A)$ and $(B, []_B)$ are n-ary Lie Banach algebras and $n \geq 3$. Suppose that the real numbers $p, q,\ p_1, \ldots, p_n$ are such that $p, q, r, s < 1$ and $\sum_{i=1}^{n} p_i < 1$. Let $f : A \longrightarrow B$ be an odd function and $\varphi : A^{4n} \to [0, \infty)$ defined by*

$$\varphi(X, Y, Z, W) = \prod_{i=1}^{n} \|x_i\|^{p_i} + \sum_{i=1}^{n} \|x_i\|^{p} + \sum_{i=1}^{n} \|y_i\|^{q} + \sum_{i=1}^{n} \|z_i\|^{r} + \sum_{i=1}^{n} \|w_i\|^{s}$$

holds the relation:

$$(2) \qquad ||C_\mu f(X, Y, Z, W)|| \leq \varphi(X, Y, Z, W)$$

for all $\mu \in \mathbb{C}$ with $|\mu| = 1$ and all X, Y, Z, W in A^n. Then φ satisfies the relations (3) and (4) as follows:

(3) $\varphi(2X, 2Y, 2Z, 2W) \leq 2\alpha\ \varphi(X, Y, Z, W)$*;* $0 < \alpha < 1$*;* $X, Y, Z, W \in A^n$*.*

(4) $\lim_{m,k\to\infty} \frac{1}{2^m} \sum_{i=1}^{k} \frac{1}{2^i} \varphi(0, 2^{i+m-1}x_2, \ldots, 2^{i+m-1}x_n, O, O, O) = 0$*;* $x_2, \ldots, , x_n \in A$*.*

Furthermore, there exists a unique an n-ary Lie homomorphism $H : A \longrightarrow B$ such that

$$(5) \qquad ||f(x) - H(x)|| \leq (\tfrac{n+1}{n})^p \tfrac{1}{2-2^p} (\|x\|^p + \textstyle\sum_{j=2}^{n-1} \|x_j\|^p)$$

for all $x_2, \ldots, x_{n-1}, x_n \in A$ satisfying $(1 + 2n)x = -(x_2+, \ldots, +x_{n-1})$.

Proof. Note that for all $X \in A^n$, we have

$$\sum_{i=1}^{k} 2^{-i-m} \varphi(0, 2^{i+m-1}x_2, \ldots, 2^{i+m-1}x_n, O, O, O)$$

$$= 2^{-m+(m-1)p} \left(\sum_{j=2}^{n} \|x_j\|^p \right) \sum_{i=1}^{k} 2^{i(p-1)},$$

which tends to 0 as $m, k \to \infty$. So φ holds in the relation (4). Note that $\varphi(2X, 2Y, 2Z, 2W) \leq 2\alpha\varphi(X, Y, Z, W)$ where $\alpha = 2^{\beta-1}$ and $\beta = \frac{1}{2}\max\{p, q, r, s,$

$p_1+, \ldots, +p_n\}$. So, φ satisfies also the inequality (3). In (2), put $\mu = 1$, $x_1 = 0, Y = Z = W = O$. Then we get

$$\|C_1 \mathrm{f}(0, x_2, \ldots, x_{n-1}, x_n, O, O, O)\| \leq \varphi(0, x_2, \ldots, x_{n-1}, x_n, O, O, O)$$

for all $x_2, \ldots, x_{n-1}, x_n \in A$ satisfying $(1+2n)x_n = -(x_2+, \ldots, +x_{n-1})$. Hence

$$\|\mathrm{f}(\frac{n}{n+1}x_n) - \frac{1}{2}\mathrm{f}(2(\frac{n}{n+1}x_n))\| \leq \frac{1}{2}\varphi(0, x_2, \ldots, x_n, O, O, O)$$

for all $x_2, \ldots, x_{n-1}, x_n \in A$ satisfying $(1+2n)x_n = -(x_2+, \ldots, +x_{n-1})$. By proceeding in this way, we obtain

$$(6) ||f(\frac{n}{n+1}x_n) - \frac{1}{2^k}\mathrm{f}(2^k(\frac{n}{n+1}x_n))|| \leq \sum_{i=1}^{k} \frac{1}{2^i}\varphi(0, 2^{i-1}x_2, \ldots, 2^{i-1}x_n, O, O, O)$$

for all $x_2, \ldots, x_{n-1}, x_n \in A$ satisfying $(1+2n)x_n = -(x_2+, \ldots, +x_{n-1})$.

Replacing x_n by $2^m x_n$ in (6) and then dividing by 2^m, we get

$$(7) \quad \begin{aligned} &\|\frac{1}{2^m}\mathrm{f}(2^m(\frac{n}{n+1} \quad x_n \quad)) - \frac{1}{2^{m+k}}\mathrm{f}(2^{m+k}(\frac{n}{n+1}x_n))\| \\ &\leq \quad \frac{1}{2^m}\sum_{i=1}^{k}\frac{1}{2^i}\varphi(0, 2^{i+m-1}x_2, \ldots, 2^{i+m-1}x_n, O, O, O) \end{aligned}$$

For all integers m, k and all $x_2, \ldots, , x_{n-1}, x_n \in A$ satisfying $(1+2n)x_n = -(x_2+, \ldots, +x_{n-1})$. By the relations (4) and (7), the sequence

$$\{\frac{1}{2^m}\mathrm{f}(2^m(\frac{n}{n+1})x_n)\}_m$$

is a Cauchy sequence in B, for all $x_n \in A$ and so it is convergent. Thus there exists $H : A \to B$ such that $H(x) = \lim_{m\to\infty} \frac{1}{2^m} f(2^m x)$ for all $x \in A$. In (6), let $k \to \infty$, then we get

$$\|\mathrm{f}(\frac{n}{n+1}x_n) - H(\frac{n}{n+1}x_n)\| \leq \sum_{i=1}^{\infty}\frac{1}{2^i}\varphi(0, 2^{i-1}x_2, \ldots, 2^{i-1}x_n, O, O, O)$$

for all $x_2, \ldots, , x_{n-1}, x_n \in A$ satisfying $(1+2n)x_n = -(x_2+, \ldots, +x_{n-1})$. Since $(1+2n)(\frac{n+1}{n})x_n = -((\frac{n+1}{n})x_2+, \ldots, +(\frac{n+1}{n})x_{n-1})$, thus

$$||f(x) - H(x)|| \leq (\frac{n+1}{n})^p \frac{1}{2-2^p}(\|x\|^p + \sum_{j=2}^{n-1}\|x_j\|^p)$$

for all $x, x_2, \dots, x_{n-1} \in A$ satisfying $(1+2n)x = -(x_2+, \dots, +x_{n-1})$. So (5) holds. By using (3), we have $\|C_\mu H(X,O,O,O)\| \leq \lim_{m\to\infty} r^m \varphi(X,O,O,O) = 0$ for all $X \in A^n$. In the relation $C_\mu H(X,O,O,O) = 0$, put $\mu = 1, x_1 = 0, w = \frac{n}{n+1} x_n$ and $z = \frac{1}{n+1}\sum_{i=2}^n x_i$, then we get $H(z+w) = H(z) + H(w)$. So, H is additive. Also, it is now easy to see that $H(\mu x) = \mu H(x)$ for all $\mu \in \mathbb{C}$ with $|\mu| = 1$ and all x in A. Thus H is indeed $\mathbb{C}$ -linear. To show that H is an n-ary homomorphism, by using (3), we prove that $C_\mu H(O,Y,O,O) = 0$ for all Y in A^n. We have $\|C_\mu H(O,Y,O,O)\| \leq \lim_{m\to\infty} 2^{m-nm} r^m \varphi(O,Y,O,O) = 0$ for all $Y \in A^n$. Also, we note that

$$\|C_\mu H(O,O,Z,W)\| \leq \lim_{m\to\infty} 2^{m-2mn} r^m \varphi(O,O,Z,W) = 0,$$

so indeed H is an n-ary Lie homomorphism. This completes the proof. □

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