STABILITY OF LIE HOMOMORPHISMS ON LIE BANACH ALGEBRAS

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Abstract: In this paper we prove the Hyers-Ulam stability of $n$-ary Lie homomorphisms on $n$-ary Lie Banach algebras associated to a generalized functional equation.

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1. Introduction

A classical question in the theory of functional equations is: When is it true that a function which approximately satisfies a functional equation must be close to an exact solution of the equation? If the problem accepts a solution, we say that the equation is stable. For some sources see [1]-[3].

Definition 1.1. An $n$-ary algebra $A$ is a complex linear space, endowed with an $n$-array product $(x_1, \ldots, x_n) \to [x_1, \ldots, x_n]_A$ from $A^n$ into $A$ such that

$$[[x_1, \ldots, x_n]Ay_1, \ldots, y_{n-1}]_A = [x_1[x_2, \ldots, x_ny_1]Ay_2, \ldots, y_{n-1}]_A = \cdots = [x_1, \ldots, x_{n-1}[x_ny_1, \ldots, y_{n-1}]_A],$$

for all $x_1, \ldots, x_n, y_1, \ldots, y_{n-1}$ in $A$. 

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Definition 1.2. A Banach algebra \( A \), endowed with the Lie product 
\([x, y] := (xy - yx)/2\) on \( A \), is called a Lie Banach algebra.

Definition 1.3. Assume that the algebras \( A \) and \( B \) are complex \( n \)-ary algebras. A linear mapping \( H : A \to B \) is said to be an \( n \)-ary Lie homomorphism if

\[
H([x_1, \ldots, x_n]_A) = [H(x_1), \ldots, H(x_n)]_B,
\]

\[
H([x_1, \ldots, x_n]_B, [y_1, \ldots, y_n]_B) = [H[x_1, \ldots, x_n]_A, H[y_1, \ldots, y_n]_A]
\]

hold for all \( x_1, \ldots, x_n \) in \( A \).

We want to investigate the Hyers-Ulam stability of \( n \)-ary Lie homomorphisms.

2. Hyers-Ulam Stability of \( n \)-ary Lie Homomorphisms

In this section, we want to investigate the Hyers-Ulam stability of \( n \)-ary Lie homomorphisms acting on \( n \)-ary Lie Banach algebras associated with the following generalized functional equation:

\[
\left( \sum_{i=1}^{n} \frac{x_i}{n+1} \right) + f \left( \frac{nx_1 - \sum_{i=2}^{n-1} x_i - (n+1)x_n}{n+1} \right) + f \left( \frac{(n+1)x_1 + nx_n}{n+1} \right) - 2f(x_1) = 0.
\]

Let \( A \) be an \( n \)-ary Banach algebra. For simplicity, for \( x_1, \ldots, x_n \in A \) and \( \alpha \in \mathbb{C} \), we will denote the element \([x_1, \ldots, x_n]_A\) and \( n \)-tuples \((x_1, \ldots, x_n)\) and \((\alpha x_1, \ldots, \alpha x_n)\), respectively by \([X]_A\), \( X \) and \( \alpha X \). The same notations can be defined for \( Y, Z, W, \alpha Y, \alpha Z, \alpha W, [Y]_A, [Z]_A, [W]_A \). Also, \( O \) denotes the zero \( n \)-tuple \((0, \ldots, 0)\) in \( A^n \).

For a given mapping \( g : A \to B \), we define

\[
C_\mu g(X, Y, Z, W) = \frac{1}{\mu} g \left( \frac{\mu x_2 + \sum_{i=1, i \neq 2}^{n} x_i}{n+1} \right) + g \left( \frac{nx_1 - \sum_{i=2}^{n-1} x_i - (n+1)x_n}{n+1} \right) + g \left( \frac{(n+1)x_1 + nx_n}{n+1} \right) - 2g(x_1) + g ([Y]_A) - [g(y_1), \ldots, g(y_n)]_B + g ([Z]_A, [W]_A) - [g[Z]_A, g[W]_A].
\]
Theorem 2.1. Assume that \((A, \| A \|)\) and \((B, \| B \|)\) are \(n\)-ary Lie Banach algebras and \(n \geq 3\). Suppose that the real numbers \(p, q, p_1, \ldots, p_n\) are such that \(p, q, r, s < 1\) and \(\sum_{i=1}^{n} p_i < 1\). Let \(f : A \rightarrow B\) be an odd function and \(\varphi : A^{4n} \rightarrow [0, \infty)\) defined by

\[
\varphi(X, Y, Z, W) = \prod_{i=1}^{n} \|x_i\|^{p_i} + \sum_{i=1}^{n} \|x_i\|^p + \sum_{i=1}^{n} \|y_i\|^q + \sum_{i=1}^{n} \|z_i\|^r + \sum_{i=1}^{n} \|w_i\|^s
\]

holds the relation:

\[
(2) \quad ||C_{\mu}f(X, Y, Z, W)|| \leq \varphi(X, Y, Z, W)
\]

for all \(\mu \in \mathbb{C}\) with \(|\mu| = 1\) and all \(X, Y, Z, W\) in \(A^n\). Then \(\varphi\) satisfies the relations (3) and (4) as follows:

\[
(3) \quad \varphi(2X, 2Y, 2Z, 2W) \leq 2\alpha \varphi(X, Y, Z, W); \quad 0 < \alpha < 1; \quad X, Y, Z, W \in A^n.
\]

\[
(4) \quad \lim_{m,k \to \infty} \frac{1}{2^m} \sum_{i=1}^{k} \frac{1}{2^i} \varphi(0, 2^{i+m-1}x_2, \ldots, 2^{i+m-1}x_n, O, O, O) = 0; \quad x_2, \ldots, x_n \in A.
\]

Furthermore, there exists a unique an \(n\)-ary Lie homomorphism \(H : A \rightarrow B\) such that

\[
(5) \quad ||f(x) - H(x)|| \leq (\frac{n+1}{n})p \frac{1}{2^{2p}} \sum_{j=2}^{n} \|x_j\|^p + \sum_{j=2}^{n-1} \|x_j\|^p
\]

for all \(x_2, \ldots, x_{n-1}, x_n \in A\) satisfying \((1 + 2n)x = -(x_2 +, \ldots, +x_{n-1})\).

Proof. Note that for all \(x \in A^n\), we have

\[
\sum_{i=1}^{k} 2^{-i-m} \varphi(0, 2^{i+m-1}x_2, \ldots, 2^{i+m-1}x_n, O, O, O) = 2^{-m+(n-1)p} \left( \sum_{j=2}^{n} \|x_j\|^p \right) \sum_{i=1}^{k} 2^{i(p-1)},
\]

which tends to 0 as \(m, k \to \infty\). So \(\varphi\) holds in the relation (4). Note that \(\varphi(2X, 2Y, 2Z, 2W) \leq 2\alpha \varphi(X, Y, Z, W)\) where \(\alpha = 2^{\beta-1}\) and \(\beta = \frac{1}{2} \max\{p, q, r, s,\).
For all $x_2, \ldots, x_{n-1}, x_n \in A$ satisfying $(1 + 2n)x_n = -(x_2 + \ldots, + x_{n-1})$. Hence

\[
\|f\left(\frac{n}{n+1}x_n\right) - \frac{1}{2}f\left(\frac{n}{n+1}x_n\right)\| \leq \frac{1}{2}\varphi(0, 2^{i-1}x_2, \ldots, 2^{i-1}x_n, O, O, O)
\]

for all $x_2, \ldots, x_{n-1}, x_n \in A$ satisfying $(1 + 2n)x_n = -(x_2 + \ldots, + x_{n-1})$. By proceeding in this way, we obtain

\[
(6)\|f\left(\frac{n}{n+1}x_n\right) - \frac{1}{2^k}f\left(\frac{n}{n+1}x_n\right)\| \leq \sum_{i=1}^{k} \frac{1}{2^i}\varphi(0, 2^{i-1}x_2, \ldots, 2^{i-1}x_n, O, O, O)
\]

for all $x_2, \ldots, x_{n-1}, x_n \in A$ satisfying $(1 + 2n)x_n = -(x_2 + \ldots, + x_{n-1})$.

Replacing $x_n$ by $2^m x_n$ in (6) and then dividing by $2^m$, we get

\[
(7)\quad \left\| \frac{1}{2^m}f\left(2^m\left(\frac{n}{n+1}x_n\right)\right) - \frac{1}{2^{m+k}}f\left(2^{m+k}\left(\frac{n}{n+1}x_n\right)\right) \right\| \leq \frac{1}{2^m} \sum_{i=1}^{k} \frac{1}{2^i}\varphi(0, 2^{i+m-1}x_2, \ldots, 2^{i+m-1}x_n, O, O, O)
\]

For all integers $m, k$ and all $x_2, \ldots, x_{n-1}, x_n \in A$ satisfying $(1 + 2n)x_n = -(x_2 + \ldots, + x_{n-1})$. By the relations (4) and (7), the sequence

\[
\left\{ \frac{1}{2^m}f\left(2^m\left(\frac{n}{n+1}x_n\right)\right) \right\}_m
\]

is a Cauchy sequence in $B$, for all $x_n \in A$ and so it is convergent. Thus there exists $H : A \to B$ such that $H(x) = \lim_{m \to \infty} \frac{1}{2^m}f\left(2^m x\right)$ for all $x \in A$. In (6), let $k \to \infty$, then we get

\[
\|f\left(\frac{n}{n+1}x_n\right) - H\left(\frac{n}{n+1}x_n\right)\| \leq \sum_{i=1}^{\infty} \frac{1}{2^i}\varphi(0, 2^{i-1}x_2, \ldots, 2^{i-1}x_n, O, O, O)
\]

for all $x_2, \ldots, x_{n-1}, x_n \in A$ satisfying $(1 + 2n)x_n = -(x_2 + \ldots, + x_{n-1})$. Since $(1 + 2n)\left(\frac{n+1}{n}\right)x_n = -\left((\frac{n+1}{n})x_2 + \ldots, +(\frac{n+1}{n})x_{n-1}\right)$, thus

\[
\|f(x) - H(x)|| \leq \left(\frac{n+1}{n}\right)^p \frac{1}{2 - 2^p} (\|x\|^p + \sum_{j=2}^{n-1} \|x_j\|^p)
\]
for all \(x, x_2, \ldots, x_{n-1} \in A\) satisfying \((1 + 2n)x = -(x_2 + \ldots + x_{n-1})\). So (5) holds. By using (3), we have \(\|C_\mu H(X, O, O, O)\| \leq \lim_{m \to \infty} r^m \varphi(X, O, O, O) = 0\) for all \(X \in A^n\). In the relation \(C_\mu H(X, O, O, O) = 0\), put \(\mu = 1, x_1 = 0, w = \frac{n}{n+1} x_n\) and \(z = \frac{1}{n+1} \sum_{i=2}^{n} x_i\), then we get \(H(z + w) = H(z) + H(w)\). So, \(H\) is additive. Also, it is now easy to see that \(H(\mu x) = \mu H(x)\) for all \(\mu \in \mathbb{C}\) with \(|\mu| = 1\) and all \(x\) in \(A\). Thus \(H\) is indeed \(\mathbb{C}\)-linear. To show that \(H\) is an \(n\)-ary homomorphism, by using (3), we prove that \(C_\mu H(O, Y, O, O) = 0\) for all \(Y \in A^n\). We have \(\|C_\mu H(O, Y, O, O)\| \leq \lim_{m \to \infty} 2^{m-nm} r^m \varphi(O, Y, O, O) = 0\) for all \(Y \in A^n\). Also, we note that

\[
\|C_\mu H(O, O, Z, W)\| \leq \lim_{m \to \infty} 2^{m-2mn} r^m \varphi(O, O, Z, W) = 0,
\]

so indeed \(H\) is an \(n\)-ary Lie homomorphism. This completes the proof. \(\square\)

References


