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ON THE DIOPHANTINE EQUATION $143^x + 143^{2s}n^y = z^{2t}$ WHERE s, t, n ARE NON-NEGATIVE INTEGERS AND $n \equiv 5 \pmod{20}$

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Abstract: Let s, t, n be non-negative integers and $n \equiv 5 \pmod{20}$. In this paper, we found that all non-negative integer solutions (x, y, z) of the Diophantine equation $143^x + 143^{2s}n^y = z^{2t}$ are in the following form:

$$(x, y, z) = \begin{cases} (1 + 2s, 0, 12(143)^s) & ; \quad t = 1, \\ \text{no solution} & ; \quad \text{otherwise.} \end{cases}$$

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1. Introduction

There are many mathematicians solved all non-negative integer solutions (x, y, z) of the Diophantine equation $a^x + b^y = z^2$ where a, b are positive integers and x, y, z are non-negative integers. For example, in 2013, Chotchaisthit [1] showed that $(p, x, y, z) = (7, 0, 1, 3)$ and $(p, x, y, z) = (3, 2, 2, 5)$ are the only solutions of the Diophantine equation $p^x + (p + 1)^y = z^2$ where x, y, z are non-negative integers and p is the Mersenne prime.

In 2012-2014, there are many papers of Sroysang, for examples, in [4, 5, 6, 8]

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solved non-negative integer solutions (x, y, z) of the Diophantine equations $3^x + 5^y = z^2$, $3^x + 17^y = z^2, 3^x + 85^y = z^2$ and $3^x + 45y^2 = z^2$, respectively. He found that the Diophantine equations have exactly one non-negative integer solution, namely, $(x, y, z) = (1, 0, 2)$.

In recently 2014, Sroysang [7] showed that there is a unique non-negative integer solution $(x, y, z) = (1, 0, 12)$ for the Diophantine equation $143^x + 145^y = z^2$. Now we find all possible non-negative integer solutions (x, y, z) of the Diophantine equation $143^x + 143^{2s}n^y = z^{2t}$ where $n \equiv 5 \pmod{20}$, which is a generalization of the Diophantine equation $143^x + 145^y = z^2$ when $s = 0$ and $n = 145$.

2. Preliminaries

Throughout this paper, we assume that n is a non-negative integer such that $n \equiv 5 \pmod{20}$. It is clear that $n \equiv 1 \pmod 4$ and $n \equiv 0 \pmod 5$.

Corollary 2.1. *[3] Let a and b be non-zero integers and c be an integer. If $a|c$ and $b|c$, with $\gcd(a, b) = 1$, then $ab|c$.*

Theorem 2.2. *[3] Let a, b, p be integers. If p is a prime and $p|ab$, then $p|a$ or $p|b$.*

Lemma 2.3. *[7] $(1, 12)$ is a unique non-negative integer solution (x, z) for the Diophantine equation $143^x + 1 = z^2$.*

Let p be an odd prime and a be a positive integer where $gcd(a, p) = 1$. If the quadratic congruence $x^2 \equiv a \pmod p$ has a solution, then a is said to be *a quadratic residue of p*. Otherwise, a is called *a quadratic non-residue of p*. In 1798, Adrien-Marie Legendre introduced *the Legendre symbol* $\left(\frac{a}{p}\right)$ which is defined by

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & ;\text{if } a \text{ is a quadratic residue of } p, \\ -1 & ;\text{if } a \text{ is a quadratic non-residue of } p. \end{cases}$$

In the present paper, we need the following well-known facts about the Legendre symbols.

Theorem 2.4. *[3] If p is an odd prime, then*

$$\left(\frac{2}{p}\right) = \begin{cases} 1 & ;\textit{if } p \equiv 1 \pmod 8 \textit{ or } p \equiv 7 \pmod 8, \\ -1 & ;\textit{if } p \equiv 3 \pmod 8 \textit{ or } p \equiv 5 \pmod 8. \end{cases}$$

Theorem 2.5. *[3] If $p \neq 3$ is an odd prime, then*

$$\left(\frac{3}{p}\right) = \begin{cases} 1 & ; \textit{if } p \equiv \pm 1 \pmod{12}, \\ -1 & ; \textit{if } p \equiv \pm 5 \pmod{12}. \end{cases}$$

3. Main Results

First, we consider all non-negative integer solutions (x, y, z) of the Diophantine equation $143^x + n^y = z^2$ where x, y, z are non-negative integers.

Theorem 3.1. $(1, 0, 12)$ *is a unique non-negative integer solution* (x, y, z) *for the Diophantine equation* $143^x + n^y = z^2$.

Proof. Since $n \equiv 5 \pmod{20}$, n is an odd integer. This implies that z is an even integer. Thus $z^2 \equiv 0 \pmod{4}$. Now we consider the Diophantine equation $143^x + n^y = z^2$ in the following cases:

Case 1. $y = 0$. By Lemma 2.3, we have $(x, y, z) = (1, 0, 12)$ is a unique non-negative integer solution of the Diophantine equation $143^x + n^y = z^2$.

Case 2. $y \geq 1$. Assume that there exists a non-negative integer solution (x, y, z) for the Diophantine equation $143^x + n^y = z^2$. Since $n \equiv 5 \pmod{20}$, $n \equiv 1 \pmod{4}$ and $n \equiv 0 \pmod{5}$. Then we have $0 \equiv z^2 = 143^x + n^y \equiv 3^x + 1 \pmod{4}$. Thus $3^x \equiv -1 \pmod{4}$. This implies that x is odd. It follows that $3^x \equiv 2 \pmod{5}$ or $3^x \equiv 3 \pmod{5}$. Since $n^y \equiv 0 \pmod{5}$, we obtain $z^2 = 143^x + n^y = 3^x + 0 \equiv 2$ or $3 \pmod{5}$. By assumption, we have $\left(\frac{2}{5}\right) = 1$ and $\left(\frac{3}{5}\right) = 1$. That are contradictions to Theorem 2.4 and Theorem 2.5, respectively. In this case, there is no non-negative integer solution.

It can be seen that $(1, 0, 12)$ is a unique non-negative integer solution (x, y, z) for the Diophantine equation $143^x + n^y = z^2$. This completes the proof. □

We know that $n = 145 \equiv 5 \pmod{20}$. The following example is the main theorem of Sroysang [7] which is a special case of Theorem 3.1.

Example 3.2. $(1, 0, 12)$ *is a unique non-negative integer solution* (x, y, z) *for the Diophantine equation* $143^x + 145^y = z^2$.

Lemma 3.3. *[2] Let m be a positive integer with $m \equiv 1 \pmod{4}$. The Diophantine equation $1 + mn^y = z^2$ has no non-negative integer solution (y, z).*

Lemma 3.4. *Let m be an integer greater than 1. The Diophantine equation $143 + 143^m n^y = z^2$ has no non-negative integer solution (y, z).*

Proof. Suppose that there are non-negative integers y, z such that $143 + 143^m n^y = z^2$. We have $143 \mid z^2$. Since $11|143$ and $13|143$, this implies $11|z^2$ and $13|z^2$. By Corollary 2.1 and Theorem 2.2, we conclude that $143|z$ and hence $z = 143r$ for some integer r. Substituting $z = 143r$ in the Diophantine equation $143+143^m n^y = z^2$, then $143+143^m n^y = 143^2 r^2$ and thus $1+143^{m-1}n^y = 143r^2$. We get $1 = 143(r^2 - 143^{m-2}n^y)$. So that $143|1$. This is a contradiction. Hence $143 + 143^m n^y = z^2$ has no non-negative integer solution (y, z). □

The next theorem is main result of this paper.

Theorem 3.5. *Let s be a non-negative integer. Then $(1+2s, 0, 12(143)^s)$ is a unique non-negative integer solution (x, y, z) for the Diophantine equation $143^x + 143^{2s} n^y = z^2$.*

Proof. We will prove by induction on s.
Let $P(s)$: The Diophantine equation $143^x + 143^{2s} n^y = z^2$ has a unique non-negative integer solution $(x, y, z) = (1 + 2s, 0, 12(143)^s)$.
When $s = 0$, we have $P(0)$ is true by Theorem 3.1.

Suppose that $P(k)$ is true, that is the Diophantine equation $143^x+143^{2k}n^y = z^2$ has a unique non-negative integer solution $(x, y, z) = (1 + 2k, 0, 12(143)^k)$.

Consider the Diophantine equation $143^x + 143^{2(k+1)} n^y = z^2$ into the following cases:

Case $x = 0$. Since $143^{2(k+1)} \equiv 1 \pmod 4$ and by Lemma 3.3, we obtain that $1 + 143^{2(k+1)} n^y = z^2$ has no non-negative integer solution.

Case $x = 1$. By Lemma 3.4, the Diophantine equation $143+143^{2(k+1)} n^y = z^2$ has no non-negative integer solution.

Case $x \geq 2$. Note that $143^x + 143^{2(k+1)} n^y = z^2$ can be written as $143^{x-2} + 143^{2k} n^y = \left(\frac{z}{143}\right)^2$ such that $x - 2, \frac{z}{143}$ are non-negative integers. Let $u = x - 2$ and $v = \frac{z}{143}$. By assumption $P(k)$ is true, we obtain that $143^u + 143^{2k} n^y = v^2$ has a unique non-negative integer solution $(u, y, v) = (1 + 2k, 0, 12(143)^k)$. That is $u = 1 + 2k$ and $v = 12(143)^k$. Thus $(x, y, z) = (1 + 2(k + 1), 0, 12(143)^{k+1})$ is a unique non-negative integer solution of the Diophantine equation $143^x + 143^{2(k+1)} n^y = z^2$. Therefore $P(k + 1)$ is true.

By mathematical induction, $P(s)$ is true for all non-negative integer s. That is the Diophantine equation $143^x + 143^{2s} n^y = z^2$ has a unique non-negative integer solution $(x, y, z) = (1 + 2s, 0, 12(143)^s)$. □

Corollary 3.6. *Let s, t be non-negative integers such that $t \geq 2$. The Diophantine equation $143^x + 143^{2s} n^y = z^{2t}$ has no non-negative integer solution (x, y, z).*

Proof. Suppose that (x, y, z) is a non-negative integer solution of the Diophantine equation $143^x + 143^{2s}n^y = z^{2t}$. Thus (x, y, z^t) is a non-negative integer solution of the Diophantine equation $143^x + 143^{2s}n^y = z^2$. By Theorem 3.5, we have $(x, y, z^t) = (1 + 2s, 0, 12(143)^s)$. Therefore, $z^t = 12(143)^s$. This is a contradiction. Hence, the Diophantine equation $143^x + 143^{2s}n^y = z^{2t}$ has no non-negative integer solution (x, y, z). □

The Diophantine equation $143^x + 143^{2s}n^y = z^{2t}$, when s, t are non-negative integers, is a generalization of the Diophantine equation $143^x + 145^y = z^2$. It can verify that the Diophantine equation $143^x + 143^{2s}n^y = z^{2t}$ has no non-negative integer solution when $t = 0$. Theorem 3.5 and Corollary 3.6 give the following result.

Theorem 3.7. *Let n, s, t be any non-negative integers such that $n \equiv 5 \pmod{20}$. All non-negative integer solutions (x, y, z) of the Diophantine equation $143^x + 143^{2s}n^y = z^{2t}$ are the following:*

$$(x, y, z) = \begin{cases} (1 + 2s, 0, 12(143)^s) & ; \quad t = 1, \\ \textit{no solution} & ; \quad \textit{otherwise.} \end{cases}$$

Using Theorem 3.7, it is easy to verify the following examples.

Example 3.8. *For $n = 5, s = 0, 1, 2, 3, 4, ...$*

s	$143^x + 143^{2s}5^y = z^2$	*solution* (x, y, z)
$s = 0$	$143^x + 5^y = z^2$	$(1, 0, 12)$
$s = 1$	$143^x + (143)^2 5^y = z^2$	$(3, 0, 1716)$
$s = 2$	$143^x + (143)^4 5^y = z^2$	$(5, 0, 245388)$
$s = 3$	$143^x + (143)^6 5^y = z^2$	$(7, 0, 35090484)$
$s = 4$	$143^x + (143)^8 5^y = z^2$	$(9, 0, 5017939212)$
$\vdots$	$\vdots$	$\vdots$

For $n = 25, s = 0, 1, 2, 3, 4, ...$

s	$143^x + 143^{2s}25^y = z^2$	*solution* (x, y, z)
$s = 0$	$143^x + 25^y = z^2$	$(1, 0, 12)$
$s = 1$	$143^x + (143)^2 25^y = z^2$	$(3, 0, 1716)$
$s = 2$	$143^x + (143)^4 25^y = z^2$	$(5, 0, 245388)$
$s = 3$	$143^x + (143)^6 25^y = z^2$	$(7, 0, 35090484)$
$s = 4$	$143^x + (143)^8 25^y = z^2$	$(9, 0, 5017939212)$
$\vdots$	$\vdots$	$\vdots$

For $n = 45, s = 0, 1, 2, 3, 4, ...$

s	$143^x + 143^{2s}45^y = z^2$	*solution* (x, y, z)
$s = 0$	$143^x + 45^y = z^2$	$(1, 0, 12)$
$s = 1$	$143^x + (143)^2 45^y = z^2$	$(3, 0, 1716)$
$s = 2$	$143^x + (143)^4 45^y = z^2$	$(5, 0, 245388)$
$s = 3$	$143^x + (143)^6 45^y = z^2$	$(7, 0, 35090484)$
$s = 4$	$143^x + (143)^8 45^y = z^2$	$(9, 0, 5017939212)$
$\vdots$	$\vdots$	$\vdots$

Moreover, for any $n \equiv 5 \pmod{20}$ we obtain the non-negative integer solutions (x, y, z) of the Diophantine equation of the form $143^x + 143^{2s}n^y = z^{2t}$ in a similar way.

We can apply Theorem 3.7 to solve all non-negative solutions of many Diophantine equations, which related with the Diophantine equation $143^x + 143^{2s}n^y = z^{2t}$. We show as the following examples.

Example 3.9. $(x, y, z) = (5, 0, 22308)$ *is a unique non-negative integer solution* (x, y, z) *for the Diophantine equation* $143^x + 418161601(135)^y = 121z^2$.

Proof. Note that the Diophantine equation $143^x + 418161601(135)^y = 121z^2$ can be written as $143^x + (143)^4(135)^y = (11z)^2$. By Theorem 3.7, we have $(x, y, 11z) = (5, 0, 12(143)^2)$. Thus $11z = 12(143)^2$ or $z = 22308$. Therefore, $(x, y, z) = (5, 0, 22308)$ is a unique non-negative integer solution. □

Example 3.10. $(x, y, z) = (2, 0, 1)$ *is a unique non-negative integer solution* (x, y, z) *for the Diophantine equation* $143^x + 143(65)^y = 20592z^2$.

Proof. Note that the Diophantine equation $143^x + 143(65)^y = 20592z^2$ can be written as $143^x + 143(65)^y = 143(12z)^2$. If $x = 0$, then $143|1$. This is a contradiction. Suppose $x \geq 1$. So we have $143^{x-1} + 65^y = (12z)^2$. By Theorem 3.7, we have $(x - 1, y, 12z) = (1, 0, 12)$. Thus $x - 1 = 1$ and $12z = 12$ or $x = 2$ and $z = 1$. Therefore, $(x, y, z) = (2, 0, 1)$ is a unique non-negative integer solution of the Diophantine equation $143^x + 143(65)^y = 20592z^2$. □

Example 3.11. *The Diophantine equation* $143^x + 2924207(45)^y = 2288z^4$ *has no non-negative integer solution* (x, y, z).

Proof. Note that the Diophantine equation $143^x + 2924207(45)^y = 2288z^4$ can be written as $143^x + (143)^3(45)^y = 143(2z)^4$. If $x = 0$, then $143|1$. This is a contradiction. Suppose $x \geq 1$. So we have $143^{x-1} + 143^2(45)^y = (2z)^4$. By Theorem 3.7, this equation has no non-negative integer solution (x, y, z). □

Example 3.12. *The Diophantine equation* $143^x + 143(25)^y = 3575z^2$ *has no non-negative integer solution* (x, y, z).

Proof. Note that the Diophantine equation $143^x + 143(25)^y = 3575z^2$ can be written as $143^x + 143(25)^y = 143(5z)^2$. If $x = 0$, then $143|1$. This is a contradiction. Suppose $x \geq 1$. So we have $143^{x-1} + 25^y = (5z)^2$. By Theorem 3.7, we have $(x-1, y, 5z) = (1, 0, 12)$. This implies that $5z = 12$, i.e., $z = \frac{12}{5}$. This is a contradiction since z is integer. Thus this equation has no non-negative integer solution (x, y, z). □

Example 3.13. $(x, y, z) = (2, 1, 6)$ *is a unique non-negative integer solution* (x, y, z) *for the Diophantine equation* $165(143)^x + 143(165)^y = 94380z^2$.

Proof. Note that the Diophantine equation $165(143)^x + 143(165)^y = 94380z^2$ can be written as $165(143)^x + 143(165)^y = (143)(165)(2z)^2$. If $x = 0$, then $143|165$. Similarly, if $y = 0$, then $165|143$ which are impossible. Suppose $x \geq 1$ and $y \geq 1$. So we have $143^{x-1} + 165^{y-1} = (2z)^2$. By Theorem 3.7, we have $(x-1, y-1, 2z) = (1, 0, 12)$. Therefore, $(x, y, z) = (2, 1, 6)$ is a unique non-negative integer solution of the Diophantine equation $165(143)^x + 143(165)^y = 94380z^2$. □

4. Open Problem

It is important to note that all solutions (x, y, z) of the Diophantine equation $143^x + n^y = z^2$, where n is any positive integer and $n \not\equiv 5 \pmod{20}$, are still an open problem. For example, it is not known how to find all non-negative integer solutions (x, y, z) of the Diophantine equation $143^x + n^y = z^2$ where $n = 2, 3, 6, 7, ...$ etc.

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References

[1] S. Chotchaisthit, On the Diophantine equation $p^x + (p+1)^y = z^2$ where p is a Mersenne prime, *Int. J. Pure Appl. Math.*, **88**, No. 2 (2013), 169-172, **doi:** http://dx.doi.org/10.12732/ijpam.v88i2.2

[2] N. Sarasit and S. Chotchaisthit, On the Diophantine equation $3^x+3^{2s}n^y = z^{2t}$ where $n \equiv 0 \pmod 5$, *Int. J. Pure Appl. Math.*, **97**, No. 2 (2014), 211-218,
doi: http://dx.doi.org/10.12732/ijpam.v97i2.9

[3] David M. Burton, *Elementary Number Theory*, 6th ed., McGraw-Hill, Singapore (2007).

[4] B. Sroysang, On the Diophantine equation $3^x + 5^y = z^2$, *Int. J. Pure Appl. Math.*, **81**, No. 4 (2012), 605-608.

[5] B. Sroysang, On the Diophantine equation $3^x + 17^y = z^2$, *Int. J. Pure Appl. Math.*, **89**, No. 1 (2013), 111-114,
doi: http://dx.doi.org/10.12732/ijpam.v89i1.13

[6] B. Sroysang, More on the Diophantine equation $3^x + 85^y = z^2$, *Int. J. Pure Appl. Math.*, **91**, No. 1 (2014), 131-134,
doi: http://dx.doi.org/10.12732/ijpam.v91i1.13

[7] B. Sroysang, On the Diophantine equation $143^x + 145^y = z^2$, *Int. J. Pure Appl. Math.*, **91**, No. 2 (2014), 265-268,
doi: http://dx.doi.org/10.12732/ijpam.v91i2.13

[8] B. Sroysang, On the Diophantine equation $3^x + 45^y = z^2$, *Int. J. Pure Appl. Math.*, **91**, No. 2 (2014), 269-272,
doi: http://dx.doi.org/10.12732/ijpam.v91i2.14