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DYNAMICS OF A SPECIAL COMBINATION OF WEIGHTED COMPOSITION OPERATORS ON HILBERT FUNCTION SPACES

Fariba Ershad[1], Bahmann Yousefi[2 §]
[1,2]Department of Mathematics
Payame Noor University
P.O. Box 19395-3697, Tehran, IRAN

Abstract: In this paper we give some sufficient conditions for the adjoint of a combination of weighted composition operators, acting on some function spaces, satisfying the hypercyclicity criterion.

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1. Introduction

The holomorphic self maps of the open unit disk $\mathbb{D}$ are divided into classes of elliptic and non-elliptic. The elliptic type is an automorphism and has a fixed point in $\mathbb{D}$. It is well known that this map is conjugate to a rotation $z \to \lambda z$ for some complex number λ with $|\lambda| = 1$. A non-elliptic holomorphic self-map of $\mathbb{D}$ with Denjoy-Wolff point in $\mathbb{D}$ is called a dilation type.

Let $\mathcal{H}$ be a Hilbert space of functions analytic on the open unit disc $\mathbb{D}$ such that for each λ in $\mathbb{D}$ the linear functional of evaluation at λ given by $f \longrightarrow f(\lambda)$ is a bounded linear functional on $\mathcal{H}$.

A complex-valued function ψ on $\mathbb{D}$ is called a multiplier of a Hilbert space $\mathcal{H}$ if $\psi\mathcal{H} \subset \mathcal{H}$. The operator of multiplication by ψ is denoted by M_ψ and is

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[§]Correspondence author

given by $f \longrightarrow \psi f$.

If w is a multiplier of $\mathcal{H}$ and φ is a mapping from $\mathbb{D}$ into $\mathbb{D}$ such that $f \circ \varphi \in \mathcal{H}$ for all $f \in \mathcal{H}$, then C_φ (defined on $\mathcal{H}$ by $C_\varphi f = f \circ \varphi$) and $M_w C_\varphi$ are called composition and weighted composition operator, respectively.

By an n-tuple of operators we mean a finite sequence of length n of commuting continuous linear operators on a locally convex space X. If $T = (T_1, T_2, ..., T_n)$ is an n-tuple of operators, then we will let

$$\mathcal{F} = \{{T_1}^{k_1}{T_2}^{k_2}, ..., {T_n}^{k_n} : k_i \geq 0\}$$

be the semigroup generated by T. For $x \in X$, the orbit of x under the tuple T is the set $orb(T, x) = \{Sx : S \in \mathcal{F}\}$. A vector x is called a hypercyclic vector for T if $orb(T, x)$ is dense in X and in this case the tuple T is called hypercyclic. The vector x is called supercyclic for T if $\mathbb{C}orb(T, x)$ is dense in X. Also a supercyclic tuple is one that has a supercyclic vector. When $n = 1$, then orbits of single operators have been studied widely. For some works on the topics of hypercyclicity and supercyclicity we refer to [1 – 4].

2. Main Results

In this section we give some sufficient conditions for the adjoint of a combination of weighted composition operators, acting on a Hilbert space of analytic functions, satisfying the hypercyclicity criterion. First, we state the hypercyclicity criterion in the weak form due to Gethner and Shapiro ([2]):

Theorem 2.1. (Hypercyclicity Criterion) *Let X be a separable Frechet space and T a continuous linear operator on X. If there exist dense subsets X_0, Y_0 of X, an increasing sequence $\{n_k\}_k$ of positive integers and mapping $S : Y_0 \longrightarrow X$ such that $T^{n_k}x \longrightarrow 0$ and $S^{n_k}y \longrightarrow 0$ for every $x \in X_0$, $y \in Y_0$, and $T \circ S = I_{Y_0}$, the identity on Y_0, then the operator T is hypercyclic.*

In the following, we give the hypercyclicity criterion for a 2-tuples that can be extended similarly to any k-tuple.

Theorem 2.2. (The Hypercyclicity Criterion for Tuples) *Suppose X is a separable Banach space and $T = (T_1, T_2)$ is a pair of continuous linear mappings on X. If there exist two dense subsets Y and Z in X and two strictly increasing sequences $\{n_j\}$ and $\{k_j\}$ such that:*

1. $T_1^{n_j} T_2^{k_j} y \to 0$ for every $y \in Y$, and

2. There exist functions $S_j : Z \to X$ such that for every $z \in Z, S_j z \to 0$, and $T_1^{n_j} T_2^{k_j} S_j z \to z$, then T is hypercyclic.

As earlier, we suppose that $\mathcal{H}$ is a separable Hilbert space of analytic functions on the open unit disc $\mathbb{D}$ such that $\mathcal{H}$ contains constants and the functional of evaluation at λ is bounded for all λ in $\mathbb{D}$, and $\mathcal{H}$ is automorphism invariant. The proof of the following theorem follows by the same method used in the proof of Theorem 3.3 in [4].

Theorem 2.3. *Let $k \geq 2$ and $\psi_1, \psi_2, ..., \psi_k$ be elliptic automorphisms with common interior fixed point p and let the multipliers $\varphi_1, \varphi_2, ..., \varphi_k : \mathbb{D} \to \mathbb{C}$ satisfy the inequality:*

$$\prod_{j=1}^{k} |\varphi_j(p)| < 1 < \lim_{|z| \to 1^-} \inf \prod_{j=1}^{k} |\varphi_j(z)|.$$

Then the adjoint of the operator $C_{\varphi_1,\psi_1} C_{\varphi_2,\psi_2} ... C_{\varphi_k,\psi_k}$ satisfies the hypercyclicity criterion.

Proof. Without loss of generality suppose that $p = 0$. Then

$$\prod_{j=1}^{k} |\varphi_j(0)| < 1 < \lim_{|z| \to 1^-} \inf \prod_{j=1}^{k} |\varphi_j(z)|.$$

Therefore, there exist a constant λ and a positive number $\delta < 1$ such that $\prod_{j=1}^{k} |\varphi_j(z)| < \lambda < 1$ when $|z| < \delta$, and $\prod_{j=1}^{k} |\varphi_j(z)| > 1$ when $|z| > 1 - \delta$. Set

$$W = \prod_{j=1}^{k} \varphi_j \circ \psi_{j-1} \circ ... \circ \psi_1 \circ \psi_0,$$

and

$$H = \psi_k \circ \psi_{k-1} \circ ... \circ \psi_1 \circ \psi_0.$$

Then $|W(z)| > 1$ for all $z \in \mathbb{D}$ with $|z| > 1 - \delta$. Also, $|W(z)| < \lambda < 1$ when $|z| < \delta$. Set

$$\mathcal{H}_1 = span\{k_z : |z| < \delta\}$$

and $\mathcal{H}_2 = span\{k_z : |z| > 1 - \delta\}$, where span $\{.\}$ is the set of finite linear combinations of $\{.\}$. So $\mathcal{H}_1$ and $\mathcal{H}_2$ are dense subsets of $\mathcal{H}$. Put $L = C^*_{\varphi_k,\psi_k} C^*_{\varphi_{k-1},\psi_{k-1}} ... C^*_{\varphi_1,\psi_1}$. Then for every $z \in \mathbb{D}$, we get $L(k_z) = \overline{W(z)} k_{H(z)}$.

Define the linear map : $\mathcal{H}_2 \to \mathcal{H}_2$ by $S(k_z) = \overline{(W \circ H^{-1}(z))^{-1}} k_{H^{-1}(z)}$ for every $z \in \mathbb{D}$. Now by the same method used in [4] we can see that: $L^n k_z \to 0$, $S^n k_y \to 0$, $LS(k_y) = k_y$ for all $k_z \in \mathcal{H}_1$ and $k_y \in \mathcal{H}_2$. Therefore L satisfies the hypothesis of hypercyclicity criterion and so the proof is complete. □

Corollary 2.4. *Under the conditions of Theorem 2.3, if*

$$\varphi_m.\varphi_n \circ \psi_m = \varphi_n.\varphi_m \circ \psi_n$$

for all $m, n \in \{1, 2, ..., k\}$, then the k-tuple $(C^*_{\varphi_1,\psi_1}, C^*_{\varphi_2,\psi_2}, ..., C^*_{\varphi_k,\psi_k})$ satisfies the hypercyclicity criterion.

Proof. First note that by the assumption

$$\varphi_m.\varphi_n \circ \psi_m = \varphi_n.\varphi_m \circ \psi_n; \quad m, n \in \{1, 2, ..., k\},$$

$C^*_{\varphi_i,\psi_i}$ commutes with $C^*_{\varphi_j,\psi_j}$ for all $i, j \in \{1, ..., k\}$, and indeed T is a k-tuple. Now clearly the proof is complete. □

Proposition 2.5. *Let $k \geq 2$ and $\varphi_m.\varphi_n \circ \psi_m = \varphi_n.\varphi_m \circ \psi_n$ for all $m, n \in \{1, 2, ..., k\}$. If the k-tuple $(C_{\varphi_1,\psi_1}, C_{\varphi_2,\psi_2}, ..., C_{\varphi_k,\psi_k})$ is hypercyclic, then $\varphi_j(z) \neq 0$ for all $z \in \mathbb{D}$ and all $j = 1, 2, ..., k$. Also, $|\varphi_j(p)| > 1$ for at least some $j \in \{1, 2, ..., k\}$ whenever $\psi_1, \psi_2, ..., \psi_k$ are elliptic or dilation types with common fixed point p.*

Proof. For all $n_i \in \mathbb{N}$ $(i = 1, .., k)$, set

$$L = C^{*n_1}_{\varphi_1,\psi_1} C^{*n_2}_{\varphi_2,\psi_2} ... C^{*n_k}_{\varphi_k,\psi_k}.$$

if there exists $j \in \{1, 2, ..., k\}$ such that $\varphi_j(z) = 0$ for some $z \in \mathbb{D}$, then $C^*_{\varphi_j,\psi_j} k_z = \overline{\varphi_j(z)} k_{\psi_j}(z) = 0$. This implies that $L_n(k_z) = 0$ that is a contradiction since it is well-known that every nonzero orbit of the adjoint of a hypercyclic tuple should be unbounded. Also, note that

$$L(k_p) = \left[\prod_{i=0}^{n-1} \overline{w_i(z)}\right] k_p = \left[\prod_{i=1}^{k} (\overline{\varphi_i(p)})^{n_i}\right] k_p.$$

Now if $|\varphi_i(p)| \leq 1$ for all $i \in \{1, 2, ..., k\}$, then we get a contradiction. This completes the proof. □

Proposition 2.6. *Let $\psi_i(p) = p$ for all $i \in \{1, 2, ..., k\}$. If $\prod_{i=1}^k \varphi_i(p) \neq 0$, then $\prod_{i=1}^k \overline{\varphi_i(p)}$ is an eigenvalue for the adjoint of the multiple C_{φ_1,ψ_1} $C_{\varphi_2,\psi_2} ... C_{\varphi_k,\psi_k}$. Hence, $C_{\varphi_1,\psi_1} C_{\varphi_2,\psi_2} ... C_{\varphi_k,\psi_k}$ is not hypercyclic.*

Proof. Put $M = C_{\varphi_1,\psi_1} C_{\varphi_2,\psi_2} ... C_{\varphi_k,\psi_k}$. Then

$$M^*(k_p) = \left[\prod_{i=1}^{k} \overline{\varphi_i(p)} \right] k_p,$$

and so $\prod_{i=1}^{k} \overline{\varphi_i(p)}$ is an eigenvalue of M^*, as desired. But it is well known that the adjoint of a hypercyclic operatorcan not have an eigenvector, hence the operator M fails to be hypercyclic and so the proof is complete. □

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