SOMEWHAT r -CONTINUOUS AND SOMEWHAT r -OPEN FUNCTIONS

N. Anuradha^{1§}, Baby Chacko²

¹Government Engineering College
West Hill, Kozhikode-5, Kerala, INDIA

²St. Joseph's College
Devagiri, Kozhikode, Kerala, INDIA

Abstract: In this paper new classes of functions namely somewhat r -continuous and somewhat r -open functions are introduced and studied by making use of regular open sets and regular closed sets. Relationship between the new classes and other classes of functions like somewhat continuous, completely continuous, almost completely continuous etc., are established besides giving examples, counter examples, properties and characterisations.

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1. Introduction

Gentre and Hoyle[4] studied the concept of somewhat continuous functions and somewhat open functions. In this paper a new type of somewhat continuous and open functions namely somewhat r -continuous and somewhat r -open func-

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[§]Correspondence author

tions is introduced. These types of functions were discussed in [2] and [3]. In section 2 basic definitions are introduced. In section 3 we define somewhat r -continuous functions and study its properties. Characterizations of somewhat r -continuous functions are given and its relation with some other types of functions is also studied. In section 4, r -equivalent topologies for somewhat r -continuity is discussed. Section 5 deals with definition of somewhat r -open function and its properties.

2. Preliminary Notes

In this article, by (X, τ) , we mean a topological space X with a topology τ on it. Similarly (Y, σ) , denotes a topological space Y with a topology σ on it. We abbreviate (X, τ) and (Y, σ) as X and Y respectively. For a set A , $\text{Cl}(A)$ denotes the closure of A and $\text{int}(A)$ denotes its interior. A subset A of X is said to be regular open if $A = \text{Int}(\text{Cl}(A))$, regular closed if $A = \text{Cl}(\text{Int}(A))$.

Definition 1. A function $f : X \rightarrow Y$ is said to be somewhat continuous [4] if for $U \in \sigma$ and $f^{-1}(U) \neq \phi$, there exists an open set V in X such that $V \neq \phi$ and $V \subset f^{-1}(U)$.

Definition 2. A function $f : X \rightarrow Y$ is said to be Cl-supercontinuous [8] ($\equiv$ Clopen continuous[6]) if for each $x \in X$ and each open set V containing $f(x)$ there exists clopen set U containing x such that $f(U) \subset V$.

Definition 3. A function $f : X \rightarrow Y$ is said to be δ -continuous [6] if for each $x \in X$ and for each regular open set V containing $f(x)$ there exists a regular open set U containing x such that $f(U) \subset V$.

Definition 4. A function $f : X \rightarrow Y$ is said to be completely continuous [1] if $f^{-1}(V)$ is a regular open set in X , for every open set $V \subset Y$.

Definition 5. A function $f : X \rightarrow Y$ is said to be almost completely continuous [5] if $f^{-1}(V)$ is a regular open set in X for every regular open set V in Y .

3. Somewhat r -Continuous Functions

Definition 6. Let X and Y be any two topological spaces. A function $f : X \rightarrow Y$ is said to be somewhat r -continuous if for $U \in \sigma$ and $f^{-1}(U) \neq \phi$, there exists a regular open set V in X such that $V \neq \phi$ and $V \subset f^{-1}(U)$.

Example 7. Let

$$X = Y = \{a, b, c\}, \quad \tau = \{X, \phi, \{a\}, \{b, c\}\},$$

$$\sigma = \{X, \phi, \{b, c\}\}.$$

Define $f : X \rightarrow Y$ by $f(a) = a, f(b) = c, f(c) = b$.

Then f is somewhat r -continuous.

Theorem 8. *Every Somewhat r -continuous function is somewhat continuous.*

Proof. The proof follows from the result, regular open sets are open. $\square$

Remark 9. The opposite proposition does not hold.

Example 10. Let

$$X = \{a, b, c, d\}, \quad Y = \{p, q, r\}, \quad \tau = \{X, \phi, \{a, c\}, \{d\}\{c\}, \{c, d\}, \{a, c, d\}\},$$

$$\sigma = \{Y, \phi, \{r\}, \{q\}, \{r, q\}\}.$$

Define $f : X \rightarrow Y$ by $f(a) = f(d) = p, f(c) = f(b) = r$.

Then f is somewhat continuous, but not somewhat r -continuous.

Definition 11. A topological space is locally indiscrete [9] if every open set is closed.

Theorem 12. *Let $f : X \rightarrow Y$ be somewhat continuous and X be locally indiscrete. Then f is somewhat r -continuous.*

Proof. The proof follows from the result, clopen sets are regular open. $\square$

Theorem 13. *Every Cl-supercontinuous function is somewhat r -continuous.*

Proof. The proof follows from the result, clopen sets are regular open. $\square$

Remark 14. The opposite proposition does not hold.

Example 15. Let $X = \{a, b, c, d\}, Y = \{a, b, c\}$,

$$\tau = \{X, \phi, \{a, b\}, \{a\}\{b\}, \{a, b, d\}\},$$

$$\sigma = \{Y, \phi, \{a\}, \{b\}\}.$$

Let $f : X \rightarrow Y$ be the identity function. Then f is somewhat r -continuous, but f is not Cl-super continuous.

Theorem 16. *Let $f : X \rightarrow Y$ be somewhat r -continuous where X is locally indiscrete. Then f is Cl -super continuous.*

Proof. The proof follows from the result that regular open set is open and open set in a locally indiscrete space is clopen. $\square$

Theorem 17. *Every completely continuous function is somewhat r -continuous.*

Proof. The proof is obvious. $\square$

Remark 18. The opposite proposition does not hold.

Example 19. Let $X = Y = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{b, c\}\}$,

$$\sigma = \{Y, \phi, \{a, b\}\}.$$

Let $f : X \rightarrow Y$ be the identity function. Then f is somewhat r -continuous, but f is not completely continuous.

Theorem 20. *If X is a discrete space and $f : X \rightarrow Y$ is somewhat r -continuous, f is completely continuous.*

Proof. Proof follows from the result that finite union of regular open sets in a discrete space is regular open. $\square$

Corollary 21. *If X is finite, T_1 and $f : X \rightarrow Y$ is somewhat r -continuous, f is completely continuous.*

Theorem 22. *Every somewhat r -continuous function is δ -continuous.*

Proof. Let $f : X \rightarrow Y$ be somewhat r -continuous. Let V be non empty regular open set in Y . Then it is open. Since f is somewhat r -continuous there exists a regular open set U such that $U \subset f^{-1}(V)$. So f is δ -continuous. $\square$

Remark 23. The opposite proposition does not hold.

Example 24. Let $X = Y = \{a, b, c\}$,

$$\tau = \{X, \phi, \{a, b\}, \{c\}\},$$

$$\sigma = \{Y, \phi, \{a\}, \{b\}, \{b, c\}\}.$$

Define $f : X \rightarrow Y$ by $f(a) = b, f(b) = c, f(c) = a$. Then f is δ -continuous, but not somewhat r -continuous.

Theorem 25. *If $f : X \rightarrow Y$ is δ -continuous and Y is locally indiscrete, f is somewhat r -continuous.*

Proof. Let V be open in Y . Since Y is locally indiscrete, V is clopen and so regular open. Since f is δ -continuous there exists regular open set U such that $U \subset f^{-1}(V)$. That is corresponding to open set V in Y , there exists regular open set U in X such that $U \subset f^{-1}(V)$. So f is somewhat r -continuous. $\square$

Theorem 26. *If $f : X \rightarrow Y$ is almost completely continuous and Y is locally indiscrete, f is somewhat r -continuous.*

Proof. Let V be open in Y . Since Y is locally indiscrete, V is clopen and so regular open. Since f is almost completely continuous $f^{-1}(V) = U$ is regular open. That is there exists a regular open set U such that $U \subset f^{-1}(V)$ for each open set V in Y . Therefore f is somewhat r -continuous. $\square$

Theorem 27. *If $f : X \rightarrow Y$ is somewhat r -continuous and X is a discrete space, then f is almost completely continuous.*

Proof. Proof follows from the result that finite union of regular open sets in a discrete space is regular open. $\square$

Corollary 28. *If $f : X \rightarrow Y$ is somewhat r -continuous and X is finite and T_1 , then f is almost completely continuous.*

Proof. Proof follows from the result that finite union of regular open sets in a finite T_1 space is regular open. $\square$

Remark 29. The following diagram shows the relationship between somewhat continuous, somewhat r -continuous, completely continuous, closure super continuous and δ -continuous functions.

Theorem 30. *Let $f : (X, \tau) \rightarrow (Y, \sigma)$, $g : (Y, \sigma) \rightarrow (Z, \eta)$ be any two functions. If f is somewhat r -continuous and g is continuous, $g \circ f$ is somewhat r -continuous.*

Proof. Let $V \subset Z$ be open. Since g is continuous $g^{-1}(V)$ is open in Y . Since f is somewhat r -continuous, there exists regular open set U such that $U \subset f^{-1}(g^{-1}(V))$. Hence $g \circ f$ is somewhat r -continuous. $\square$

Theorem 31. *If $f : X \rightarrow Y$ is continuous, $g : X \rightarrow Y$ is somewhat r -continuous then $g \circ f : X \rightarrow Z$ is continuous.*

Proof. Let $V \subset Z$ be open. Since g is somewhat r -continuous there exists regular open set U such that $U \subset g^{-1}(V)$. Since every regular open set is open, U is open. Since f is continuous, $f^{-1}(U)$ is open. Now $f^{-1}(U) \subset (g \circ f)^{-1}(V)$. So $g \circ f$ is continuous. $\square$

Definition 32. Let M be a subset of a topological space (X, τ) . Then M is said to be r -dense in X if there is no regular closed set C in X such that $M \subset C \subset X$.

Theorem 33. *Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be an injective function. Then the following are equivalent.*

1. f is somewhat r -continuous
2. If C is a closed subset of Y such that $f^{-1}(C) \neq X$ then there is a proper regular closed subset D of X such that $D \supset f^{-1}(C)$.
3. If M is an r -dense subset of X , then $f(M)$ is a dense subset of Y .

Proof. (1) $\Rightarrow$ (2) Let C be a closed subset of Y such that $f^{-1}(C) \neq X$. Then $Y - C$ is open in Y such that $f^{-1}(Y - C) = X - f^{-1}(C) \neq \emptyset$. Since f is somewhat r -continuous there exists regular open set V such that $V \subset X - f^{-1}(C)$. This implies $f^{-1}(C) \subset X - V$. Since V is regular open, $X - V = D$ is regular closed. That is there exists regular closed set D such that $D \supset f^{-1}(C)$.

(2) $\Rightarrow$ (3) Let M be an r -dense subset of X . Suppose $f(M)$ is not dense in Y . Then there exists a proper closed set C in Y such that $f(M) \subset C \subset Y$. Clearly $f^{-1}(C) \neq X$. Hence by (2) there exists proper regular closed set D such that $D \supset f^{-1}(C)$. That is $M \subset f^{-1}(C) \subset D \subset X$. This contradicts the fact that M is r -dense in X . So $f(M)$ is dense in Y .

(3) $\Rightarrow$ (2) Suppose (2) is not true. This means that there exists a closed set C in Y such that $f^{-1}(C) \neq X$, but there is no proper r -closed set D in X such that $f^{-1}(C) \subset D$. This means $f^{-1}(C)$ is r -dense in X . But by (3) $f(f^{-1}(C)) = C$ must be dense in Y , a contradiction to choice of C . So (2) is true.

(2) $\Rightarrow$ (1) Let $U \in \sigma$ and $f^{-1}(U) \neq \emptyset$. Then $Y - U$ is closed in Y and $f^{-1}(Y - U) = X - f^{-1}(U) \neq X$. So by (2) there exists a proper regular closed subset D of

X such that $D \supset f^{-1}(Y - U) = X - f^{-1}(U)$. That is $X - D \subset f^{-1}(U)$ and $X - D$ is a non empty regular open subset. So f is somewhat r -continuous. $\square$

Theorem 34. Let (X, τ) and (Y, σ) be any two topological spaces. Let A be a regular open set of X and $f : (A, \tau/A) \rightarrow (Y, \sigma)$ be somewhat r -continuous such that $f(A)$ is dense in Y . Then any extension F of f is somewhat r -continuous.

Proof. Let U be any open set in (Y, σ) such that $F^{-1}(U) \neq \phi$. Since $f(A) \subset Y$ is dense in Y , $U \cap f(A) \neq \phi$. So $F^{-1}(U) \cap A \neq \phi$. So $f^{-1}(U) \cap A \neq \phi$. Since f is somewhat r -continuous by hypothesis there exists a regular open set V such that $V \subset f^{-1}(U)$. Since V is a regular open subset of A and A is a regular open subset of X , V is a regular open subset of X . Hence $V \subset F^{-1}(U)$. That is corresponding to the open set U in Y such that $F^{-1}(U) \neq \phi$, there exists a regular open set V such that $V \subset F^{-1}(U)$. So F is somewhat r -continuous. $\square$

Theorem 35. Let (X, τ) and (Y, σ) be any two topological spaces. $X = A \cup B$ where A and B are regular open subsets of X . Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a function such that f/A and f/B are somewhat r -continuous. Then f is a somewhat r -continuous function.

Proof. Let U be any open set in (Y, σ) such that $f^{-1}(U) \neq \phi$. Then either $(f/A)^{-1}(U) \neq \phi$ or $(f/B)^{-1}(U) \neq \phi$ or both $(f/A)^{-1}(U)$ and $(f/B)^{-1}(U) \neq \phi$.

Case (1): $(f/A)^{-1}(U) \neq \phi$.

Since f/A is somewhat r -continuous there exists regular open set V in A such that $V \neq \phi$ and $V \subset (f/A)^{-1}(U) \subset f^{-1}(U)$. Since V is regular open in A and A is regular open in X , V is regular open in X . So f is somewhat r -continuous.

Case (2): $(f/B)^{-1}(U) \neq \phi$.

This can be proved by using the same argument as in case (1).

Case (3): $(f/A)^{-1}(U)$ and $(f/B)^{-1}(U) \neq \phi$.

The proof follows from the proofs of case(1) and (2). $\square$

Definition 36. A topological space X is said to be r -separable if there exists a countable subset B of X which is r -dense in X .

Example 37. Let $X = Y = \{a, b, c\}$, $\tau = \{X, \phi, \{b, c\}\}$. Then $\{b, c\}$ is r -dense in X . So X is r -separable.

Theorem 38. *If f is somewhat r -continuous function from X onto Y and if X is r -separable, Y is separable.*

Proof. Let $f: X \rightarrow Y$ be somewhat r -continuous function such that X is r -separable. Then there exists a countable set B of X which is r -dense in X . Then $f(B)$ is dense in Y . Since B is countable and f is onto, $f(B)$ is countable. So Y is separable. $\square$

4. r -Weakly Equivalent Topologies

Definition 39. If X is a set and τ and σ are topologies for X then τ is said to be weakly equivalent [4] to σ provided if $U \in \tau$ and $U \neq \phi$, then there is an open set V in (X, σ) such that $V \neq \phi$ and $V \subset U$ and if U in σ and $U \neq \phi$ then there is an open set V in (X, τ) such that $V \neq \phi$ and $V \subset U$.

Definition 40. If X is a set and τ and σ are topologies for X then τ is said to be r -weakly equivalent to σ provided if $U \in \tau$ and $U \neq \phi$, then there is a regular open set V in (X, σ) such that $V \neq \phi$ and $V \subset U$ and if U in σ and $U \neq \phi$ then there is a regular open set V in (X, τ) such that $V \neq \phi$ and $V \subset U$.

Example 41. Let $X = \{a, b, c\}$,
 $\tau = \{X, \phi, \{a\}, \{b, c\}, \{a, b\}, \{b\}, \{a, c\}\}$
 $\sigma = \{X, \phi, \{a\}, \{a, b\}, \{b\}\}$. Then τ is r -weakly equivalent to σ .

Theorem 42. *Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be a somewhat continuous function and let τ be a topology for X which is r -weakly equivalent to τ . Then the function $f: (X, \tau) \rightarrow (Y, \sigma)$ is somewhat r -continuous.*

Proof. Let U be any open set in (Y, σ) such that $f^{-1}(U) \neq \phi$. Since f is somewhat continuous there exists open set V in X such that $V \neq \phi$ and $V \subset f^{-1}(U)$. Since τ is r -equivalent to τ , there exists a regular open set V_1 in (X, τ) such that $V_1 \neq \phi$ and $V_1 \subset V$. That is $V_1 \subset V \subset f^{-1}(U)$. So f is somewhat r -continuous. $\square$

Theorem 43. *Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be a somewhat r -continuous function. Let σ be a topology for Y which is weakly equivalent to σ . Then $f: (X, \tau) \rightarrow (Y, \sigma)$ is somewhat r -continuous.*

Proof. Let U be open set in (Y, σ^*) such that $f^{-1}(U) \neq \phi$. Then $U \neq \phi$. Since σ and σ^* are weakly equivalent there exists an open set W in (Y, σ) such that $W \neq \phi$ and $W \subset U$. Then $f^{-1}(W) \neq \phi$. Since $f: (X, \tau) \rightarrow (Y, \sigma)$

is somewhat r -continuous there exists regular open set $V \neq \phi$ such that $V \subset f^{-1}(W)$. Now $W \subset U$ implies $f^{-1}(W) \subset f^{-1}(U)$. So $V \subset f^{-1}(W) \subset f^{-1}(U)$. That is $V \subset f^{-1}(U)$. So $f : (X, \tau) \rightarrow (Y, \sigma)$ is somewhat r -continuous. $\square$

5. Somewhat r -Open Functions

Definition 44. A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be somewhat open [4] provided for $U \in \tau$ and $U \neq \phi$ there exists an open set V in Y such that $V \neq \phi$ and $V \subset f(U)$.

Definition 45. A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be somewhat r -open provided for $U \in \tau$ and $U \neq \phi$ there exists a regular open set V in Y such that $V \neq \phi$ and $V \subset f(U)$.

Example 46. Let $X=Y=\{a,b,c\}$, $\tau=\{X,\phi,\{a\}\}$,
 $\sigma=\{Y,\phi,\{b\},\{c\},\{b,c\}\}$.

Define a function $f : (X, \tau) \rightarrow (Y, \sigma)$ by $f(a) = c, f(b) = a, f(c) = b$. Then f is somewhat r -open.

Definition 47. A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be somewhat clopen provided for $U \in \tau$ there exists clopen set V in Y such that $V \subset f(U)$.

Example 48. Let $X=Y=\{a,b,c\}$ $\tau=\{X,\phi,\{a\}\}$,
 $\sigma=\{Y,\phi,\{b\},\{a,c\}\}$. Define a function $f : (X, \tau) \rightarrow (Y, \sigma)$ by $f(a) = b, f(b) = c, f(c) = a$. Then f is somewhat clopen.

Theorem 49. Every somewhat clopen function is somewhat r -open.

Proof. The proof follows from the result that clopen sets are regular open. $\square$

Remark 50. The opposite proposition does not hold.

Example 51. Let $X=\{a,b,c,d\}, Y=\{p,q,r\}$,
 $\tau=\{X,\phi,\{a,c\},\{d\},\{c\},\{c,d\},\{a,c,d\}\}$, $\sigma=\{Y,\phi,\{r\},\{q\},\{r,q\}\}$.
 Define $f : X \rightarrow Y$ by $f(a) = f(d) = p, f(c) = f(b) = r$.

Then f is somewhat continuous, but not somewhat r -continuous.

Definition 52. A topological space is locally indiscrete [9] if every open set is closed.

Theorem 53. Let $f : X \rightarrow Y$ be somewhat continuous and X be locally indiscrete. Then f is somewhat r -continuous.

Proof. The proof follows from the result, clopen sets are regular open. $\square$

Theorem 54. *Every Cl-supercontinuous function is somewhat r -continuous.*

Proof. The proof follows from the result, clopen sets are regular open. $\square$

Remark 55. The opposite proposition does not hold.

Example 56. Let $X=\{a,b,c,d\}, Y=\{a,b,c\}$,
 $\tau=\{X,\phi,\{a,b\},\{a\}\{b\},\{a,b,d\}\}$, $\sigma=\{Y,\phi,\{a\},\{b\}\}$. Let $f : X \rightarrow Y$ be the identity function. Then f is somewhat r -continuous, but f is not Cl-super continuous.

Theorem 57. *Let $f : X \rightarrow Y$ be somewhat r -continuous where X is locally indiscrete. Then f is Cl-super continuous.*

Proof. The proof follows from the result that regular open set is open and open set in a locally indiscrete space is clopen. $\square$

Theorem 58. *Every completely continuous function is somewhat r -continuous.*

Proof. The proof is obvious. $\square$

Remark 59. The opposite proposition does not hold.

Example 60. Let $X=Y=\{a,b,c\}$, $\tau=\{X,\phi,\{a\},\{b,c\}\}$,
 $\sigma=\{Y,\phi,\{a,b\}\}$. Let $f : X \rightarrow Y$ be the identity function. Then f is somewhat r -continuous, but f is not completely continuous.

Theorem 61. *If X is a discrete space and $f : X \rightarrow Y$ is somewhat r -continuous, f is completely continuous.*

Proof. Proof follows from the result that finite union of regular open sets in a discrete space is regular open. $\square$

Corollary 62. *If X is finite, T_1 and $f : X \rightarrow Y$ is somewhat r -continuous, f is completely continuous.*

Theorem 63. *Every somewhat r -continuous function is δ -continuous.*

Proof. Let $f : X \rightarrow Y$ be somewhat r -continuous. Let V be non empty regular open set in Y . Then it is open. Since f is somewhat r -continuous there exists a regular open set U such that $U \subset f^{-1}(V)$. So f is δ -continuous. $\square$

Remark 64. The opposite proposition does not hold.

Example 65. Let $X=Y=\{a,b,c\}$, $\tau=\{X,\phi,\{a,b\},\{c\}\}$, $\sigma=\{Y,\phi,\{a\},\{b\},\{b,c\}\}$. Define $f : X \rightarrow Y$ by $f(a) = b, f(b) = c, f(c) = a$. Then f is δ -continuous, but not somewhat r -continuous.

Theorem 66. If $f : X \rightarrow Y$ is δ -continuous and Y is locally indiscrete, f is somewhat r -continuous.

Proof. Let V be open in Y . Since Y is locally indiscrete, V is clopen and so regular open. Since f is δ -continuous there exists regular open set U such that $U \subset f^{-1}(V)$. That is corresponding to open set V in Y , there exists regular open set U in X such that $U \subset f^{-1}(V)$. So f is somewhat r -continuous. $\square$

Theorem 67. If $f : X \rightarrow Y$ is almost completely continuous and Y is locally indiscrete, f is somewhat r -continuous.

Proof. Let V be open in Y . Since Y is locally indiscrete, V is clopen and so regular open. Since f is almost completely continuous $f^{-1}(V) = U$ is regular open. That is there exists a regular open set U such that $U \subset f^{-1}(V)$ for each open set V in Y . Therefore f is somewhat r -continuous. $\square$

Theorem 68. If $f : X \rightarrow Y$ is somewhat r -continuous and X is a discrete space, then f is almost completely continuous.

Proof. Proof follows from the result that finite union of regular open sets in a discrete space is regular open. $\square$

Corollary 69. If $f : X \rightarrow Y$ is somewhat r -continuous and X is finite and T_1 , then f is almost completely continuous.

Proof. Proof follows from the result that finite union of regular open sets in a finite T_1 space is regular open. $\square$

Remark 70. The following diagram shows the relationship between somewhat continuous, somewhat r -continuous, completely continuous, closure super continuous and δ -continuous functions.

Theorem 71. *Let $f : (X, \tau) \rightarrow (Y, \sigma)$, $g : (Y, \sigma) \rightarrow (Z, \eta)$ be any two functions. If f is somewhat r -continuous and g is continuous, $g \circ f$ is somewhat r -continuous.*

Proof. Let $V \subset Z$ be open. Since g is continuous $g^{-1}(V)$ is open in Y . Since f is somewhat r -continuous, there exists regular open set U such that $U \subset f^{-1}(g^{-1}(V))$. Hence $g \circ f$ is somewhat r -continuous. $\square$

Theorem 72. *If $f : X \rightarrow Y$ is continuous, $g : Y \rightarrow Z$ is somewhat r -continuous then $g \circ f : X \rightarrow Z$ is continuous.*

Proof. Let $V \subset Z$ be open. Since g is somewhat r -continuous there exists regular open set U such that $U \subset g^{-1}(V)$. Since every regular open set is open, U is open. Since f is continuous, $f^{-1}(U)$ is open. Now $f^{-1}(U) \subset (g \circ f)^{-1}(V)$. So $g \circ f$ is continuous. $\square$

Definition 73. Let M be a subset of a topological space (X, τ) . Then M is said to be r -dense in X if there is no regular closed set C in X such that $M \subset C \subset X$.

Theorem 74. *Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be an injective function. Then the following are equivalent.*

1. f is somewhat r -continuous
2. If C is a closed subset of Y such that $f^{-1}(C) \neq X$ then there is a proper regular closed subset D of X such that $D \supset f^{-1}(C)$.
3. If M is an r -dense subset of X , then $f(M)$ is a dense subset of Y .

Proof. (1) $\Rightarrow$ (2) Let C be a closed subset of Y such that $f^{-1}(C) \neq X$. Then $Y - C$ is open in Y such that $f^{-1}(Y - C) = X - f^{-1}(C) \neq \phi$. Since f is somewhat r -continuous there exists regular open set V such that $V \subset X - f^{-1}(C)$. This implies $f^{-1}(C) \subset X - V$. Since V is regular open, $X - V = D$ is regular closed. That is there exists regular closed set D such that $D \supset f^{-1}(C)$.

(2) $\Rightarrow$ (3) Let M be an r -dense subset of X . Suppose $f(M)$ is not dense in Y . Then there exists a proper closed set C in Y such that $f(M) \subset C \subset Y$. Clearly $f^{-1}(C) \neq X$. Hence by (2) there exists proper regular closed set D such that $D \supset f^{-1}(C)$. That is $M \subset f^{-1}(C) \subset D \subset X$. This contradicts the fact that M is r -dense in X . So $f(M)$ is dense in Y .

(3) $\Rightarrow$ (2) Suppose (2) is not true. This means that there exists a closed set C in Y such that $f^{-1}(C) \neq X$, but there is no proper r -closed set D in X such

that $f^{-1}(C) \subset D$. This means $f^{-1}(C)$ is r -dense in X . But by (3) $f(f^{-1}(C)) = C$ must be dense in Y , a contradiction to choice of C . So (2) is true.

(2) $\Rightarrow$ (1) Let $U \in \sigma$ and $f^{-1}(U) \neq \phi$. Then $Y - U$ is closed in Y and $f^{-1}(Y - U) = X - f^{-1}(U) \neq X$. So by (2) there exists a proper regular closed subset D of X such that $D \supset f^{-1}(Y - U) = X - f^{-1}(U)$. That is $X - D \subset f^{-1}(U)$ and $X - D$ is a non empty regular open subset. So f is somewhat r -continuous. $\square$

Theorem 75. *Let (X, τ) and (Y, σ) be any two topological spaces. Let A be a regular open set of X and $f : (A, \tau/A) \rightarrow (Y, \sigma)$ be somewhat r -continuous such that $f(A)$ is dense in Y . Then any extension F of f is somewhat r -continuous.*

Proof. Let U be any open set in (Y, σ) such that $F^{-1}(U) \neq \phi$. Since $f(A) \subset Y$ is dense in Y , $U \cap f(A) \neq \phi$. So $F^{-1}(U) \cap A \neq \phi$. So $f^{-1}(U) \cap A \neq \phi$. Since f is somewhat r -continuous by hypothesis there exists a regular open set V such that $V \subset f^{-1}(U)$. Since V is a regular open subset of A and A is a regular open subset of X , V is a regular open subset of X . Hence $V \subset F^{-1}(U)$. That is corresponding to the open set U in Y such that $F^{-1}(U) \neq \phi$, there exists a regular open set V such that $V \subset F^{-1}(U)$. So F is somewhat r -continuous. $\square$

Theorem 76. *Let (X, τ) and (Y, σ) be any two topological spaces. $X = A \cup B$ where A and B are regular open subsets of X . Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a function such that f/A and f/B are somewhat r -continuous. Then f is a somewhat r -continuous function.*

Proof. Let U be any open set in (Y, σ) such that $f^{-1}(U) \neq \phi$. Then either $(f/A)^{-1}(U) \neq \phi$ or $(f/B)^{-1}(U) \neq \phi$ or both $(f/A)^{-1}(U)$ and $(f/B)^{-1}(U) \neq \phi$.

Case (1): $(f/A)^{-1}(U) \neq \phi$.

Since f/A is somewhat r -continuous there exists regular open set V in A such that $V \neq \phi$ and $V \subset (f/A)^{-1}(U) \subset f^{-1}(U)$. Since V is regular open in A and A is regular open in X , V is regular open in X . So f is somewhat r -continuous.

Case (2): $(f/B)^{-1}(U) \neq \phi$.

This can be proved by using the same argument as in case (1).

Case (3): $(f/A)^{-1}(U)$ and $(f/B)^{-1}(U) \neq \phi$.

The proof follows from the proofs of case(1) and (2). $\square$

Definition 77. A topological space X is said to be r -separable if there exists a countable subset B of X which is r -dense in X .

Example 78. Let $X=Y=\{a,b,c\}$, $\tau=\{X,\phi,\{b,c\}\}$. Then $\{b,c\}$ is r -dense in X . So X is r -separable.

Theorem 79. If f is somewhat r -continuous function from X onto Y and if X is r -separable, Y is separable.

Proof. Let $f: X \rightarrow Y$ be somewhat r -continuous function such that X is r -separable. Then there exists a countable set B of X which is r -dense in X . Then $f(B)$ is dense in Y . Since B is countable and f is onto, $f(B)$ is countable. So Y is separable. $\square$

6. r -Weakly Equivalent Topologies

Definition 80. If X is a set and τ and σ are topologies for X then τ is said to be weakly equivalent [4] to σ provided if $U \in \tau$ and $U \neq \phi$, then there is an open set V in (X, σ) such that $V \neq \phi$ and $V \subset U$ and if U in σ and $U \neq \phi$ then there is an open set V in (X, τ) such that $V \neq \phi$ and $V \subset U$.

Definition 81. If X is a set and τ and σ are topologies for X then τ is said to be r -weakly equivalent to σ provided if $U \in \tau$ and $U \neq \phi$, then there is a regular open set V in (X, σ) such that $V \neq \phi$ and $V \subset U$ and if U in σ and $U \neq \phi$ then there is a regular open set V in (X, τ) such that $V \neq \phi$ and $V \subset U$.

Example 82. Let $X=\{a,b,c\}$,
 $\tau=\{X,\phi,\{a\},\{b,c\},\{a,b\},\{b\},\{a,c\}\}$
 $\sigma=\{X,\phi,\{a\},\{a,b\},\{b\}\}$. Then τ is r -weakly equivalent to σ .

Theorem 83. Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be a somewhat continuous function and let τ be a topology for X which is r -weakly equivalent to τ . Then the function $f: (X, \tau) \rightarrow (Y, \sigma)$ is somewhat r -continuous.

Proof. Let U be any open set in (Y, σ) such that $f^{-1}(U) \neq \phi$. Since f is somewhat continuous there exists open set V in X such that $V \neq \phi$ and $V \subset f^{-1}(U)$. Since τ is r -equivalent to τ , there exists a regular open set V_1 in (X, τ) such that $V_1 \neq \phi$ and $V_1 \subset V$. That is $V_1 \subset V \subset f^{-1}(U)$. So f is somewhat r -continuous. $\square$

Theorem 84. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a somewhat r -continuous function. Let σ be a topology for Y which is weakly equivalent to σ . Then $f : (X, \tau) \rightarrow (Y, \sigma)$ is somewhat r -continuous.

Proof. Let U be open set in (Y, σ^*) such that $f^{-1}(U) \neq \emptyset$. Then $U \neq \emptyset$. Since σ and σ^* are weakly equivalent there exists an open set W in (Y, σ) such that $W \neq \emptyset$ and $W \subset U$. Then $f^{-1}(W) \neq \emptyset$. Since $f : (X, \tau) \rightarrow (Y, \sigma)$ is somewhat r -continuous there exists regular open set $V \neq \emptyset$ such that $V \subset f^{-1}(W)$. Now $W \subset U$ implies $f^{-1}(W) \subset f^{-1}(U)$. So $V \subset f^{-1}(W) \subset f^{-1}(U)$. That is $V \subset f^{-1}(U)$. So $f : (X, \tau) \rightarrow (Y, \sigma)$ is somewhat r -continuous. $\square$

7. Somewhat r -Open Functions

Definition 85. A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be somewhat open [4] provided for $U \in \tau$ and $U \neq \emptyset$ there exists an open set V in Y such that $V \neq \emptyset$ and $V \subset f(U)$.

Definition 86. A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be somewhat r -open provided for $U \in \tau$ and $U \neq \emptyset$ there exists a regular open set V in Y such that $V \neq \emptyset$ and $V \subset f(U)$.

Example 87. Let $X=Y=\{a,b,c\}$, $\tau=\{X,\emptyset,\{a\}\}$,
 $\sigma=\{Y,\emptyset,\{b\},\{c\},\{b,c\}\}$.

Define a function $f : (X, \tau) \rightarrow (Y, \sigma)$ by $f(a) = c, f(b) = a, f(c) = b$. Then f is somewhat r -open.

Definition 88. A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be somewhat clopen provided for $U \in \tau$ there exists clopen set V in Y such that $V \subset f(U)$.

Example 89. Let $X=Y=\{a,b,c\}$ $\tau=\{X,\emptyset,\{a\}\}$,
 $\sigma=\{Y,\emptyset,\{b\},\{a,c\}\}$. Define a function $f : (X, \tau) \rightarrow (Y, \sigma)$ by $f(a) = b, f(b) = c, f(c) = a$. Then f is somewhat clopen.

Theorem 90. Every somewhat clopen function is somewhat r -open.

Proof. The proof follows from the result that clopen sets are regular open. $\square$

Remark 91. The opposite proposition does not hold.

Example 92. Let $X=\{a,b,c\}, Y=\{a,b,c\}$
 $\tau=\{X,\emptyset,\{a\}\}, \sigma=\{Y,\emptyset,\{b\},\{c\},\{b,c\}\}$.

Define a function $f : (X, \tau) \rightarrow (Y, \sigma)$ by $f(a) = c, f(b) = a, f(c) = b$. Then f is somewhat r -open, but not somewhat clopen.

Theorem 93. *If $f : X \rightarrow Y$ is somewhat r -open, where X is locally indiscrete, f is somewhat clopen.*

Proof. Let U be open in X . Since f is somewhat r -open there exists a regular open set V in X such that $V \subset f(U)$. But regular open sets in a locally indiscrete space are clopen. So f is somewhat clopen. $\square$

Theorem 94. *If $f : X \rightarrow Y$ is somewhat r -open, f is somewhat open.*

Proof. The proof follows from the result regular open sets are open. $\square$

Remark 95. The opposite proposition does not hold.

Example 96. Let $X=Y=\{a,b,c\}$
 $\tau=\{X, \phi, \{a\}, \{a,b\}\}, \sigma=\{Y, \phi, \{c\}, \{b,c\}\}.$

Define a function $f : (X, \tau) \rightarrow (Y, \sigma)$ by $f(a) = c, f(b) = a, f(c) = b$. Then f is somewhat open, but not somewhat r -open.

Theorem 97. *If $f : X \rightarrow Y$ is somewhat open and X is locally indiscrete, then f is somewhat r -open.*

Theorem 98. *If $f : (X, \tau) \rightarrow (Y, \sigma)$ is an open map and $g : (Y, \sigma) \rightarrow (Z, \eta)$ is a somewhat r -open map, $g \circ f : (X, \tau) \rightarrow (Z, \eta)$ is a somewhat r -open map.*

Proof. Let $U \in \tau$ and $U \neq \phi$. Since f is an open map $f(U)$ is open. Also $f(U) \neq \phi$. Since g is somewhat regular open and $f(U) \in \sigma$ with $f(U) \neq \phi$, there exists regular open set V in η such that $V \subset g(f(U))$. So $g \circ f$ is somewhat r -open. $\square$

Theorem 99. *If $f : (X, \tau) \rightarrow (Y, \sigma)$ is a one-one and onto mapping then the following are equivalent.*

1. f is a somewhat r -open map
2. If C is a closed subset of X such that $f(C) \neq Y$, then there is a regular closed subset D of Y such that $D \neq Y$ and $D \supset f(C)$

Proof. (1) $\Rightarrow$ (2) Let C be a closed subset of X such that $f(C) \neq Y$. Then $X-C$ is open in X and $X-C \neq \phi$. Since f is somewhat r -open there exists regular open set $V \neq \phi$ such that $V \subset f(X-C)$. Put $D=Y-V$. Clearly D is regular closed in Y . We claim that $D \neq Y$; for if $D=Y, V=\phi$, a contradiction. So $V \subset f(X-C)$ implies $D=Y-V \supset Y-[f(X-C)]=f(C)$.

(2) $\Rightarrow$ (1) Let U be a non empty open set in X . Put $C=X-U$. Then C is a closed subset of X and $f(C) = f(X - U) = Y - f(U)$. This implies $f(C) \neq \phi$. So by (2) there exists a regular closed subset D of Y such that $D \neq Y$ and $D \supset f(C)$. Let $V=Y-D$. Then V is regular open and non empty. Also $V=Y-D \subset Y-f(C)=Y-(Y-f(U))=f(U)$. So f is somewhat r -open. $\square$

Theorem 100. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a somewhat r -open function. Let A be any open subset of X . Then $f/A : (A, \tau/A) \rightarrow (Y, \sigma)$ is also somewhat r -open.

Proof. Let $U \in \tau/A$ and $U \neq \phi$. Since U is open in A and A is open in X , U is open in X . Since $f : (X, \tau) \rightarrow (Y, \sigma)$ is somewhat r -open, there exists regular open set V in Y such that $V \subset f(U)$. Thus for any open set U in τ/A with $U \neq \phi$, there exists regular open set V in Y such that $V \subset (f/A(U))$. So f/A is somewhat r -open. $\square$

Theorem 101. Let (X, τ) and (Y, σ) be any two topological spaces and $X=A \cup B$ where A and B are open subsets of X . Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a function such that f/A and f/B are somewhat r -open. Then f is also somewhat r -open.

Proof. The proof follows from the argument used in proof of Theorem 35. $\square$

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