**SPANNED VECTOR BUNDLES ON $X_1 \times \mathbb{P}^2$,
 X_1 A SURFACE, WITH $c_1 = R \boxtimes \mathcal{O}_{\mathbb{P}^2}(\infty)$**

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Abstract: Let X_1 be a smooth surface. We study rank r spanned vector bundles $\mathcal{E}$ on $X_1 \times \mathbb{P}^2$ whose determinant is of the form $L \boxtimes \mathcal{O}_{\mathbb{P}^2}(\infty)$ with L a spanned line bundle on X_1 . We divide them according the behaviour of the dependency locus of $r - 1$ general sections of $\mathcal{E}$ with respect to the projection $X_1 \times \mathbb{P}^2 \rightarrow X_1$.

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1. Introduction

Let X_1, X_2 be smooth and connected projective surfaces (usually we will take $X_2 = \mathbb{P}^2$). Set $X := X_1 \times X_2$ and call $\pi_i : X \rightarrow X_i$, $i = 1, 2$, the projections. For all line bundles $\mathcal{L}_i$ on X_i , $i = 1, 2$, set $\mathcal{L}_\infty \boxtimes \mathcal{L}_\infty := \pi_\infty^*(\mathcal{L}_\infty) \otimes \pi_\infty^*(\mathcal{L}_\infty)$. Assume that $\mathcal{L}_\infty$ is spanned. We study spanned bundles $\mathcal{E}$ on $X = X_1 \times \mathbb{P}^2$ with $\det(\mathcal{E}) \cong \mathcal{L}_\infty \boxtimes \mathcal{O}_{\mathbb{P}^2}(\infty)$. Let $r \geq 2$ be the rank of $\mathcal{E}$. We study the possible $\mathcal{E}$ in terms of the dependency locus S of $r - 1$ general sections of $\mathcal{E}$. S is normal and hence the connected component $S_1, \dots, S_c$ of S are irreducible (see §2). Fix $j \in \{1, \dots, c\}$.

For an arbitrary X_2 and $\mathcal{L}_\infty$ we introduce the following definition. We say that the component S_j has *fiber type* with respect to π_1 if $\pi_1(S_j)$ is a curve. We say that S_j is *constant type* with respect to π_1 if $\pi_1(S_j)$ is a point, i.e. $S_j = \pi_1^{-1}(O)$ for some $O \in X_1$. If $\pi_1|_{S_j} : S_j \rightarrow X_1$ is surjective, then we say that S_j is of *dominant type*.

Now assume $X_2 = \mathbb{P}^2$ and $\mathcal{L}_\infty = \mathcal{O}_{\mathbb{P}^2}(\infty)$. In this case if S_j is dominant with respect to π_1 , then $c = 1$, $\pi_1|_{S_1}$ is birational onto its image, each fiber of π_1 is either a point or a line (scheme-theoretically) and at most finitely many fibers of $\pi_1|_{S_1}$ are not a single point (see §2). We say that S_j is of *graph type* with respect to π_1 if $\pi_1|_{S_j}$ is an isomorphism (equivalently if S_1 is the graph of a morphism $f : X_1 \rightarrow \mathbb{P}^2$) and that it is of *blowing up type* with respect to π_1 if it is of birational type, but not of graph type, with respect to π_1 .

Theorem 1. Fix a smooth projective surface X_1 with $h^1(\mathcal{O}_{X_\infty}) = \iota$, a non-constant morphism $f : X_1 \rightarrow \mathbb{P}^2$ and a spanned line bundle $\mathcal{L}_\infty$ on X_1 . Set $X := X_1 \times \mathbb{P}^2$ and $\mathcal{L} := \mathcal{L}_\infty \boxtimes \mathcal{O}_{\mathbb{P}^2}(\infty)$. Set $M := f^*(\mathcal{O}_{\mathbb{P}^2}(\infty)) \otimes \mathcal{L}_\infty^\vee$ and $\alpha := h^0(X_1, M)$.

(a) There is a rank 2 spanned bundle $\mathcal{E}$ on X with $c_1(\mathcal{E}) = \mathcal{L}$ and the graph S of f as one of its dependency loci (or as a component of one of its dependency loci) if and only if $\mathcal{L}_\infty \cong \mathcal{O}_{\mathbb{P}^2}(\infty)$. Moreover, $\mathcal{E}$ is unique.

(b) Fix an integer $r \geq 4$. There is a rank r spanned bundle $\mathcal{E}$ on X with $c_1(\mathcal{E}) = \mathcal{L}$, no trivial factor, and the graph S of f as one of its dependency loci (or as a component of one of its dependency loci) if and only if M is spanned and $r \leq \alpha + 1$. Moreover any $\mathcal{E}$ as above is uniquely determined by a linear subspace $V \subseteq H^0(M)$ with $\dim(V) = r - 1$ and V spanning M .

(c) There is a rank 3 spanned bundle $\mathcal{E}$ on X with $c_1(\mathcal{E}) = \mathcal{L}$, no trivial factor, and the graph S of f as one of its dependency loci (or as a component of one of its dependency loci) if and only if M is spanned, not trivial, and there is a linear subspace $V \subseteq H^0(M)$ with $\dim(V) = 2$ and V spanning M .

(d) There is a rank $\alpha + 1$ spanned bundle $\mathcal{E}$ on X with $c_1(\mathcal{E}) = \mathcal{L}$, no trivial factor, and the graph S of f as one of its dependency loci (or as a component of one of its dependency loci) if and only if M is spanned. In this case $\mathcal{E}$ is unique.

Case (c) of Theorem 1 never occurs if there is no non-constant morphism $X_1 \rightarrow \mathbb{P}^1$ (e.g. if $\text{Pic}(X) \otimes \mathbb{Q} \cong \mathbb{Q}$ or if X is birational to a subvariety of an abelian variety). If $h^1(\mathcal{O}_{X_\infty}) > \iota$, then the conditions listed in Theorem 1 are necessary conditions for the existence of a spanned $\mathcal{E}$. We work over an algebraically closed field $\mathbb{K}$ with characteristic zero.

2. The Proof

Since $h^0(\mathcal{O}_{X_\infty}) = \infty$, [7, Base change theorem, page 11] and the projection formula gives the following lemma.

Lemma 1. *Let $\mathcal{E}$ be a spanned bundle on X with $\det(\mathcal{E}) \cong \pi_{\infty}^*(\mathcal{M})$ for some $M \in \text{Pic}(X_2)$. Then there is a unique vector bundle $\mathcal{F}$ on X_2 such that $\mathcal{E} \cong \pi_{\infty}^*(\mathcal{F})$. Moreover, $\mathcal{F} \cong \pi_{\infty*}(\mathcal{E})$, $\det(\mathcal{F}) \cong \mathcal{M}$ and $h^0(\mathcal{E}) = \langle'(\mathcal{F})$. If $h^1(\mathcal{O}_{X_\infty}) = \iota$, then $h^1(\mathcal{E}) = \langle^\infty(\mathcal{F})$.*

Let $\mathcal{E}$ be a rank $r \geq 2$ spanned vector bundle on a smooth 4-fold X . Set $\mathcal{L} := \det(\mathcal{E})$. Fix $r - 1$ general sections of $\mathcal{E}$ and let $S \subseteq X$ be their dependency locus. $S = \emptyset$ if and only if $\mathcal{E}$ is an extension of $\mathcal{L}$ by $\mathcal{O}_X^{\oplus(\nabla - \infty)}$. Now assume $S \neq \emptyset$. By [4] or [2] S has pure dimension 2 and it has isolated singularity. Since S has pure codimension 2 and X is smooth, it is locally Cohen-Macaulay ([3, page 13]). Hence S is normal. In particular the connected components $S_1, \dots, S_c$ of S are irreducible. Now assume $X = X_1 \times X_2$ and take $\mathcal{L} = \mathcal{L}_\infty \boxtimes \mathcal{L}_\infty$.

Lemma 2. *Take arbitrary X_2 . Let $\mathcal{E}$ be a rank $r \geq 2$ spanned vector bundle on X and $S = S_1 \sqcup \dots \sqcup S_c$ be its general dependency locus. If $\mathcal{L}_\infty \neq \mathcal{O}_{X_\infty}$, then no S_j has constant type*

Proof. Let S_j be a component of S of constant type with respect to π_1 , i.e. $S_j = \{O\} \times X_2$ for some $O \in X_1$. Then S_j is smooth and with trivial normal bundle. We have $\mathcal{L}|_{S_j} \cong \mathcal{L}_\infty$ (up to the identification of $\{O\} \times X_2$ with X_2). Since S_j is a connected component of a dependency locus one arrow of the Serre correspondence gives the existence of $r - 1$ sections of $\mathcal{L}_\infty^\vee$ with no common zero, contradicting the assumption $\mathcal{L}_\infty \neq \mathcal{O}_{X_\infty}$ and the spannedness of $\mathcal{L}_\infty$. $\square$

Proposition 1. *Assume $X_2 = \mathbb{P}^2$ and $\mathcal{L}_\infty = \mathcal{O}_{\mathbb{P}^2}(\infty)$. Let $\mathcal{E}$ be a rank $r \geq 2$ spanned vector bundle on X and $S = S_1 \sqcup \dots \sqcup S_c$ be its general dependency locus. If S_1 is not of fiber type, then $c = 1$.*

Proof. By Lemma 2 $\pi_1|_{S_1}$ is dominant and its general fiber is a single point. Since $\dim(S_1) = 2$, at most over finitely many points of X_1 the scheme $\pi_1^{-1}(P) \cap S_1$ is a line. Since $S_j \cap S_1 = \emptyset$ for each $j > 1$ and each fiber of $\pi_1|_S$ is non-empty and connected, we get $c = 1$ and that $\pi_1|_{S_1}$ is birational onto its image. $\square$

Proposition 2. *Assume $X_2 = \mathbb{P}^2$ and $\mathcal{L}_\infty = \mathcal{O}_{\mathbb{P}^2}(\infty)$. Let $\mathcal{E}$ be a rank $r \geq 2$ spanned vector bundle on X and $S = S_1 \sqcup \dots \sqcup S_c$ be its general dependency locus. Assume that $\pi_1^{-1}(P) \cap S_1 = \emptyset$ for some $P \in X_1$. Then each S_j is of fiber*

type, each $C_j := \pi_1(S_j)$ is a smooth curve of X_1 , $C_i \cap C_j = \emptyset$ for all $i \neq j$, $\pi_1|_{S_j} : S_j \rightarrow C_j$ is a geometrically ruled surface and S is smooth.

Proof. By Lemma 2 S_1 is of fiber type. By Proposition 1 each component S_j of S is of fiber type. Set $C_j := \pi_1(S_j)$, $j = 1, \dots, c$. By Lemma 2 C_j is an integral curve and over each point $P \in C_j$ the scheme-theoretic fiber of $\pi_1|_{S_1}$ is isomorphic to $\mathbb{P}^1$ and a line with respect to any ample line bundle $\mathcal{R}_\infty \otimes \mathcal{L}_\infty$, $\mathcal{R}_\infty$ ample line bundle on X_1 . Since C_1 is reduced, we get that $\pi_1|_{S_j} : S_j \rightarrow C_j$ is flat ([5, Theorem III.9.9]). Since each fiber of $\pi_1|_{S_j}$ is smooth, $\pi_1|_{S_j} : S_j \rightarrow C_j$ is smooth. Since S_j has only isolated singularities, we get that C_j and S_j are smooth. Hence S is smooth. Since C_j is a smooth curve, Tsen theorem gives that each $\pi_1|_{S_j} : S_j \rightarrow C_j$ is a $\mathbb{P}^1$ -bundle. Since $S_i \cap S_j = \emptyset$ for each $i \neq j$ and each fiber of $\pi_1|_S$ is either empty or connected, we get $C_i \cap C_j = \emptyset$ for all $i \neq j$. $\square$

Remark 1. In the set-up of Proposition 2 we have $c = 1$ if any two curve on X_1 meets (e.g. if $\text{Pic}(X_1) \cong \mathbb{Z}$).

Fix any smooth curve $C \subset X_1$ (not necessarily connected) and set $\mathcal{L}_\infty := \mathcal{O}_{\mathcal{X}}(\mathcal{C})$. Assume $\mathcal{O}_{\mathcal{X}}(\mathcal{C})$ is spanned and set $\mathcal{L} := \mathcal{O}_{\mathcal{X}_\infty}(\mathcal{C}) \boxtimes \mathcal{O}_{\mathbb{P}^\infty}(\infty)$. Fix a line $L \subset \mathbb{P}^2$ and set $S := C \times L \subset X$. S is a dependency locus of fiber type of two section of the spanned bundle $\pi_1^*(\mathcal{O}_{\mathcal{X}_\infty}(\mathcal{C})) \oplus \pi^*(\mathcal{O}_{\mathbb{P}^\infty}(\infty))$.

Proof of Theorem 1. Since $h^0(\mathcal{O}_{\mathbb{P}^\infty}(-\infty)) = \langle^\infty(\mathcal{O}_{\mathbb{P}^\infty}(-\infty)) = \iota$, the Künneth formula gives $h^1(X, \mathcal{L}^\vee) = \iota$. Hence the condition for the uniqueness part of [1, Theorem 1.1] is satisfied. The isomorphism $\pi_1|_S : S \rightarrow X_1$ identifies ω_S and ω_{X_1} . Since S is the graph of f , up to the identification of S and X_1 obtained by $\pi_1|_S$, for any $R \in \text{Pic}(X_1)$ and any $k \in \mathbb{Z}$ we have $R \boxtimes \mathcal{O}_{\mathbb{P}^\infty}(\|) \cong \mathcal{R} \otimes \{^*(\mathcal{O}_{\mathbb{P}^\infty}(\|))\}$. Hence $\pi_1|_S$ identifies $\omega_S^\vee \otimes (\omega_X \otimes \mathcal{L})|_S$ with $f^*(\mathcal{O}_{\mathbb{P}^\infty}(\infty))$. By Proposition 1 we have $c = 1$ for any dependency loci with S as a component.

Since $\dim(X_1) = 2$, for each spanned line bundle R on X_1 there is a linear subspace of $V \subseteq H^1(R)$ with $\dim(V) \leq 3$ and V spanning M . If $r = \alpha + 1$, there is uniqueness (part (d)), because in the set-up of (b) we need to take $V = H^0(M)$. $\square$

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