ON THE DIOPHANTINE EQUATION $483^x + 483^{2s}n^y = z^{2t}$,
WHERE $s, t, n$ ARE NON-NEGATIVE INTEGERS
AND $n \equiv 5 \pmod{20}$

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Abstract: Let $s, t, n$ be non negative integers such that $n \equiv 5 \pmod{20}$. In this paper, we found that all non-negative integer solutions $(x, y, z)$ of the Diophantine equation $483^x + 483^{2s}n^y = z^{2t}$ are in the following form:

$$(x, y, z) = \begin{cases} 
(1 + 2s, 0, 22(483)^s) & ; \ t = 1, \\
no solution & ; \ otherwise.
\end{cases}$$

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1. Introduction

There are many mathematicians solved all non-negative integer solutions $(x, y, z)$ of the Diophantine equation $a^x + b^y = z^2$ where $a, b$ are positive integers and $x, y, z$ are non-negative integers. For example, in 2013, Chotchaisthit [1] showed that $(p, x, y, z) = (7, 0, 1, 3)$ and $(p, x, y, z) = (3, 2, 2, 5)$ are the only solutions of the Diophantine equation $p^x + (p + 1)^y = z^2$ where $x, y, z$ are non-negative integers and $p$ is the Mersenne prime.

In 2013-2014, there are numbers of paper of Sroysang, for examples, in [4, 5]
solved all non-negative integer solutions \((x, y, z)\) of the Diophantine equations 
\[3^x + 5^y = z^2, \quad 3^x + 17^y = z^2\]
and he found that both Diophantine equations have exactly one non-negative integer solution, namely, \((x, y, z) = (1, 0, 2)\).

In recently 2014, Sroysang [6] showed that there is a unique non-negative integer solution \((x, y, z) = (1, 0, 22)\) for the Diophantine equation \(483^x + 485^y = z^2\). Now we find all possible non-negative integer solutions \((x, y, z)\) of the Diophantine equation \(483^x + 483^{2n}y = z^2\) where \(n \equiv 5 \pmod{20}\), which is a generalization of the Diophantine equation \(483^x + 485^y = z^2\) where \(s = 0\) and \(n = 485\).

2. Preliminaries

Throughout this paper, we assume that \(n\) is a non-negative integer such that \(n \equiv 5 \pmod{20}\). It is clear that \(n \equiv 1 \pmod{4}\) and \(n \equiv 0 \pmod{5}\).

**Corollary 2.1.** [3] Let \(a\) and \(b\) be nonzero integers and \(c\) be an integer. If \(a\mid c\) and \(b\mid c\), with \(\gcd(a, b) = 1\), then \(ab\mid c\).

**Theorem 2.2.** [3] Let \(a, b, p\) be integers. If \(p\) is a prime and \(p\mid ab\), then \(p\mid a\) or \(p\mid b\).

**Lemma 2.3.** [6] \((1, 22)\) is a unique non-negative integer solution \((x, z)\) for the Diophantine equation \(483^x + 1 = z^2\).

Let \(p\) be an odd prime and \(a\) be a positive integer where \(\gcd(a, p) = 1\). If the quadratic congruence \(x^2 \equiv a \pmod{p}\) has a solution, then \(a\) is said to be a quadratic residue of \(p\). Otherwise, \(a\) is called a quadratic non-residue of \(p\). In 1798 Adrien-Marie Legendre introduced the Legendre symbol \(\left( \frac{a}{p} \right)\) which is defined by

\[
\left( \frac{a}{p} \right) = \begin{cases} 
1 & \text{if } a \text{ is a quadratic residue of } p, \\
-1 & \text{if } a \text{ is a quadratic non-residue of } p.
\end{cases}
\]

In the present paper, we need the following well-known facts about the Legendre symbols.

**Theorem 2.4.** [3] If \(p\) is an odd prime, then

\[
\left( \frac{2}{p} \right) = \begin{cases} 
1 & \text{if } p \equiv 1 \pmod{8} \text{ or } p \equiv 7 \pmod{8}, \\
-1 & \text{if } p \equiv 3 \pmod{8} \text{ or } p \equiv 5 \pmod{8}.
\end{cases}
\]
ON THE DIOPHANTINE EQUATION $483^x + 483^2n^y = z^{2t}$...

Theorem 2.5. [3] If $p \neq 3$ is an odd prime, then

$$\left( \frac{3}{p} \right) = \begin{cases} 1 & \text{if } p \equiv \pm 1 \pmod{12}, \\ -1 & \text{if } p \equiv \pm 5 \pmod{12}. \end{cases}$$

3. Main Results

First, we find all non-negative integer solutions $(x, y, z)$ of the Diophantine equation $483^x + n^y = z^2$ where $x, y, z$ are non-negative integers.

Theorem 3.1. $(1, 0, 22)$ is a unique non-negative integer solution $(x, y, z)$ for the Diophantine equation $483^x + n^y = z^2$.

Proof. Note that $z$ is an even integer. Then $z^2 \equiv 0 \pmod{4}$.

Case 1. $y = 0$. By Lemma 2.3, we have $(x, y, z) = (1, 0, 22)$ is a unique non-negative integer solution of the Diophantine equation $483^x + n^y = z^2$.

Case 2. $y \geq 1$. Suppose that there exists a non-negative integer solution $(x, y, z)$ for the Diophantine equation $483^x + n^y = z^2$. Since $n \equiv 5 \pmod{20}$, $n \equiv 1 \pmod{4}$ and $n \equiv 0 \pmod{5}$. Then we have $0 \equiv z^2 = 483^x + n^y \equiv 3^x + 1 \pmod{4}$. Thus $3^x \equiv -1 \pmod{4}$. This implies that $x$ is odd. Using $x$ is odd, we have $3^x \equiv 2 \pmod{5}$ or $3^x \equiv 3 \pmod{5}$. Since $n^y \equiv 0 \pmod{5}$, we obtain $z^2 = 483^x + n^y = 3^x + 0 \equiv 2$ or $3 \pmod{5}$. That is $\left( \frac{2}{5} \right) = 1$ and $\left( \frac{3}{5} \right) = 1$. This is a contradiction to Theorem 2.4 and Theorem 2.5, respectively. In this case, there is no non-negative integer solution.

It can be seen that $(1, 0, 22)$ is a unique non-negative integer solution $(x, y, z)$ for the Diophantine equation $483^x + n^y = z^2$. This completes the proof.

We know that $n = 485 \equiv 5 \pmod{20}$. The following example is the main theorem of Sroysang [6] which is a special case of Theorem 3.1.

Example 3.2. $(1, 0, 12)$ is a unique non-negative integer solution $(x, y, z)$ for the Diophantine equation $483^x + 485^y = z^2$.

Lemma 3.3. [2] Let $m$ be a positive integer with $m \equiv 1 \pmod{4}$. The Diophantine equation $1 + mn^y = z^2$ has no non-negative integer solution where $y, z$ are non-negative integers.
Lemma 3.4. Let $m$ be an integer greater than 1. The Diophantine equation $483 + 483^m n^y = z^2$ has no non-negative integer solution $(y, z)$.

Proof. Suppose that there are non-negative integers $y, z$ such that $483 + 483^m n^y = z^2$. This implies $483|z^2$. Since $3|z^2$, $7|z^2$, $23|z^2$ such that $gcd(3, 7) = 1$ and $gcd(21, 23) = 1$, we conclude $483|z$ by Corollary 2.1 and Theorem 2.2. We write $z = 483r$ for some integers $r$. Substituting $z = 483r$ in the Diophantine equation $483 + 483^m n^y = z^2$, then $483 + 483^m n^y = 483^2 r^2$ and thus $1 + 483^{m-1} n^y = 483r^2$. We get $1 = 483(\frac{r^2}{483} - 483^{m-2} n^y)$. So that $483|1$. This is a contradiction. Hence $483 + 483^m n^y = z^2$ has no non-negative integer solution.

The next theorem is main result of this paper.

Theorem 3.5. Let $s$ be a non-negative integer. Then $(1 + 2s, 0, 22(483)^s)$ is a unique non-negative integer solution $(x, y, z)$ for the Diophantine equation $483^x + 483^{2s} n^y = z^2$.

Proof. We prove by induction on $s$.

Let $P(s)$ : The Diophantine equation $483^x + 483^{2s} n^y = z^2$ has a unique non-negative integer solution $(x, y, z) = (1 + 2s, 0, 22(483)^s)$.

By Theorem 3.1, $P(0)$ is true.

Suppose that $P(k)$ is true, that is the Diophantine equation $483^x + 483^{2k} n^y = z^2$ has a unique non-negative integer solution $(x, y, z) = (1 + 2k, 0, 22(483)^k)$. Consider the Diophantine equation $483^x + 483^{2(k+1)} n^y = z^2$ into the following cases:

Case $x = 0$. Since $483^{2(k+1)} \equiv 1 \pmod{4}$ and by Lemma 3.3, we obtain that $1 + 483^{2(k+1)} n^y = z^2$ has no non-negative integer solution.

Case $x = 1$. By Lemma 3.4, the Diophantine equation $483 + 483^{2(k+1)} n^y = z^2$ has no non-negative integer solution.

Case $x \geq 2$. Note that the Diophantine equation $483^x + 483^{2(k+1)} n^y = z^2$ can be written as $483^{x-2} + 483^{2k} n^y = \left(\frac{z}{483}\right)^2$ and $x - 2, \frac{z}{483}$ are non-negative integers. Let $u = x - 2$ and $v = \frac{z}{483}$. By assumption $P(k)$ is true, we obtain that $483^u + 483^{2k} n^y = v^2$ has a unique non-negative integer solution $(u, y, v) = (1 + 2k, 0, 22(483)^k)$. That is $u = 1 + 2k$ and $v = 22(483)^k$. Thus $(x, y, z) = (1 + 2(k+1), 0, 22(483)^{k+1})$ is a unique non-negative integer solution
of the Diophantine equation $483^x + 483^{2(k+1)}ny = z^2$. Therefore $P(k + 1)$ is true.

By mathematical induction, the Diophantine equation $483^x + 483^{2s}ny = z^2$ has a unique non-negative integer solution $(x, y, z) = (1 + 2s, 0, 22(483)^s)$. □

As a consequence of Theorem 3.5, we obtain:

**Corollary 3.6.** Let $s, t$ be non-negative integers such that $t \geq 2$. The Diophantine equation $483^x + 483^{2s}ny = z^2$ has no non-negative integer solution $(x, y, z)$.

**Proof.** Suppose that $(x, y, z)$ is a non-negative integer solution of the Diophantine equation $483^x + 483^{2s}ny = z^2$. Thus $(x, y, z^t)$ is a non-negative integer solution of the Diophantine equation $483^x + 483^{2s}ny = z^2$. By Theorem 3.5, we have $(x, y, z^t) = (1 + 2s, 0, 22(483)^s)$. Thus $z^t = 22(483)^s$ which is a contradiction. Therefore, the Diophantine equation $483^x + 483^{2s}ny = z^2$ has no non-negative integer solution $(x, y, z)$. □

The Diophantine equation $483^x + 483^{2s}ny = z^2t$, when $s, t$ are non-negative integers, is a generalization of the Diophantine equation $483^x + 485^y = z^2$. It easy to verify that the Diophantine equation $483^x + 483^{2s}ny = z^2$ has no non-negative integer solution when $t = 0$. Theorem 3.5 and Corollary 3.6 give the following result.

**Theorem 3.7.** Let $n, s, t$ be any non-negative integers. All non-negative integer solutions $(x, y, z)$ of the Diophantine equation $483^x + 483^{2s}ny = z^{2t}$ are the following:

$$(x, y, z) = \begin{cases} 
(1 + 2s, 0, 22(483)^s) & ; \ t = 1, \\
no solution & ; \ otherwise.
\end{cases}$$

Using Theorem 3.7, it is easy to verify the following examples.

**Example 3.8.** For $n = 5, s = 0, 1, 2, 3, 4, ...$
For $n = 25$, $s = 0, 1, 2, 3, 4, ...$

<table>
<thead>
<tr>
<th>$s$</th>
<th>$483^x + 483^{2s}5^y = z^2$</th>
<th>solution $(x, y, z)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s = 0$</td>
<td>$483^x + 5^y = z^2$</td>
<td>$(1, 0, 22)$</td>
</tr>
<tr>
<td>$s = 1$</td>
<td>$483^x + (233289)5^y = z^2$</td>
<td>$(3, 0, 10626)$</td>
</tr>
<tr>
<td>$s = 2$</td>
<td>$483^x + (54423757521)5^y = z^2$</td>
<td>$(5, 0, 5132358)$</td>
</tr>
<tr>
<td>$s = 3$</td>
<td>$483^x + (12696463968316569)5^y = z^2$</td>
<td>$(7, 0, 2478928914)$</td>
</tr>
<tr>
<td>$s = 4$</td>
<td>$483^x + (2961945382704604065441)5^y = z^2$</td>
<td>$(9, 0, 1197322665462)$</td>
</tr>
</tbody>
</table>

For $n = 45$, $s = 0, 1, 2, 3, 4, ...$

<table>
<thead>
<tr>
<th>$s$</th>
<th>$483^x + 483^{2s}25^y = z^2$</th>
<th>solution $(x, y, z)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s = 0$</td>
<td>$483^x + 25^y = z^2$</td>
<td>$(1, 0, 22)$</td>
</tr>
<tr>
<td>$s = 1$</td>
<td>$483^x + (233289)25^y = z^2$</td>
<td>$(3, 0, 10626)$</td>
</tr>
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<td>$(9, 0, 1197322665462)$</td>
</tr>
</tbody>
</table>

Moreover, for any $n \equiv 5 \pmod{20}$, we obtain the non-negative integer solutions $(x, y, z)$ of the Diophantine equation $483^x + 483^{2s}5^y = z^2$ in a similar way.

We can apply Theorem 3.7 to solve non-negative solutions of many Diophantine equations, which related with the Diophantine equation $483^x + 483^{2s}5^y = z^2$, and we show as the following examples:

**Example 3.9.** $(x, y, z) = (4, 0, 483)$ is a unique non-negative integer solution $(x, y, z)$ for the Diophantine equation $483^x + 112678587(125)^y = 233772z^2$.

**Proof.** Note that the Diophantine equation $483^x + 112678587(125)^y = 233772z^2$ can be written as $483^x + (483)^3(125)^y = 483(22z)^2$. If $x = 0$, then $483|1$. This
is a contradiction. Suppose $x \geq 1$. So we have $483^{x-1} + 483^2(125)^y = (22z)^2$. By Theorem 3.7, we have $(x-1, y, 22z) = (3, 0, 22(483))$. Thus $x-1 = 3$ and $22z = 22(483)$ or $x = 4$ and $z = 483$. Therefore, $(x, y, z) = (4, 0, 483)$ is a unique non-negative integer solution.

Example 3.10. $(x, y, z) = (4, 0, 966)$ is a unique non-negative integer solution $(x, y, z)$ for the Diophantine equation $483^x + 112678587(65)^y = 58443z^2$.

Proof. Note that the Diophantine equation $483^x + 112678587(65)^y = 58443z^2$ can be written as $483^x + (483)^3(65)^y = 483(11z)^2$. If $x = 0$, then $483|1$. This is a contradiction. Suppose $x \geq 1$. So we have $483^{x-1} + 483^2(65)^y = (11z)^2$. By Theorem 3.7, we have $(x-1, y, 11z) = (3, 0, 22(65))$. Thus $x-1 = 3$ and $11z = 22(483)$ or $x = 4$ and $z = 966$. Therefore, $(x, y, z) = (4, 0, 966)$ is a unique non-negative integer solution.

Example 3.11. The Diophantine equation $483^x + 112678587(45)^y = 30912z^6$ has no non-negative integer solution $(x, y, z)$.

Proof. Note that the Diophantine equation $483^x + 112678587(45)^y = 30912z^6$ can be written as $483^x + (483)^3(45)^y = 483(2z)^6$. If $x = 0$, then $483|1$. This is a contradiction. Suppose $x \geq 1$. So we have $483^{x-1} + 483^2(45)^y = (2z)^6$. By Theorem 3.7, this equation has no non-negative integer solution $(x, y, z)$.

Example 3.12. The Diophantine equation $483^x + 483(25)^y = 4347z^2$ has no non-negative integer solution $(x, y, z)$.

Proof. Note that the Diophantine equation $483^x + 483(25)^y = 4347z^2$ can be written as $483^x + 483(25)^y = 483(3z)^2$. If $x = 0$, then $483|1$. This is a contradiction. Suppose $x \geq 1$. So we have $483^{x-1} + 25^y = (3z)^2$. By Theorem 3.7, we have $(x-1, y, 3z) = (1, 0, 22)$. This implies that $3z = 22$, i.e., $z = \frac{22}{3}$. This is a contradiction since $z$ is integer. Therefore, this equation has no non-negative integer solution $(x, y, z)$.
4. Open Problem

It is important to note that all non-negative solutions \((x, y, z)\) of the Diophantine equation \(483^x + n^y = z^2\), where \(n\) is any positive integer and \(n \not\equiv 5 \pmod{20}\), are still an open problem. For example, it is not known how to find all non-negative integer solutions \((x, y, z)\) of the Diophantine equation \(483^x + n^y = z^2\) where \(n = 2, 3, 6, 7, \ldots\) etc.

Acknowledgments

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References


[2] N. Sarasit, S. Chotchaisthit, On the Diophantine equation \(3^x + 3^{2x}n^y = z^{2t}\) where \(n \equiv 0 \pmod{5}\), *Int. J. Pure Appl. Math.*, 97, No. 2 (2014), 211-218, doi: http://dx.doi.org/10.12732/ijpam.v97i2.9.


