ON THE DIOPHANTINE EQUATION $483^x + 483^{2s}n^y = z^{2t}$,
WHERE s, t, n ARE NON-NEGATIVE INTEGERS
AND $n \equiv 5 \pmod{20}$

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Abstract: Let s, t, n be non negative integers such that $n \equiv 5 \pmod{20}$. In this paper, we found that all non-negative integer solutions (x, y, z) of the Diophantine equation $483^x + 483^{2s}n^y = z^{2t}$ are in the following form:

$$(x, y, z) = \begin{cases} (1 + 2s, 0, 22(483)^s) & ; \quad t = 1, \\ \text{no solution} & ; \quad \text{otherwise.} \end{cases}$$

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1. Introduction

There are many mathematicians solved all non-negative integer solutions (x, y, z) of the Diophantine equation $a^x + b^y = z^2$ where a, b are positive integers and x, y, z are non-negative integers. For example, in 2013, Chotchaisthit [1] showed that $(p, x, y, z) = (7, 0, 1, 3)$ and $(p, x, y, z) = (3, 2, 2, 5)$ are the only solutions of the Diophantine equation $p^x + (p + 1)^y = z^2$ where x, y, z are non-negative integers and p is the Mersenne prime.

In 2013-2014, there are numbers of paper of Sroysang, for examples, in [4, 5]

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solved all non-negative integer solutions (x, y, z) of the Diophantine equations $3^x + 5^y = z^2$, $3^x + 17^y = z^2$ and he found that both Diophantine equations have exactly one non-negative integer solution, namely, $(x, y, z) = (1, 0, 2)$.

In recently 2014, Sroysang [6] showed that there is a unique non-negative integer solution $(x, y, z) = (1, 0, 22)$ for the Diophantine equation $483^x + 485^y = z^2$. Now we find all possible non-negative integer solutions (x, y, z) of the Diophantine equation $483^x + 483^{2s}n^y = z^{2t}$ where $n \equiv 5 \pmod{20}$, which is a generalization of the Diophantine equation $483^x + 485^y = z^2$ where $s = 0$ and $n = 485$.

2. Preliminaries

Throughout this paper, we assume that n is a non-negative integer such that $n \equiv 5 \pmod{20}$. It is clear that $n \equiv 1 \pmod{4}$ and $n \equiv 0 \pmod{5}$.

Corollary 2.1. [3] *Let a and b be nonzero integers and c be an integer. If $a|c$ and $b|c$, with $\gcd(a, b) = 1$, then $ab|c$.*

Theorem 2.2. [3] *Let a, b, p be integers. If p is a prime and $p|ab$, then $p|a$ or $p|b$.*

Lemma 2.3. [6] *$(1, 22)$ is a unique non-negative integer solution (x, z) for the Diophantine equation $483^x + 1 = z^2$.*

Let p be an odd prime and a be a positive integer where $\gcd(a, p) = 1$. If the quadratic congruence $x^2 \equiv a \pmod{p}$ has a solution, then a is said to be a *quadratic residue* of p . Otherwise, a is called a *quadratic non-residue* of p . In 1798 Adrien-Marie Legendre introduced the *Legendre symbol* $\left(\frac{a}{p}\right)$ which is defined by

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & \text{; if } a \text{ is a quadratic residue of } p, \\ -1 & \text{; if } a \text{ is a quadratic non-residue of } p. \end{cases}$$

In the present paper, we need the following well-known facts about the Legendre symbols.

Theorem 2.4. [3] *If p is an odd prime, then*

$$\left(\frac{2}{p}\right) = \begin{cases} 1 & \text{; if } p \equiv 1 \pmod{8} \text{ or } p \equiv 7 \pmod{8}, \\ -1 & \text{; if } p \equiv 3 \pmod{8} \text{ or } p \equiv 5 \pmod{8}. \end{cases}$$

Theorem 2.5. [3] *If $p \neq 3$ is an odd prime, then*

$$\left(\frac{3}{p}\right) = \begin{cases} 1 & ; \text{if } p \equiv \pm 1 \pmod{12}, \\ -1 & ; \text{if } p \equiv \pm 5 \pmod{12}. \end{cases}$$

3. Main Results

First, we find all non-negative integer solutions (x, y, z) of the Diophantine equation $483^x + n^y = z^2$ where x, y, z are non-negative integers.

Theorem 3.1. *$(1, 0, 22)$ is a unique non-negative integer solution (x, y, z) for the Diophantine equation $483^x + n^y = z^2$.*

Proof. Note that z is an even integer. Then $z^2 \equiv 0 \pmod{4}$.

Case 1. $y = 0$. By Lemma 2.3, we have $(x, y, z) = (1, 0, 22)$ is a unique non-negative integer solution of the Diophantine equation $483^x + n^y = z^2$.

Case 2. $y \geq 1$. Suppose that there exists a non-negative integer solution (x, y, z) for the Diophantine equation $483^x + n^y = z^2$. Since $n \equiv 5 \pmod{20}$, $n \equiv 1 \pmod{4}$ and $n \equiv 0 \pmod{5}$. Then we have $0 \equiv z^2 = 483^x + n^y \equiv 3^x + 1 \pmod{4}$. Thus $3^x \equiv -1 \pmod{4}$. This implies that x is odd. Using x is odd, we have $3^x \equiv 2 \pmod{5}$ or $3^x \equiv 3 \pmod{5}$. Since $n^y \equiv 0 \pmod{5}$, we obtain $z^2 = 483^x + n^y = 3^x + 0 \equiv 2$ or $3 \pmod{5}$. That is $\left(\frac{2}{5}\right) = 1$ and $\left(\frac{3}{5}\right) = 1$. This is a contradiction to Theorem 2.4 and Theorem 2.5, respectively. In this case, there is no non-negative integer solution.

It can be seen that $(1, 0, 22)$ is a unique non-negative integer solution (x, y, z) for the Diophantine equation $483^x + n^y = z^2$. This completes the proof. $\square$

We know that $n = 485 \equiv 5 \pmod{20}$. The following example is the main theorem of Sroysang [6] which is a special case of Theorem 3.1.

Example 3.2. *$(1, 0, 12)$ is a unique non-negative integer solution (x, y, z) for the Diophantine equation $483^x + 485^y = z^2$.*

Lemma 3.3. [2] *Let m be a positive integer with $m \equiv 1 \pmod{4}$. The Diophantine equation $1 + mn^y = z^2$ has no non-negative integer solution where y, z are non-negative integers.*

Lemma 3.4. *Let m be an integer greater than 1. The Diophantine equation $483 + 483^m n^y = z^2$ has no non-negative integer solution (y, z) .*

Proof. Suppose that there are non-negative integers y, z such that $483 + 483^m n^y = z^2$. This implies $483|z^2$. Since $3|z^2, 7|z^2, 23|z^2$ such that $\gcd(3, 7) = 1$ and $\gcd(21, 23) = 1$, we conclude $483|z$ by Corollary 2.1 and Theorem 2.2. We write $z = 483r$ for some integers r . Substituting $z = 483r$ in the Diophantine equation $483 + 483^m n^y = z^2$, then $483 + 483^m n^y = 483^2 r^2$ and thus $1 + 483^{m-1} n^y = 483r^2$. We get $1 = 483(r^2 - 483^{m-2} n^y)$. So that $483|1$. This is a contradiction. Hence $483 + 483^m n^y = z^2$ has no non-negative integer solution. $\square$

The next theorem is main result of this paper.

Theorem 3.5. *Let s be a non-negative integer. Then $(1 + 2s, 0, 22(483)^s)$ is a unique non-negative integer solution (x, y, z) for the Diophantine equation $483^x + 483^{2s} n^y = z^2$.*

Proof. We prove by induction on s .

Let $P(s)$: The Diophantine equation $483^x + 483^{2s} n^y = z^2$ has a unique non-negative integer solution $(x, y, z) = (1 + 2s, 0, 22(483)^s)$.

By Theorem 3.1, $P(0)$ is true.

Suppose that $P(k)$ is true, that is the Diophantine equation $483^x + 483^{2k} n^y = z^2$ has a unique non-negative integer solution $(x, y, z) = (1 + 2k, 0, 22(483)^k)$.

Consider the Diophantine equation $483^x + 483^{2(k+1)} n^y = z^2$ into the following cases:

Case $x = 0$. Since $483^{2(k+1)} \equiv 1 \pmod{4}$ and by Lemma 3.3, we obtain that $1 + 483^{2(k+1)} n^y = z^2$ has no non-negative integer solution.

Case $x = 1$. By Lemma 3.4, the Diophantine equation $483 + 483^{2(k+1)} n^y = z^2$ has no non-negative integer solution.

Case $x \geq 2$. Note that the Diophantine equation $483^x + 483^{2(k+1)} n^y = z^2$ can be written as $483^{x-2} + 483^{2k} n^y = \left(\frac{z}{483}\right)^2$ and $x - 2, \frac{z}{483}$ are non-negative integers. Let $u = x - 2$ and $v = \frac{z}{483}$. By assumption $P(k)$ is true, we obtain that $483^u + 483^{2k} n^y = v^2$ has a unique non-negative integer solution $(u, y, v) = (1 + 2k, 0, 22(483)^k)$. That is $u = 1 + 2k$ and $v = 22(483)^k$. Thus $(x, y, z) = (1 + 2(k + 1), 0, 22(483)^{k+1})$ is a unique non-negative integer solution

of the Diophantine equation $483^x + 483^{2(k+1)}n^y = z^2$. Therefore $P(k+1)$ is true.

By mathematical induction, the Diophantine equation $483^x + 483^{2s}n^y = z^2$ has a unique non-negative integer solution $(x, y, z) = (1 + 2s, 0, 22(483)^s)$. $\square$

As a consequence of Theorem 3.5, we obtain:

Corollary 3.6. *Let s, t be non-negative integers such that $t \geq 2$. The Diophantine equation $483^x + 483^{2s}n^y = z^{2t}$ has no non-negative integer solution (x, y, z) .*

Proof. Suppose that (x, y, z) is a non-negative integer solution of the Diophantine equation $483^x + 483^{2s}n^y = z^{2t}$. Thus (x, y, z^t) is a non-negative integer solution of the Diophantine equation $483^x + 483^{2s}n^y = z^2$. By Theorem 3.5, we have $(x, y, z^t) = (1 + 2s, 0, 22(483)^s)$. Thus $z^t = 22(483)^s$ which is a contradiction. Therefore, the Diophantine equation $483^x + 483^{2s}n^y = z^{2t}$ has no non-negative integer solution (x, y, z) . $\square$

The Diophantine equation $483^x + 483^{2s}n^y = z^{2t}$, when s, t are non-negative integers, is a generalization of the Diophantine equation $483^x + 485^y = z^2$. It easy to verify that the Diophantine equation $483^x + 483^{2s}n^y = z^{2t}$ has no non-negative integer solution when $t = 0$. Theorem 3.5 and Corollary 3.6 give the following result.

Theorem 3.7. *Let n, s, t be any non-negative integers. All non-negative integer solutions (x, y, z) of the Diophantine equation $483^x + 483^{2s}n^y = z^{2t}$ are the following:*

$$(x, y, z) = \begin{cases} (1 + 2s, 0, 22(483)^s) & ; \quad t = 1, \\ \text{no solution} & ; \quad \text{otherwise.} \end{cases}$$

Using Theorem 3.7, it is easy to verify the following examples.

Example 3.8. For $n = 5, s = 0, 1, 2, 3, 4, \dots$

s	$483^x + 483^{2s}5^y = z^2$	solution (x, y, z)
$s = 0$	$483^x + 5^y = z^2$	$(1, 0, 22)$
$s = 1$	$483^x + (233289)5^y = z^2$	$(3, 0, 10626)$
$s = 2$	$483^x + (54423757521)5^y = z^2$	$(5, 0, 5132358)$
$s = 3$	$483^x + (12696463968316569)5^y = z^2$	$(7, 0, 2478928914)$
$s = 4$	$483^x + (2961945382704604065441)5^y = z^2$	$(9, 0, 1197322665462)$
$\vdots$	$\vdots$	$\vdots$

For $n = 25, s = 0, 1, 2, 3, 4, \dots$

s	$483^x + 483^{2s}25^y = z^2$	solution (x, y, z)
$s = 0$	$483^x + 25^y = z^2$	$(1, 0, 22)$
$s = 1$	$483^x + (233289)25^y = z^2$	$(3, 0, 10626)$
$s = 2$	$483^x + (54423757521)25^y = z^2$	$(5, 0, 5132358)$
$s = 3$	$483^x + (12696463968316569)25^y = z^2$	$(7, 0, 2478928914)$
$s = 4$	$483^x + (2961945382704604065441)25^y = z^2$	$(9, 0, 1197322665462)$
$\vdots$	$\vdots$	$\vdots$

For $n = 45, s = 0, 1, 2, 3, 4, \dots$

s	$483^x + 483^{2s}45^y = z^2$	solution (x, y, z)
$s = 0$	$483^x + 45^y = z^2$	$(1, 0, 22)$
$s = 1$	$483^x + (233289)45^y = z^2$	$(3, 0, 10626)$
$s = 2$	$483^x + (54423757521)45^y = z^2$	$(5, 0, 5132358)$
$s = 3$	$483^x + (12696463968316569)45^y = z^2$	$(7, 0, 2478928914)$
$s = 4$	$483^x + (2961945382704604065441)45^y = z^2$	$(9, 0, 1197322665462)$
$\vdots$	$\vdots$	$\vdots$

Moreover, for any $n \equiv 5 \pmod{20}$, we obtain the non-negative integer solutions (x, y, z) of the Diophantine equation $483^x + 483^{2s}n^y = z^{2t}$ in a similar way.

We can apply Theorem 3.7 to solve non-negative solutions of many Diophantine equations, which related with the Diophantine equation $483^x + 483^{2s}n^y = z^{2t}$, and we show as the following examples:

Example 3.9. $(x, y, z) = (4, 0, 483)$ is a unique non-negative integer solution (x, y, z) for the Diophantine equation $483^x + 112678587(125)^y = 233772z^2$.

Proof. Note that the Diophantine equation $483^x + 112678587(125)^y = 233772z^2$ can be written as $483^x + (483)^3(125)^y = 483(22z)^2$. If $x = 0$, then $483 \mid 1$. This

is a contradiction. Suppose $x \geq 1$. So we have $483^{x-1} + 483^2(125)^y = (22z)^2$. By Theorem 3.7, we have $(x-1, y, 22z) = (3, 0, 22(483))$. Thus $x-1 = 3$ and $22z = 22(483)$ or $x = 4$ and $z = 483$. Therefore, $(x, y, z) = (4, 0, 483)$ is a unique non-negative integer solution. $\square$

Example 3.10. $(x, y, z) = (4, 0, 966)$ is a unique non-negative integer solution (x, y, z) for the Diophantine equation $483^x + 112678587(65)^y = 58443z^2$.

Proof. Note that the Diophantine equation $483^x + 112678587(65)^y = 58443z^2$ can be written as $483^x + (483)^3(65)^y = 483(11z)^2$. If $x = 0$, then $483|1$. This is a contradiction. Suppose $x \geq 1$. So we have $483^{x-1} + 483^2(65)^y = (11z)^2$. By Theorem 3.7, we have $(x-1, y, 11z) = (3, 0, 22(65))$. Thus $x-1 = 3$ and $11z = 22(483)$ or $x = 4$ and $z = 966$. Therefore, $(x, y, z) = (4, 0, 966)$ is a unique non-negative integer solution. $\square$

Example 3.11. The Diophantine equation $483^x + 112678587(45)^y = 30912z^6$ has no non-negative integer solution (x, y, z) .

Proof. Note that the Diophantine equation $483^x + 112678587(45)^y = 30912z^6$ can be written as $483^x + (483)^3(45)^y = 483(2z)^6$. If $x = 0$, then $483|1$. This is a contradiction. Suppose $x \geq 1$. So we have $483^{x-1} + 483^2(45)^y = (2z)^6$. By Theorem 3.7, this equation has no non-negative integer solution (x, y, z) . $\square$

Example 3.12. The Diophantine equation $483^x + 483(25)^y = 4347z^2$ has no non-negative integer solution (x, y, z) .

Proof. Note that the Diophantine equation $483^x + 483(25)^y = 4347z^2$ can be written as $483^x + 483(25)^y = 483(3z)^2$. If $x = 0$, then $483|1$. This is a contradiction. Suppose $x \geq 1$. So we have $483^{x-1} + 25^y = (3z)^2$. By Theorem 3.7, we have $(x-1, y, 3z) = (1, 0, 22)$. This implies that $3z = 22$, i.e., $z = \frac{22}{3}$. This is a contradiction since z is integer. Therefore, this equation has no non-negative integer solution (x, y, z) . $\square$

4. Open Problem

It is important to note that all non-negative solutions (x, y, z) of the Diophantine equation $483^x + n^y = z^2$, where n is any positive integer and $n \not\equiv 5 \pmod{20}$, are still an open problem. For example, it is not known how to find all non-negative integer solutions (x, y, z) of the Diophantine equation $483^x + n^y = z^2$ where $n = 2, 3, 6, 7, \dots$ etc.

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