ON THE HYPERCYCLICITY AND WEIGHTED
COMPOSITION OPERATORS ON
BANACH FUNCTION SPACES

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Abstract: In this paper we investigate the hypercyclicity of adjoint of a special weighted composition operators on a Banach function space.

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1. Introduction

Let $\mathcal{B}(X)$ denotes the set of all bounded linear operators on a Banach space X . An operator $T \in \mathcal{B}(X)$ is said to be hypercyclic if there exists a vector $x \in X$ for which the orbit $Orb(T, x) = \{T^n x : n \in \mathbb{N}\}$ is dense in X and in this case we refer to x as a hypercyclic vector for T . The holomorphic self maps of the open unit disk U are divided into classes of elliptic and non-elliptic. The elliptic type is an automorphism and has a fixed point in U . It is well known that this map is conjugate to a rotation $z \rightarrow \lambda z$ for some complex number λ with $|\lambda| = 1$. The maps of that are not elliptic are called of non-elliptic type. The iterate of a non-elliptic map can be characterized by the Grand Iteration Theorem ([3, p. 78]). By $\psi(w)$ we denote the angular derivative of ψ at $w \in \partial U$. Note that if $w \in U$, then $\psi(w)$ has the natural meaning of derivative. Also, by ψ_n we mean the n th iteration of the function ψ .

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Proposition 1.1. (see [6]) *Suppose ψ is a holomorphic self-map of U that is not an elliptic automorphism. If ψ has a fixed point $w \in U$, then $\psi_n \rightarrow w$ uniformly on compact subsets of U and $|\psi(w)| < 1$.*

The unique attracting point w in the above proposition is called the *Denjoy-Wolff point* of ψ .

Consider the weighted composition operator $C_{\varphi, \psi}$ on a Banach space X of analytic functions defined by

$$C_{\varphi, \psi} f = \varphi \cdot f \circ \psi \quad (f \in X).$$

We will investigate the hypercyclicity of the operator $C_{\varphi, \psi}$. For some sources we refer to [1–4].

2. Main Results

Let X be a Banach space of analytic functions on the open unit disk U . For each $\lambda \in U$, the evaluation function $e_\lambda : X \rightarrow \mathbb{C}$ is defined by $e_\lambda(f) = f(\lambda)$, $f \in X$. A complex valued function φ on U for which $\varphi X \subseteq X$ is called a multiplier of X . The set of all multipliers of X is denoted by $M(X)$ and we have $M(X) \subseteq H^\infty(U)$.

It is well known that if the adjoint of a continuous operator T on a Banach space has an eigenvector, then T fails to be hypercyclic (see [2]).

Lemma 2.1. *Let X be a separable Banach space of analytic functions on the open unit disk U such that $1 \in X$, and for each $\lambda \in U$, $e_\lambda \in X$. Also, let $z \in M(X)$ and the map $\varphi \rightarrow M_\varphi$ from $M(X)$ into $\mathcal{B}(X)$ is an isometry. Then $M(X) = H^\infty(U) \subset X$.*

Proof. Let $f \in H^\infty(U)$. By the Farrell-Rubel-Shields Theorem [1, Theorem 5.1, p.151], there is a sequence $\{p_n\}_n$ of polynomials converging to f such that for all n , $\|p_n\|_U \leq c_0$ for some $c_0 > 0$. So we obtain $\|M_{p_n}\| = \|p_n\|_U \leq c_0$ for all n . But ball $B(X)$ is compact in the weak operator topology and so by passing to a subsequence if necessary, we may assume that for some $A \in \mathcal{B}(X)$, $M_{p_n} \rightarrow A$ in the weak operator topology. Using the fact that $M_{p_n} \rightarrow A$ in the weak operator topology and acting these operators on e_λ we get $p_n(\lambda)e_\lambda = M_{p_n}e_\lambda \rightarrow Ae_\lambda$ weakly. Since $p_n(\lambda) \rightarrow f(\lambda)$ we see that $Ae_\lambda = f(\lambda)e_\lambda$. Because the closed linear span of $\{e_\lambda : \lambda \in U\}$ is dense in X , we conclude that $A = M_f$ and this implies that $f \in M(X)$. Indeed $f \in X$, since X contains the constant functions. $\square$

Theorem 2.2. *Let X be a separable Banach space of analytic functions on the open unit disk U such that $1 \in X$, and for each $\lambda \in U$, $e_\lambda \in X$. Also, let $z \in M(X)$ and the map $\varphi \rightarrow M_\varphi$ from $M(X)$ into $\mathcal{B}(X)$ is an isometry. Let $0 < \lambda < 1$ and $\varphi \in M(X)$ be such that $\varphi(0) \neq 0$. Then $(C_{\varphi, \lambda z})$ is not hypercyclic.*

Proof. Since φ is bounded, an application of Schwartz's Lemma shows that

$$|\varphi(0) - \varphi(z)| < 2\|\varphi\|_U |z|$$

for all z in U . But $\varphi(0) \neq 0$, thus

$$\left|1 - \frac{1}{\varphi(0)}\varphi(z)\right| < \frac{2\|\varphi\|_U}{|\varphi(0)|}|z| \quad (z \in U).$$

By substituting $\lambda^i z$ instead of z in the above inequality, we get

$$\left|1 - \frac{1}{\varphi(0)}\varphi(\psi_i(z))\right| < \frac{2\|\varphi\|_U}{|\varphi(0)|}|\lambda|^i|z|.$$

This implies that

$$\sum_{i=0}^{\infty} \left|1 - \frac{1}{\varphi(0)}\varphi(\lambda^i z)\right|$$

and consequently

$$\prod_{i=0}^{\infty} \frac{1}{\varphi(0)}\varphi(\lambda^i z)$$

converges uniformly on compact subsets of U . Set

$$g(z) = \prod_{i=0}^{\infty} \frac{1}{\varphi(0)}\varphi(\lambda^i z),$$

then g is a nonzero holomorphic function on U . Also, note that

$$|\varphi(z)| < 2\|\varphi\|_U |z| + |\varphi(0)|$$

for every $z \in U$. So it follows that if $z \in U$, then

$$|\varphi(\lambda^n z)| < 2\|\varphi\|_U |\lambda|^n |z| + |\varphi(0)|.$$

But $\varphi(0) \neq 0$, thus

$$\frac{|\varphi(\lambda^n z)|}{|\varphi(0)|} < 2\|\varphi\|_U |\lambda|^n \frac{|z|}{|\varphi(0)|} + 1$$

$$\begin{aligned}
&\leq \exp(2\|\varphi\|_U |\lambda|^n \frac{|z|}{|\varphi(0)|}) \\
&\leq \exp(\frac{2\|\varphi\|_U}{|\varphi(0)|} |\lambda|^n).
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
\prod_{n=0}^{\infty} |\frac{1}{\varphi(0)} \varphi(\lambda^n z)| &\leq \exp(\sum_{n=0}^{\infty} (\frac{2\|\varphi\|_U}{|\varphi(0)|} |\lambda|^n)) \\
&= \exp(\frac{2\|\varphi\|_U}{|\varphi(0)|} \frac{1}{1-|\lambda|})
\end{aligned}$$

for every $z \in U$. Hence, indeed g belongs to $H(U)$ and so by Lemma 2.1, $g \in X$. But $\varphi \cdot g \circ \lambda z = \varphi(0)g$ and so $\varphi(0)$ is an eigenvalue for the operator $M_\varphi C_\psi$. Now since the operator $C_{\varphi, \lambda z}$ has a non-zero eigenvalue, then $(C_{\varphi, \lambda z})$ fails to be hypercyclic. This completes the proof. $\square$

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