DYNAMICS OF HYPERBOLIC WEIGHTED COMPOSITION OPERATORS

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Abstract: In the present paper we investigate conditions under which a hyperbolic self-map of the open unit disk induces a hypercyclic weighted composition operator in the space of holomorphic functions on the unit ball in \( C^N \).

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1. Introduction

For \( z = (z_1, ..., z_N) \) and \( w = (w_1, ..., w_N) \) in \( C^N \), write \( < z, w > \) for the Euclidean inner product \( \sum_{j=1}^{N} z_j \bar{w}_j \) and let \( |z| =< z, z >^{1/2} \). With this notation, the unit ball in \( C^N \) is the set \( B_N = \{ z \in C^N : |z| < 1 \} \) and the unit sphere in \( C^N \) is the set \( S_N = \{ z \in C^N : |z| = 1 \} \), analogously to the unit disc and circle for \( N = 1 \). The space \( H(B_N) \), is the set of all holomorphic fuctions on \( B_N \), can be made into a F-space by a complete metric for which a sequence \( \{f_n\} \) in \( H(B_N) \) converges to \( f \in H(B_N) \) if and only if \( f_n \rightarrow f \) uniformly on every compact subsets of \( B_N \). Each \( \varphi \in H(B_N) \) and holomorphic self-map \( \psi \) of \( B_N \) induces a linear weighted composition operator \( C_{\varphi,\psi} : H(B_N) \rightarrow H(B_N) \) de-
fined by $C_{\varphi,\psi}(f)(z) = \varphi(z)f(\psi(z))$ for every $f \in H(B_N)$ and $z \in B_N$. Indeed, $C_{\varphi,\psi} = M_\varphi C_\psi$ where $M_\varphi$ denotes the operator of multiplication by $\varphi$ and $C_\psi$ is a composition operator by means of the definition $C_\psi(f) = f \circ \psi$ for every $f \in H(B_N)$.

A bounded linear operator $T$ on a $F$-space $X$ is said to be hypercyclic if there exists a vector $x \in X$ for which the orbit $\text{Orb}(T, x) = \{T^n x : n \in \mathbb{N}\}$ is dense in $X$ and in this case we refer to $x$ as a hypercyclic vector for $T$.

The holomorphic self maps of $B_N$ are divided into classes of elliptic and non-elliptic. The elliptic type is an automorphism and has a fixed point in $B_N$. It is well known that this map is conjugate to a rotation.

For simplicity, throughout this paper we use the notation $" \to "$ for indicating uniform convergence on compact subsets of $B_N$. Also, by $\psi_n$ we denote the $n$th iterate of $\psi$. To state the main result of the paper, we need the following theorems from [3].

**Theorem 1.1.** (Denjoy-Wollf Iteration in $B_N$) Suppose $\psi$ is a holomorphic self-map of the open unit ball $B_N$ without interior fixed point. Then there is a point $w \in \partial B_N$ such that $\psi_n \to w$ and $0 < d(w) \leq 1$ where

$$d(w) = \lim_{|z| \to 1^-} \inf \frac{1 - |\psi(z)|^2}{1 - |z|^2}.$$ 

The boundary point $w$ is called the Denjoy-Wolff point of $\psi$.

**Theorem 1.2.** (Julia’s Lemma in $B_N$) Let $\psi$ be an analytic map of the unit ball into itself with Denjoy-Wolff point $w \in \partial B_N$. Then for every $z \in B_N$,

$$\frac{|1 - <\psi(z), w>|^2}{1 - |\psi(z)|^2} \leq d(w) \frac{|1 - <z, w>|^2}{1 - |z|^2}.$$

Recall that a holomorphic self-map $\psi$ of $B_N$ is called elliptic if $\psi$ has a fixed point in $B_N$. Also, if $\psi$ has no interior fixed point, then it is called hyperbolic whenever $d(w) < 1$, and is called parabolic if $d(w) = 1$.

For simplicity, we call a weighted composition operator $C_{\varphi,\psi}$, a hyperbolic weighted composition operator whenever the compositional symbol $\psi$ is hyperbolic.

**Definition 1.3.** We say that a mapping $\varphi : B_N \to \mathbb{C}$ is semi-nonexpansive provided there exists a neighborhood $U_w$ of $w$ such that $|\varphi(z) - \varphi(w)| \leq |z - w|$ for all $z$ in $U_w \cap B_N$. 
The next section of the present paper shows that weighted composition operators with non-constant weight function and hyperbolic compositional symbol can be hypercyclic on $H(B_N)$. For some sources see [1]-[7].

2. Main Result

In this section we investigate the hypercyclicity of a hyperbolic weighted composition operator acting on $H(B_N)$.

Proposition 2.1. Let $\varphi$ be a nonzero holomorphic map on $B_N$ and $\psi$ be a hyperbolic map of $B_N$ with $w$ the Denjoy-Wolff point such that $\varphi(w) \neq 0$. If $\varphi$ is semi-nonexpansive, then $C_{\varphi,\psi}^*$ is not hypercyclic, but $C_{\varphi,\psi}$ is hypercyclic whenever $\varphi$ never vanishes on $B_N$, $C_\psi$ is hypercyclic and $|\varphi(w)| = 1$.

Proof. Let $K$ be a compact subset of $B_N$. By Theorem 1.2, there exists a constant $c > 0$ such that

$$|1 - < \psi_n(z), w >|^2 \leq c(1 - |\psi_n(z)|^2)$$

for every $z \in K$ and every $n \in \mathbb{N}$. But $|1 - < \psi_n(z), w >|^2 = |w - \psi_n(z)|^2$, thus $|w - \psi_n(z)|^2 \leq c(1 - |\psi_n(z)|^2)$ for every $z \in K$ and every $n \in \mathbb{N}$. On the other hand, since $\varphi$ is semi-nonexpansive, there exists a neighborhood $U_w$ of $w$ satisfying $|\varphi(w) - \varphi(z)| \leq |w - z|$ for every $z$ in $U_w \cap B_N$. Since $\psi_n \rightarrow w$, there exists $N$ such that for all $n > N$, $\psi_n(z) \in U_w$. Substituting $\psi_n(z)$ instead of $z$ in the previous relation we get

$$|\varphi(w) - \varphi(\psi_n(z))| \leq |w - \psi_n(z)|$$

$$= |1 - < \psi_n(z), w >|$$

$$\leq c^{1/2}(1 - |\psi_n(z)|^2)^{1/2}, \quad (*)$$

for every $n > N$. Now we apply the techniques used in [7]. Since $\psi$ is hyperbolic, thus $0 < d(w) < 1$ and by Theorem 1.2 we have

$$\frac{|1 - < \psi(z), w >|^2}{1 - |\psi(z)|^2} \leq d(w) \frac{|1 - < z, w >|^2}{1 - |z|^2}$$

for all $z \in B_N$. By substituting $\psi_n(z)$ for $\psi(z)$ in the above inequality, we get

$$\frac{|1 - < \psi_n(z), w >|^2}{1 - |\psi_n(z)|^2} \leq d(w)^n \frac{|1 - < z, w >|^2}{1 - |z|^2}$$
for every \( z \in B_N \) and \( n \in \mathbb{N} \). Also, note that since \( K \) is compact, then there exists a constant \( \beta > 0 \) such that

\[
4 \frac{1 - |z, w|}{1 - |z|^2} < \beta
\]

for all \( z \) in \( K \). So it follows that

\[
1 - |\psi_n(z)|^2 = (1 - |\psi_n(z)|)(1 + \psi_n(z))
\]

\[
\leq 2|1 - \langle \psi_n(z), w \rangle|
\]

\[
\leq 4 \frac{|1 - \langle \psi_n(z), w \rangle|^2}{1 - |\psi_n(z)|^2}
\]

\[
\leq 4 \frac{1 - |z, w|^2}{1 - |z|^2} d(w)^n
\]

\[
< \beta d(w)^n.
\]

Hence we obtain

\[
1 - |\psi_n(z)|^2 \leq \beta d(w)^n. \quad (**)
\]

Now by using the relations (*) and (**), we get

\[
|1 - \frac{1}{\varphi(w)} \varphi(\psi_n(z))| < \frac{c_1^2}{|\varphi(w)|} (1 - |\psi_n(z)|^2)^{1/2}
\]

\[
\leq \frac{c_1^2 \beta^{1/2}}{|\varphi(w)|} d(w)^{n/2}.
\]

Since \( 0 < d(w) < 1 \), thus \( \sum_{n=0}^{\infty} 1 - \frac{1}{\varphi(w)} \varphi(\psi_n(z)) \) and so \( \prod_{n=0}^{\infty} 1/\varphi(w) \varphi(\psi_n(z)) \) converges uniformly on \( K \). Define

\[
g(z) = \prod_{n=0}^{\infty} \frac{1}{\varphi(w)} \varphi(\psi_n(z)).
\]

Since \( \varphi(w) \neq 0 \) and \( \psi_n \xrightarrow{k} w \), so there exists a neighborhood \( U_w \) of \( w \) such that \( \varphi \circ \psi_n \neq 0 \) on \( U_w \) for all \( n \) large enough. Let \( z = (z_1, z') \in B_N \) where \( z' = (z_2, ..., z_N) \in \mathbb{C}^{N-1} \). Define \( f(z_1) = g(z_1, z') \), then \( f \) is a nonzero holomorphic function. Thus \( g \) is also a nonzero holomorphic function with respect to \( z_1 \). By the same method we can see that \( g \) is holomorphic with respect to other variables \( z_2, ..., z_n \). This implies that \( g \) is a nonzero holomorphic function on \( B_N \). Clearly, \( C_{\varphi, \psi} g = \varphi(w) g \), and so \( \varphi(w) \) is an eigenvalue of \( C_{\varphi, \psi} \). But it is well-known that the adjoint of a hypercyclic operator has no eigenvector, thus \( C_{\varphi, \psi}^* \) fails to be hypercyclic. Also, note that \( C_{\varphi, \psi} M_g = M_g (\varphi(w) C_{\psi}) \), and \( g \)
has no zero in $B_N$ whenever $\varphi$ never vanishes. Thus, $M_g$ is one to one and has dense range and so $C_{\varphi,\psi}$ is quasisimilar to $\varphi(w)C_{\psi}$. Now if $|\varphi(w)| = 1$ and $C_{\psi}$ is hypercyclic, then $\varphi(w)C_{\psi}$ and so $C_{\varphi,\psi}$ is also hypercyclic on $H(B_N)$. This completes the proof.

References


