International Journal of Pure and Applied Mathematics
Volume 101 No. 1 2015, 95-99
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v101i1.10

ijpam.eu

SUBSPACE CRITERIONS OF TUPLES OF OPERATORS

Bahmann Yousefi[1 §], Elham Fathi[2]
[1,2]Department of Mathematics
Payame Noor University
P.O. Box 19395-3697, Tehran, IRAN

Abstract: In this paper, we introduce subspace hypercyclicity and transitivity of tuples of operators and we give some relations between these concepts and the subspace transitivity criterion for a tuple of operators.

AMS Subject Classification: 47B37, 47B33
Key Words: tuple, subspace hypercyclicity, subspace transitivity, hypercylicity criterion

1. Introduction

By an n-tuple of operators we mean a finite sequence of length n of commuting continuous linear operators on a Banach space X.

Definition 1.1. Let $\mathcal{T} = (T_1, T_2, ..., T_n)$ be an n-tuple of operators acting on a separable infinite dimensional Banach space X over $\mathbb{C}$ and let M be a nonzero subspace of X. We will let

$$\mathcal{F} = \{{T_1}^{k_1}{T_2}^{k_2}...{T_n}^{k_n} : k_i \geq 0, i = 1, ..., n\}$$

be the semigroup generated by $\mathcal{T}$. For $x \in X$, the orbit of x under the tuple $\mathcal{T}$ is the set $Orb(\mathcal{T}, x) = \{Sx : S \in \mathcal{F}\}$. A vector x is called a subspace-hypercyclic (or M-hypercyclic) vector for $\mathcal{T}$ if $Orb(\mathcal{T}, x) \cap M$ is dense in M and in this case

Received: February 28, 2015
© 2015 Academic Publications, Ltd.
url: www.acadpubl.eu

§Correspondence author

the tuple $\mathcal{T}$ is called subspace-hypercyclic for M. The set of all M-hypercyclic vectors of $\mathcal{T}$ is denoted by $HC(\mathcal{T}, M)$. A vector x is called a M-supercyclic vector for $\mathcal{T}$ if $\mathbb{C}Orb(\mathcal{T}, x) \cap M$ is dense in M and in this case the tuple $\mathcal{T}$ is called M-supercyclic. The set of all M-supercyclic vectors of $\mathcal{T}$ is denoted by $SC(\mathcal{T}, M)$. Also, for all $k \geq 2$, by $\mathcal{T}_d^{(k)}$ we will refer to the set of all k copies of an element of $\mathcal{F}$, i.e.

$$\mathcal{T}_d^{(k)} = \{S_1 \oplus ... \oplus S_k : S_1 = ... = S_k \in \mathcal{F}\}.$$

We say that $\mathcal{T}_d^{(k)}$ is subspace-hypercyclic, with respect to M, provided there exist $x_1, ..., x_k \in X$ such that $\{W(x_1 \oplus ... \oplus x_k) : W \in \mathcal{T}_d^{(k)}\} \cap M$ is dense in the k copies of M, $M \oplus ... \oplus M$. Similarly, we say that $\mathcal{T}_d^{(k)}$ is subspace-supercyclic, with respect to M, provided there exist $x_1, ..., x_k \in X$ such that $\mathbb{C}\{W(x_1 \oplus ... \oplus x_k) : W \in \mathcal{T}_d^{(k)}\} \cap M$ is dense in the k copies of M, $M \oplus ... \oplus M$.

Definition 1.2. Suppose that $\mathcal{T} = (T_1, T_2, ..., T_n)$ is an n-tuple of operators acting on a separable infinite dimensional Banach space X over $\mathbb{C}$ and M is a nonzero subspace of X. We say that a tuple $\mathcal{T} = (T_1, T_2, ..., T_n)$ is called M-transitive with respect to a tuple of nonnegative integer sequences

$$(\{k_{j(1)}\}_j, \{k_{j(2)}\}_j, ..., \{k_{j(n)}\}_j),$$

if for every nonempty relatively open subsets U, V of X there exists $j_0 \in \mathbb{N}$ such that $T_1^{-k_{j_0(1)}} T_2^{-k_{j_0(2)}} ... T_n^{-k_{j_0(n)}}(U) \cap V$ contains a relatively open nonempty subset of M. Also, we say that an n-tuple $\mathcal{T}$ is M-transitive if it is M-transitive with respect an n-tuple of nonnegative integer sequences.

Suprisingly, there are something that does not happen for single operators. For example, hypercyclic tuples can arise in finite dimensional, and there are operators that have somewhere dense orbits that are not everywhere dense. Also, we note that there are subspace-hypercyclic operators that are not hypercyclic. For some topics we refer to [1]-[3].

2. Main Results

In this section, we investigate subspace-transitivity and subspace-supercyclicity criterions for tuples of operators.

Theorem 2.1. (Subspace-Transitivity Criterion for Tuples) *Let $\mathcal{T} = (T_1, T_2, ..., T_n)$ be a tuple of continuous operators acting on a separable infinite dimensional Banach space X. Suppose that there exist two dense subsets Y and Z in M, and strictly increasing sequences of positive integers $\{m_{j(i)}\}_j$ for $i = 1, ..., n$ such that:*

1. $T_1^{m_{j(1)}}...T_n^{m_{j(n)}} y \to 0$ for all $y \in Y$ as $j \to 0$,

2. For every $z \in Z$, there exists a sequence $\{x_j\}_j$ in M such that $x_j \to 0$ and $T_1^{m_{j(1)}}...T_n^{m_{j(n)}} x_j \to z$,

3. M is invariant subspace for $T_1^{m_{j(1)}}...T_n^{m_{j(n)}}$ for all j.

Then $\mathcal{T}$ is subspace-transitive with respect to M.

Proof. Let U and V be two nonempty relatively open subsets of M. Choose $v \in Y \bigcap V$ and $u \in Z \cap U$. Since U and V are relatively open, there exists $\epsilon > 0$ such that $B(v, \epsilon) \cap M \subset V$ and $B(u, \epsilon) \cap M \subset U$. By the hypothesis we have:

$1'$. $T_1^{m_{j(1)}}...T_n^{m_{j(n)}} v \to 0$ for all $y \in Y$ as $j \to 0$,

$2'$. There exists a sequence $\{x_j\}_j$ in M such that $x_j \to 0$ and

$$T_1^{m_{j(1)}}...T_n^{m_{j(n)}} x_j \to u,$$

and

$3'$. M is invariant subspace for $T_1^{m_{j(1)}}...T_n^{m_{j(n)}}$ for all j.

Hence for large j, we have

$$\|T_1^{m_{j(1)}}...T_n^{m_{j(n)}} v\| < \epsilon/2, \ \|x_j\| < \epsilon, \ \|T_1^{m_{j(1)}}...T_n^{m_{j(n)}} x_j - u\| < \epsilon/2.$$

Note that since $v \in M$ and $x_j \in M$, we have $v + x_j \in M$ and $\|(v + x_j) - v\| = \|x_j\| < \epsilon$. Hence $v + x_j \in V$. Also by $3'$, $T_1^{m_{j(1)}}...T_n^{m_{j(n)}}(v + x_j) \in M$ and we have

$$\|T_1^{m_{j(1)}}...T_n^{m_{j(n)}}(v + x_j) - u\| \le \|T_1^{m_{j(1)}}...T_n^{m_{j(n)}} v\| + \|T_1^{m_{j(1)}}...T_n^{m_{j(n)}} x_j - u\| < \epsilon.$$

Thus $T_1^{m_{j(1)}}...T_n^{m_{j(n)}}(v + x_j) \in B(u, \epsilon) \cap M$ and so $T_1^{m_{j(1)}}...T_n^{m_{j(n)}}(v + x_j) \in U$. Hence $v + x_j \in T_1^{-m_{j(1)}}...T_n^{-m_{j(n)}}(U) \cap V$ and so the proof is complete. □

Corollary 2.2. *The subspace-Transitivity Criterion for tuples implies subspace-hypercyclicity.*

Theorem 2.3. (Subspace-Supercyclicity Criterion for Tuples) *Let $\mathcal{T} = (T_1, T_2, ..., T_n)$ be a tuple of continuous operators acting on a separable infinite dimensional Banach space X and M be a nonzero closed subspace of X. Suppose that there exist two dense subsets Y and Z in M, and strictly increasing sequences of positive integers $\{m_{j(i)}\}_j$ for $i = 1, ..., n$ such that:*

1. For every $z \in Z$, there exists a sequence $\{x_j\}_j$ in M such that

$$T_1^{m_{\mathrm{j}(1)}}...T_n^{m_{\mathrm{j}(\mathrm{n})}}x_j \to \ z$$

and

$$\|T_1^{m_{\mathrm{j}(1)}}...T_n^{m_{\mathrm{j}(\mathrm{n})}}y\|\|x_j\| \to 0,$$

for all $y \in Y$.

2. M is invariant subspace for $T_1^{m_{\mathrm{j}(1)}}...T_n^{m_{\mathrm{j}(\mathrm{n})}}$ for all j.

Then $\mathcal{T}$ is subspace-supercyclic with respect to M.

Proof. Let U and V be two nonempty relatively open subsets of M. Choose $v \in Y \cap V$ and $u \in Z \cap U$. Since U and V are relatively open, there exists $\epsilon > 0$ such that $B(v, \epsilon) \cap M \subset V$ and $B(u, \epsilon) \cap M \subset U$. By the hypothesis we have:

$1'$. There exists a sequence $\{x_j\}_j$ in M such that $T_1^{m_{\mathrm{j}(1)}}...T_n^{m_{\mathrm{j}(\mathrm{n})}}x_j \to \ u$, and $\|T_1^{m_{\mathrm{j}(1)}}...T_n^{m_{\mathrm{j}(\mathrm{n})}}v\|\|x_j\| \to 0$.

$2'$. M is invariant subspace for $T_1^{m_{\mathrm{j}(1)}}...T_n^{m_{\mathrm{j}(\mathrm{n})}}$ for all j.

Hence for large j, we can find $\lambda_j \in \mathbb{C}\backslash\{0\}$ such that

$$\|\lambda_j T_1^{m_{\mathrm{j}(1)}}...T_n^{m_{\mathrm{j}(\mathrm{n})}}v\| < \epsilon/2, \ \|\lambda_j^{-1}x_j\| < \epsilon, \ \|T_1^{m_{\mathrm{j}(1)}}...T_n^{m_{\mathrm{j}(\mathrm{n})}}x_j - u\| < \epsilon/2.$$

Note that since $v \in M$ and $x_j \in M$, we have $v + \lambda_j^{-1}x_j \in M$ and $\|(v + \lambda_j^{-1}x_j) - v\| = \|\lambda_j^{-1}x_j\| < \epsilon$. Hence $v + x_j \in V$. Also by $2'$, $T_1^{m_{\mathrm{j}(1)}}...T_n^{m_{\mathrm{j}(\mathrm{n})}}(v + \lambda_j^{-1}x_j) \in M$ and we have

$$\begin{aligned}&\|\lambda_j T_1^{m_{\mathrm{j}(1)}}...T_n^{m_{\mathrm{j}(\mathrm{n})}}(v + \lambda_j^{-1}x_j) - u\| \\ &\qquad \leq \|\lambda_j T_1^{m_{\mathrm{j}(1)}}...T_n^{m_{\mathrm{j}(\mathrm{n})}}v\| + \|T_1^{m_{\mathrm{j}(1)}}...T_n^{m_{\mathrm{j}(\mathrm{n})}}x_j - u\| < \epsilon.\end{aligned}$$

Thus

$$\lambda_j T_1^{m_{\mathrm{j}(1)}}...T_n^{m_{\mathrm{j}(\mathrm{n})}}(v + \lambda_j^{-1}x_j) \in B(u, \epsilon) \cap M$$

and so

$$\lambda_j T_1^{m_{\mathrm{j}(1)}}...T_n^{m_{\mathrm{j}(\mathrm{n})}}(v + \lambda_j^{-1}x_j) \in U.$$

Hence

$$v + \lambda_j^{-1}x_j \in \lambda_j^{-1}T_1^{-m_{\mathrm{j}(1)}}...T_n^{-m_{\mathrm{j}(\mathrm{n})}}(U) \cap V$$

and so the proof is complete. □

References

[1] N.S. Feldman, Hypercyclic tuples of operators and somewhere dense orbits, *J. Math. Appl.*, **346** (2008), 82-98.

[2] B.F. Madore, R.A. Martinez-Avendano, Subsapce hypercyclicity, *Journal of Mathematical Analysis and Applications*, **375**, No. 2 (2011), 502-511.

[3] B. Yousefi, Hereditarily transitive tuples, *Rend. Circ. Mat. Palermo*, Volume **2011**, **doi:** 10.1007/S12215-011-0066-y.