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ORDINARY SPECIAL WEIERSTRASS n-SEMIGROUPS

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Abstract: We study ordinary Weierstrass n-semigroups on genus g curves, which are defined as the n-semigroups such that $(a_1, \dots, a_n)$ is a gap if $0 < a_1 + \cdots + a_n < g$, and a non-gap if $a_1 + \cdots + a_n \geq g + 2$. Most of the results are for curves with general moduli.

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1. Ordinary Weierstrass n-Semigroups

Let X be a smooth and connected projective curve of genus $g \geq 3$ defined over an algebraically closed field with characteristic zero. Fix $P_1, \dots, P_n \in X$ such that $P_i \neq P_j$ for all $i \neq j$. Let $H(P_1, \dots, P_n) \subset \mathbb{N}^n$ be the set of all n-ples $(a_1, \dots, a_n) \in \mathbb{N}^n$ such that there is a rational function on X with $a_1P_1 + \cdots + a_nP_n$ as its divisor of poles ([1], [2]). The set $H(P_1, \dots, P_n)$ is a semigroup (called the Weierstrass semigroup of $P_1, \dots, P_n$) for the componentwise addition $+ : \mathbb{N}^n \times \mathbb{N}^n \to \mathbb{N}^n$. The elements of the finite set $G(P_1, \dots, P_n) := \mathbb{N}^n \setminus H(P_1, \dots, P_n)$ are called the gaps of $(P_1, \dots, P_n)$. Note that g is uniquely determined by $H(P_1, \dots, P_n)$ by the formula $g = \sharp(G(P_1, \dots, P_n) \cap \mathbb{N}^1 \times \{(0, \dots,$

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$0)\}$. For any $a = (a_1, \dots, a_n) \in \mathbb{N}^n$ set $\|a\| := a_1 + \cdots + a_n$. For any $i \in \{1, \dots, n\}$ set $e_i := (a_1, \dots, a_n)$ with $a_j = 0$ if $j \neq i$ and $e_i = 1$. For all $a = (a_1, \dots, a_n), b = (b_1, \dots, b_n) \in \mathbb{N}^n$ set $a \leq b$ if and only if $a_i \leq b_i$ for all i. Write $a < b$ if $a \leq b$ and $a \neq b$. Let $A \subset \mathbb{N}^n$ be a semigroup. We say that $a, b \in A$ are consecutive if $a < b$ and there is no $c \in A$ with $a < b < c$. In this way we get the notion of maximal chains between ta, b with $a < b$. A is said to be *catenary* if for all $a, b \in A$ with $a < b$ any two maximal chains between a and b are catenary. $H(P_1, \dots, P_n)$ is catenary because all integers $h^0(\mathcal{O}_X(a_1P_1 + \cdots + a_nP_n))$ and $h^1(\mathcal{O}_X(a_1P_1 + \cdots + a_nP_n))$, $(a_1, \dots, a_n) \in \mathbb{N}^n$ are known in terms of g, the integer $a_1 + \cdots + a_n$ and the partial relation $\leq$ in $H(P_1, \dots, P_n)$. If $a = (a_1, \dots, a_n) \neq 0$, then $h^0(\mathcal{O}_X(a_1P_1 + \cdots + a_nP_n)) - 1$ is the number of elements of $H(P_1, \dots, P_n)$ in a maximal chain from 0 to a, while $h^1(\mathcal{O}_X(a_1P_1 + \cdots + a_nP_n)) = h^0(\mathcal{O}_X(a_1P_1 + \cdots + a_nP_n)) + g - 1 - \|a\|$ (Riemann-Roch).

Let $A \subset \mathbb{N}^n$ with $\mathbb{N}^n \setminus A$ finite. We say that A has *genus* g if $\sharp(\mathbb{N}^n \setminus A) \cap \mathbb{N}e_i = g$ for all i, $A \supset \{\|a\| \geq 2g\}$, it is catenary and it satisfies the following condition ♠:

♠: if $\|a\| = 2g - 2$ and $a + e_i \notin A$ for some i, then $a \in A$ and $a + e_j \notin A$ for all $j = 1, \dots, n$.

$H(P_1, \dots, P_n)$ satisfies ♠, because $|\omega_X|$ has no base points and $a = (a_1, \dots, a_n) \in \mathbb{N}^n$ with $\|a\| = 2g - 2$ and $a + e_i \in G(P_1, \dots, P_n)$ for some $i \Leftrightarrow a_1P_1 + \cdots + a_nP_n \in |\omega_X| \Leftrightarrow a + e_i \in G(P_1, \dots, P_n)$ for for all i.

Set $w(P_1, \dots, P_n) := \sum_{(a_1, \dots, a_n) \in \mathbb{N}^n} h^1(\mathcal{O}_X(a_1P_1 + \cdots + a_nP_n)) - \binom{g+n}{n+1}$ (the *weight*) and $v(P_1, \dots, P_n) := \sum_{(a_1, \dots, a_n) \in H(P_1, \dots, P_n) \setminus \{0\}} h^1(\mathcal{O}_X(a_1P_1 + \cdots + a_nP_n))$ (the *gist*) of $H(P_1, \dots, P_n)$. The *non-special* semigroup is the semigroup $\{a \in \mathbb{N}^n : \|a\| \geq g + 1\}$. This is the only n-semigroup with $w(P_1, \dots, P_n) = 0$ and the only n-semigroup with $v(P_1, \dots, P_n) = 0$.

Definition 1. We say that $H(P, \dots, P_n)$ is *ordinary* if all $a \in \mathbb{N}^n \setminus \{0\}$ with $\|a\| < g$ are gaps, $H(P_1, \dots, P_n)$ contains all a with $\|a\| \geq g + 2$ and it is not non-special, i.e. there is some $a \in H(P_1, \dots, P_n)$ with $\|a\| = g$.

For an ordinary n-semigroup we have $w(P_1, \dots, P_n) = v(P_1, \dots, P_n) = \sharp(\{a \in H(P_1, \dots, P_n) : \|a\| = g\})$ and $H(P_1, \dots, P_n)$ is uniquely determined by the set $E(P_1, \dots, P_n) := \{a \in H(P_1, \dots, P_n) : \|a\| = g\}$ (Remark 1). The notion of ordinary n-semigroup is well-defined for an arbitrary n-semigroup, i.e. a semigroup not coming from $X, P_1, \dots, P_n$. When $n > 1$ ordinary Weierstrass n-semigroups are very restricted among the ordinary n-semigroups of genus g (see Remark 2). We say that $H(P_1, \dots, P_n)$ is *strongly ordinary* if it is ordinary and $\sharp(E(P_1, \dots, P_n)) = 1$.

Question 1. Fix g and n. Which are the possible sets $E(P_1,\dots,P_n)$ for some X of genus g and some $P_1,\dots,P_n \in X$ with $H(P_1,\dots,P_n)$ ordinary? Among all g,n and all ordinary $H(P_1,\dots,P_n)$ with X of genus g call $\alpha(g,n)$ the maximal cardinality of a set $E(P_1,\dots,P_n)$. Describe the asymptotic shape of the function $\alpha(g,n)$, either fixing g and taking $n \to +\infty$ or fixing n and taking $g \to +\infty$?

2. Properties of Ordinary n-Semigroups and their Existence

For any $a = (a_1,\dots,a_n) \in \mathbb{N}^n$ let $\mathrm{supp}(a)$ denote the set of all $i \in \{1,\dots,n\}$ such that $a_i \neq 0$.

Remark 1. Let $H(P_1,\dots,P_n)$ be an ordinary semigroup.

Observation 1. *Fix $a = (a_1,\dots,a_n)$ with $\|a\| = g$. Since $\{\|b\| \leq g-1\} \subset G(P_1,\dots,P_n)$, then: $a \in E(P_1,\dots,P_n) \Leftrightarrow h^0(\mathcal{O}_X(a_1,\dots,a_n)) = 2 \Leftrightarrow h^1(\mathcal{O}_X(a_1,\dots,a_n)) = 1$.* □

Claim 1. *Fix $a = (a_1,\dots,a_n) \in E(P_1,\dots,P_n)$. Then $a+e_i \in G(P_1,\dots,P_n)$ for all $1 \leq i \leq n$.*

Proof of Claim 1. $a+e_i \in H(P_1,\dots,P_n)$ if an only if $h^0(\mathcal{O}_X(a_1P_1+\cdots+a_nP_n+P_i)) = 3$ and $h^1(\mathcal{O}_X(a_1P_1+\cdots+a_nP_n+P_i) = 1$. Since $\|a+(x+1)e_i\| = g+1+x \geq g+2$ for all $x > 0$, we have $a+(x+1)e_i \in H(P_1,\dots,P_n)$. By induction on x we get $h^0(\mathcal{O}_X(a_1P_1+\cdots+a_nP_n+(x+1)P_i)) = 3+x$ and $h^1(\mathcal{O}_X(a_1P_1+\cdots+a_nP_n+(x+1)P_i)) = 1$, which is obviously false if $x \geq g-1$. □

Claim 2. *We have $h^1(\mathcal{O}_X(b_1P_1+\cdots,b_nP_n)) = 0$ for all $b = (b_1,\dots,b_n) \in \mathbb{N}^n$ with $b_1+\cdots+b_n = g+1$.*

Proof of Claim 2. Fix $b = (b_1,\dots,b_n)$ with $b_1+\cdots+b_n = g+1$. Take $i \in \{1,\dots,n\}$ such that $b_i \neq 0$. If $b-e_i \in G(P_1,\dots,P_n)$, then use Claim 1. If $(c_1,\dots,c_n) := b-e_i \in H(P_1,\dots,P_n)$, then $h^1(\mathcal{O}_X(c_1P_1+\cdots+c_nP_n)) = 0$ and hence $h^1(\mathcal{O}_X(b_1P_1+\cdots,b_nP_n)) = 0$. □

Claim 2 gives the following result.

Observation 2. $w(P_1,\dots,P_n) = v(P_1,\dots,P_n) = \sharp(E(P_1,\dots,P_n))$. □

Claim 3. Fix $b = (b_1, \dots, b_n)$ with $\|b\| = g + 1$. We have $b \in G(P_1, \dots, P_n)$ $\Leftrightarrow b - e_i \in G(P_1, \dots, P_n)$ for all i with $b_i > 0 \Leftrightarrow b - e_i \in G(P_1, \dots, P_n)$ for some i with $b_i > 0$.

Proof of Claim 3. *The "only if" part is true by Claim 1. For the " if " part use Observation 1.* □

We say that a genus g semigroup is *ordinary* if it satisfies all claims and observations of Remark 2.

Remark 2. The total weight of all Weierstrass points on a smooth curve X of genus g is $g^3 - g^2$. Hence if $n > g^3 - g^2$ no $H(P_1, \dots, P_n)$ may have $xe_i \in G(P_1, \dots, P_n)$ if and only if $x \le g - 1$ or $x = g$ for all i.

Theorem 1. *Fix integers $g \ge 3$ and $n \ge 2$ and take $a \in \mathbb{N}^n$ such that $\|a\| = g$. Let X be a general smooth curve of genus g. Then there are $P_1, \dots, P_n \in X$ such that $P_i \neq P_j$ for all $i \neq j$, $H(P_1, \dots, P_n)$ is strongly ordinary and $E(P_1, \dots, P_n) = \{a\}$.*

Proof. Write $a = (a_1, \dots, a_n)$.

(a) In this step we assume $\mathrm{supp}(a) = \{1, \dots, n\}$. This assumption implies $n \le g$. First assume $n = 1$. In this case we not only have the existence part, but that (X, P) is in the smooth locus of the Weierstrass subset of $\mathcal{M}_{g,1}$, i.e. in the set of all ramification points of the relative dualizing sheaf. The case $n > 1$ is obtained deforming $(gP, 0, \dots, 0)$ into $(a_1P_1, a_2P_2, \dots, a_nP_n)$ inside the ramification divisors of the relative dualizing sheaf. At the very least we get $P_1, \dots, P_n \in X$ such that $P_i \neq P_j$ for all $i \neq j$, $\mathcal{O}_X(a_1P_1 + \cdots + a_nP_n)$ is spanned, $h^0(\mathcal{O}_X(a_1P_1 + \cdots + a_nP_n)) = 2$ and that for all $i = 1, \dots, n$ we have $h^1(\mathcal{O}_X(b_1P_1 + \cdots + b_nP_n)) = 0$, where $(b_1, \dots, b_n) = a + e_i$. We also get that the set $T(a)$ of all ordinary $(X, P_1, \dots, P_n)$ with $E(P_1, \dots, P_n) \supseteq \{a\}$ has dimension $3g - 3 + n - 1$. To conclude we only need to check that $E(P_1, \dots, P_n) = \{a\}$ for a general $(X, P_1, \dots, P_n) \in T(a)$, i.e. that if $b \neq a$, then $T(a)$ and $T(b)$ have no common component of dimension $3g - 3 + n - 1$. Since $\mathrm{supp}(a) = \mathrm{supp}(b) = \{1, \dots, n)$, $a \neq b$, and $\|a\| = \|b\|$, we may, permuting the points $P_1, \dots, P_n$, to assume $a_n \neq b_n$. Fix a general $(X, P_1, \dots, P_{n-1}) \in \mathcal{M}_{g,n-1}$. It is sufficient to prove the existence of $P_n, Q_n \in X \setminus \{P_1, \dots, P_{n-1}\}$ such that $H(P_1, \dots, P_n)$ is ordinary with $a \in E(P_1, \dots, P_n)$, $H(P_1, \dots, P_{n-1}, Q_n)$ is ordinary with $b \in E(P_1, \dots, P_{n-1}, Q_n)$ and $P_n \neq Q_n$. Since $\|a\| = \|b\|$ and $a_n \neq b_n$, we have $a_1 + \cdots + a_{n-1} = g - a_n \neq g - b_n$. Set $R := \omega_X(-a_1P_1 + \cdots - a_{n-1}P_{n-1})$ and $L := \omega_X(-b_1P_1 + \cdots - b_{n-1}P_{n-1})$. Since we are in characteristic zero and $P_1, \dots, P_{n-1}$ are general, $h^1(R) = 1$ and $h^1(L) = 1$, i.e. $h^0(R) = a_n$ and

$h^0(L) = b_n$. Assume for the moment $a_n > 1$ and $b_n > 1$. By the Brill-Segre formula ([4, Theorem 15 (iv)]), R (resp. L) has $a_n((a_n-1)(g-1)+g-2+a_n)$ (resp. $b_n((b_n-1)(g-1)+g-2+b_n)$) ramification points. Since this ramification points come from deformations of the ramification points of $\omega_X(-gP+a_nP)$ and $\omega_X(-gP+b_nP)$, these ramification points are simple ramification points. Since $a_n \neq b_n$ we may find ramification points P_n of R and Q_n of L with $Q_n \neq P_n$. Now assume that $\min\{a_n, b_n\} = 1$, say $b_n = 1$. In this case $|L|$ has a unique divisor of degree $g-1$ and hence it is sufficient to use that (since $a_n > 1$) R has at least g ramification points.

(b) Now assume $\mathrm{supp}(a) \subsetneq \{1, \dots, n\}$. With no loss of generality we may assume $\mathrm{supp}(a) = \{1, \dots, m\}$. Write $a = (a', 0, \dots, 0)$ with $a' \in \mathbb{N}^m$. Take $P_1, \dots, P_m$ such that $H(P_1, \dots, P_m)$ is the only ordinary m-semigroup with $\{a'\} = E(P_1, \dots, P_m)$. Since we are in characteristic zero, by [4, Theorem 15] it is sufficient to take as $P_{m+1}, \dots, P_n$ and general $(P_{m+1}, \dots, P_n) \in X^{n-m}$. $\square$

Theorem 2. *Fix integers $g \geq 3$ and $n \geq 2$.*

(a) If $n > g^3 - g^2$, then there is no ordinary $(X, P_1, \dots, P_n)$ with $ge_i \in E(P_1, \dots, P_n)$ for all i.

(b) If $2 \leq m \leq g^3 - g^2$, $n \geq m$, and X is general, then there are $P_1, \dots, P_n \in X$, $P_i \neq P_j$ for all $i \neq j$, such that $H(P_1, \dots, P_n)$ is ordinary and $E(P_1, \dots, P_n) = \{ge_i\}_{1 \leq i \leq m}$.

Proof. Part (a) follows from the Brill-Segre formula for the canonical line bundle, which says that $g^3 - g^2$ is the total weight of all Weiestrass points of a genus g curve. Now assume that X is general. All its Weierstrass points are ordinary and $g^3 - g^2$ is their number. Fix n, m with $2 \leq m \leq g^3 - g^2$, $n \geq m$, and take as $P_1, \dots, P_m$ any m distinct Weierstrass points.

(a) Assume $n = m$. We have $h^0(\mathcal{O}_X(tP_i)) = 1$ if $0 \leq t \leq g-1$, $h^1(\mathcal{O}_X(gP_i)) = 2$, $h^1(\mathcal{O}_X(gP_i)) = 0$ and $h^1(\mathcal{O}_X(tP_i)) = 0$ for all $t > g$. Therefore to prove that $H(P_1, \dots, P_n)$ is ordinary and that $E(P_1, \dots, P_n) = \{ge_i\}_{1 \leq i \leq n}$ it is sufficient to prove that $h^1(\mathcal{O}_X(a_1P_1 + \cdots + a_nP_n) = 0$ if either $a_1 + \cdots + a_n \geq g+1$ or $a_1 + \cdots + a_n = g$ and $a_i < g$ for all i. Since $h^1(\mathcal{O}_X(gP_i)) = 0$ for all i, it is sufficient to prove that $h^1(\mathcal{O}_X(a_1P_1 + \cdots + a_nP_n)) = 0$ if $a_1 + \cdots + a_n = g$ and $a_i < g$ for all i. Assume the existence of $a = (a_1, \dots, a_n)$ with $a_1 + \cdots + a_n = g$, $a_i < g$ for all i, and $h^1(\mathcal{O}_X(a_1P_1 + \cdots + a_nP_n)) > 0$. Among these $a \in \mathbb{N}^n$ take one such that the integer $c := \sharp(\mathrm{supp}(a))$ is minimal. We have $2 \leq c \leq g$. With no loss of generality we may assume $\text{supp}(a) = \{1, \dots, c\}$. Set $L := \omega_X(-\sum_{i=1}^{c-1} a_i P_i)$. We have $\deg(L) = g - 2 + a_n$.

(a1) Assume $c = 2$ and $a_2 = 1$. Since P_1 is an ordinary Weierstrass point, we have $h^1(\mathcal{O}_X((g-1)P_1)) = 1$. Let D be the only element of $|\omega_X(-(g-1)P_1)|$. Since $h^1(\mathcal{O}_X((g-1)P_1 + P_2)) > 0$, P_2 is in the support of D. Since the monodromy group of the Weierstrass points is the full symmetric group ([3]), we get that all the Weierstrass points of X different from P_1 are in the support of D. Hence $g - 1 = \deg(D) > g^3 - g^2 - 1$, a contradiction.

(a2) Assume $c = 2$ and $a_2 > 1$. Since P_1 is ordinary, we have $h^0(L) = a_2$. Since $h^1(\mathcal{O}_X(a_1P_1 + \cdots + a_nP_n)) > 0$, P_2 is a ramification point of L. Since the monodromy group of the Weierstrass points is the full symmetric group ([3]), all the Weierstrass points of X different from P_1 are ramification points of $|L|$. The Brill-Segre formula for L gives that the ramification points of $|L|$ are at most $a_2((a_2-1)(g-1)+(g-2+a_2)) \leq (g-1)((g-2)(g-1)+2g-2) < g^3-g^2-1$, a contradiction.

(a3) Assume $c > 2$ and $a_c = 1$. Since $c > 2$, the minimality property for c shows that $h^1(\mathcal{O}_X(a_1P_1 + \cdots + (a_{c-1} + a_c)P_{c-1}) = 0$. Hence $h^0(L) = 1$. We get that P_c is in the base locus of L. Since the monodromy group of the general Weierstrass points is the full symmetric group ([3]), we get that all Weierstrass points, except at most $P_1, \dots, P_{c-1}$, are in this base locus. We get a contradiction, because $\deg(L) = g - 1 < g^3 - g^2 - (c-1)$, by our choice of the integer c.

(a4) Assume $c > 2$ and $a_c > 1$. Since $c > 2$, the minimality property for c shows that $h^1(\mathcal{O}_X(a_1P_1+\cdots+(a_{c-1}+a_c)P_{c-1})) = 0$. Hence $h^0(L) = a_c$. We use again the Brill-Segre formula. The definition of c gives $a_c \leq g+1-c$. Hence the number of ramification points of $|L|$ is at most $a_c((a_c-1)(g-1)+(g-2+a_c)) \leq (g+1-c)((g-c)(g-1)+2g-1-c)$. Since the monodromy group of the general Weierstrass points is the full symmetric group ([3]), all Weierstrass points of X, except at most $c-1$, are ramification points of $|L|$. Therefore it is sufficient to prove that for all $c = 3, \dots, g-1$ we have $(g+1-c)((g-c)(g-1)+2g-1-c) \leq g^3 - g^2 - c$. Call $u(g, c)$ the difference between the right hand side and the left hand side of the last inequality. Since $u(g, c)$ in an increasing function of c, it is sufficient to use that $u(g, 3) = g^3-g^2-3-(g-2)(g-3)(g-1)-(g-2)(2g-4) = g^3 - g^2 - 3 - g^3 + 6g^2 - g + 6 - 2g^2 + 8g - 8 > 0$.

(b) Now assume $n > m$. Take m distinct Weierstrass points $P_1, \dots, P_m$, apply to them part (a) and then take a general $(P_{m+1}, \dots, P_n) \in X^{n-m}$. Apply [4, Theorem 15]. $\square$

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