ORDINARY SPECIAL WEIERSTRASS $n$-SEMIGROUPS

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Abstract: We study ordinary Weierstrass $n$-semigroups on genus $g$ curves, which are defined as the $n$-semigroups such that $(a_1, \ldots, a_n)$ is a gap if $0 < a_1 + \cdots + a_n < g$, and a non-gap if $a_1 + \cdots + a_n \geq g + 2$. Most of the results are for curves with general moduli.

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1. Ordinary Weierstrass $n$-Semigroups

Let $X$ be a smooth and connected projective curve of genus $g \geq 3$ defined over an algebraically closed field with characteristic zero. Fix $P_1, \ldots, P_n \in X$ such that $P_i \neq P_j$ for all $i \neq j$. Let $H(P_1, \ldots, P_n) \subset \mathbb{N}^n$ be the set of all $n$-ples $(a_1, \ldots, a_n) \in \mathbb{N}^n$ such that there is a rational function on $X$ with $a_1 P_1 + \cdots + a_n P_n$ as its divisor of poles ([1], [2]). The set $H(P_1, \ldots, P_n)$ is a semigroup (called the Weierstrass semigroup of $P_1, \ldots, P_n$) for the componentwise addition $+: \mathbb{N}^n \times \mathbb{N}^n \to \mathbb{N}^n$. The elements of the finite set $G(P_1, \ldots, P_n) := \mathbb{N}^n \setminus H(P_1, \ldots, P_n)$ are called the gaps of $(P_1, \ldots, P_n)$. Note that $g$ is uniquely determined by $H(P_1, \ldots, P_n)$ by the formula $g = \#(G(P_1, \ldots, P_n) \cap \mathbb{N}^1 \times \{0, \ldots,$
0). For any \( a = (a_1, \ldots, a_n) \in \mathbb{N}^n \) set \( \|a\| := a_1 + \cdots + a_n \). For any \( i \in \{1, \ldots, n\} \) set \( e_i := (a_1, \ldots, a_n) \) with \( a_j = 0 \) if \( j \neq i \) and \( e_i = 1 \). For all \( a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n) \in \mathbb{N}^n \) set \( a \leq b \) if and only if \( a_i \leq b_i \) for all \( i \). Write \( a < b \) if \( a \leq b \) and \( a \neq b \). Let \( A \subset \mathbb{N}^n \) be a semigroup. We say that \( a, b \in A \) are consecutive if \( a < b \) and there is no \( c \in A \) with \( a < c < b \). In this way we get the notion of maximal chains between \( ta, b \) with \( a < b \). \( A \) is said to be catenary if for all \( a, b \in A \) with \( a < b \) any two maximal chains between \( a \) and \( b \) are catenary. \( H(P_1, \ldots, P_n) \) is catenary because all integers \( h^0(O_X(a_1P_1 + \cdots + a_nP_n)) \) and \( h^1(O_X(a_1P_1 + \cdots + a_nP_n)) \), \( (a_1, \ldots, a_n) \in \mathbb{N}^n \) are known in terms of \( g \), the integer \( a_1 + \cdots + a_n \) and the partial relation \( \leq \) in \( H(P_1, \ldots, P_n) \). If \( a = (a_1, \ldots, a_n) \neq 0 \), then \( h^0(O_X(a_1P_1 + \cdots + a_nP_n)) - 1 \) is the number of elements of \( H(P_1, \ldots, P_n) \) in a maximal chain from 0 to \( a \), while \( h^1(O_X(a_1P_1 + \cdots + a_nP_n)) = h^0(O_X(a_1P_1 + \cdots + a_nP_n)) + g - 1 - \|a\| \) (Riemann-Roch).

Let \( A \subset \mathbb{N}^n \) with \( \mathbb{N}^n \setminus A \) finite. We say that \( A \) has genus \( g \) if \( \sharp (\mathbb{N}^n \setminus A) \cap \mathbb{N}e_i = g \) for all \( i \), \( A \supset \{\|a\| \geq 2g\} \), it is catenary and it satisfies the following condition ♠:

♠: if \( \|a\| = 2g - 2 \) and \( a + e_i \notin A \) for some \( i \), then \( a \in A \) and \( a + e_j \notin A \) for all \( j = 1, \ldots, n \).

\( H(P_1, \ldots, P_n) \) satisfies ♠, because \( |\omega_X| \) has no base points and \( a = (a_1, \ldots, a_n) \in \mathbb{N}^n \) with \( \|a\| = 2g - 2 \) and \( a + e_i \in G(P_1, \ldots, P_n) \) for some \( i \Leftrightarrow a_1P_1 + \cdots + a_nP_n \in |\omega_X| \Leftrightarrow a + e_i \in G(P_1, \ldots, P_n) \) for all \( i \).

Set \( w(P_1, \ldots, P_n) := \sum_{(a_1, \ldots, a_n) \in \mathbb{N}^n} h^1(O_X(a_1P_1 + \cdots + a_nP_n)) - (g+n) \) (the weight) and \( v(P_1, \ldots, P_n) := \sum_{(a_1, \ldots, a_n) \in \mathbb{H}(P_1, \ldots, P_n) \setminus \{0\}} h^1(O_X(a_1P_1 + \cdots + a_nP_n)) \) (the gist) of \( H(P_1, \ldots, P_n) \). The non-special semigroup is the semigroup \( \{a \in \mathbb{N}^n : \|a\| \geq g + 1\} \). This is the only n-semigroup with \( w(P_1, \ldots, P_n) = 0 \) and the only n-semigroup with \( v(P_1, \ldots, P_n) = 0 \).

**Definition 1.** We say that \( H(P_1, \ldots, P_n) \) is ordinary if all \( a \in \mathbb{N}^n \setminus \{0\} \) with \( \|a\| < g \) are gaps, \( H(P_1, \ldots, P_n) \) contains all \( a \) with \( \|a\| \geq g + 2 \) and it is not non-special, i.e. there is some \( a \in H(P_1, \ldots, P_n) \) with \( \|a\| = g \).

For an ordinary n-semigroup we have \( w(P_1, \ldots, P_n) = v(P_1, \ldots, P_n) = \sharp \{a \in H(P_1, \ldots, P_n) : \|a\| = g\} \) and \( H(P_1, \ldots, P_n) \) is uniquely determined by the set \( E(P_1, \ldots, P_n) := \{a \in H(P_1, \ldots, P_n) : \|a\| = g\} \) (Remark 1). The notion of ordinary n-semigroup is well-defined for an arbitrary n-semigroup, i.e. a semigroup not coming from \( X, P_1, \ldots, P_n \). When \( n > 1 \) ordinary Weierstrass n-semigroups are very restricted among the ordinary n-semigroups of genus \( g \) (see Remark 2). We say that \( H(P_1, \ldots, P_n) \) is strongly ordinary if it is ordinary and \( \sharp (E(P_1, \ldots, P_n)) = 1 \).
**Question 1.** Fix $g$ and $n$. Which are the possible sets $E(P_1, \ldots, P_n)$ for some $X$ of genus $g$ and some $P_1, \ldots, P_n \in X$ with $H(P_1, \ldots, P_n)$ ordinary? Among all $g, n$ and all ordinary $H(P_1, \ldots, P_n)$ with $X$ of genus $g$ call $\alpha(g, n)$ the maximal cardinality of a set $E(P_1, \ldots, P_n)$. Describe the asymptotic shape of the function $\alpha(g, n)$, either fixing $g$ and taking $n \to +\infty$ or fixing $n$ and taking $g \to +\infty$?

**2. Properties of Ordinary $n$-Semigroups and their Existence**

For any $a = (a_1, \ldots, a_n) \in \mathbb{N}^n$ let supp$(a)$ denote the set of all $i \in \{1, \ldots, n\}$ such that $a_i \neq 0$.

**Remark 1.** Let $H(P_1, \ldots, P_n)$ be an ordinary semigroup.

**Observation 1.** Fix $a = (a_1, \ldots, a_n)$ with $\|a\| = g$. Since $\{\|b\| \leq g - 1\} \subset G(P_1, \ldots, P_n)$, then: $a \in E(P_1, \ldots, P_n) \iff h^0(\mathcal{O}_X(a_1, \ldots, a_n)) = 2 \iff h^1(\mathcal{O}_X(a_1, \ldots, a_n)) = 1$.

**Claim 1.** Fix $a = (a_1, \ldots, a_n) \in E(P_1, \ldots, P_n)$. Then $a + e_i \in G(P_1, \ldots, P_n)$ for all $1 \leq i \leq n$.

*Proof of Claim 1.* $a + e_i \in H(P_1, \ldots, P_n)$ if an only if $h^0(\mathcal{O}_X(a_1 P_1 + \cdots + a_n P_n + P_i)) = 3$ and $h^1(\mathcal{O}_X(a_1 P_1 + \cdots + a_n P_n + P_i)) = 1$. Since $\|a + (x + 1)e_i\| = g + 1 + x \geq g + 2$ for all $x > 0$, we have $a + (x + 1)e_i \in H(P_1, \ldots, P_n)$. By induction on $x$ we get $h^0(\mathcal{O}_X(a_1 P_1 + \cdots + a_n P_n + (x + 1)P_i)) = 3 + x$ and $h^1(\mathcal{O}_X(a_1 P_1 + \cdots + a_n P_n + (x + 1)P_i)) = 1$, which is obviously false if $x \geq g - 1$.

**Claim 2.** We have $h^1(\mathcal{O}_X(b_1 P_1 + \cdots, b_n P_n)) = 0$ for all $b = (b_1, \ldots, b_n) \in \mathbb{N}^n$ with $b_1 + \cdots + b_n = g + 1$.

*Proof of Claim 2.* Fix $b = (b_1, \ldots, b_n)$ with $b_1 + \cdots + b_n = g + 1$. Take $i \in \{1, \ldots, n\}$ such that $b_i \neq 0$. If $b - e_i \in G(P_1, \ldots, P_n)$, then use Claim 1. If $(c_1, \ldots, c_n) := b - e_i \in H(P_1, \ldots, P_n)$, then $h^1(\mathcal{O}_X(c_1 P_1 + \cdots + c_n P_n)) = 0$ and hence $h^1(\mathcal{O}_X(b_1 P_1 + \cdots, b_n P_n)) = 0$.

Claim 2 gives the following result.

**Observation 2.** $w(P_1, \ldots, P_n) = v(P_1, \ldots, P_n) = \sharp(E(P_1, \ldots, P_n))$. \hfill \□
Claim 3. Fix \( b = (b_1, \ldots, b_n) \) with \( \|b\| = g + 1 \). We have \( b \in G(P_1, \ldots, P_n) \) if and only if \( b - e_i \in G(P_1, \ldots, P_n) \) for all \( i \) with \( b_i > 0 \) or \( b - e_i \in G(P_1, \ldots, P_n) \) for some \( i \) with \( b_i > 0 \).

Proof of Claim 3. The “only if” part is true by Claim 1. For the “if” part use Observation 1.

We say that a genus \( g \) semigroup is ordinary if it satisfies all claims and observations of Remark 2.

Remark 2. The total weight of all Weierstrass points on a smooth curve \( X \) of genus \( g \) is \( g^3 - g^2 \). Hence if \( n > g^3 - g^2 \) no \( H(P_1, \ldots, P_n) \) may have \( x e_i \in G(P_1, \ldots, P_n) \) if and only if \( x \leq g - 1 \) or \( x = g \) for all \( i \).

Theorem 1. Fix integers \( g \geq 3 \) and \( n \geq 2 \) and take \( a \in \mathbb{N}^n \) such that \( \|a\| = g \). Let \( X \) be a general smooth curve of genus \( g \). Then there are \( P_1, \ldots, P_n \in X \) such that \( P_i \neq P_j \) for all \( i \neq j \), \( H(P_1, \ldots, P_n) \) is strongly ordinary and \( E(P_1, \ldots, P_n) = \{a\} \).

Proof. Write \( a = (a_1, \ldots, a_n) \).

(a) In this step we assume \( \text{supp}(a) = \{1, \ldots, n\} \). This assumption implies \( n \leq g \). First assume \( n = 1 \). In this case we not only have the existence part, but that \( (X, P) \) is in the smooth locus of the Weierstrass subset of \( \mathcal{M}_{g,1} \), i.e. in the set of all ramification points of the relative dualizing sheaf. The case \( n > 1 \) is obtained deforming \((gP, 0, \ldots, 0)\) into \((a_1P_1, a_2P_2, \ldots, a_nP_n)\) inside the ramification divisors of the relative dualizing sheaf. At the very least we get \( P_1, \ldots, P_n \in X \) such that \( P_i \neq P_j \) for all \( i \neq j \), \( \mathcal{O}_X(a_1P_1 + \cdots + a_nP_n) \) is spanned, \( h^0(\mathcal{O}_X(a_1P_1 + \cdots + a_nP_n)) = 2 \) and that for all \( i = 1, \ldots, n \) we have \( h^1(\mathcal{O}_X(b_1P_1 + \cdots + b_nP_n)) = 0 \), where \( (b_1, \ldots, b_n) = a + e_i \). We also get that the set \( T(a) \) of all ordinary \((X, P_1, \ldots, P_n)\) with \( E(P_1, \ldots, P_n) \supseteq \{a\} \) has dimension \( 3g - 3 + n - 1 \). To conclude we only need to check that \( E(P_1, \ldots, P_n) = \{a\} \) for a general \((X, P_1, \ldots, P_n) \in T(a) \), i.e. that if \( b \neq a \), then \( T(a) \) and \( T(b) \) have no common component of dimension \( 3g - 3 + n - 1 \). Since \( \text{supp}(a) = \text{supp}(b) = \{1, \ldots, n\}, a \neq b \), and \( \|a\| = \|b\| \), we may, permuting the points \( P_1, \ldots, P_n \), to assume \( a_n \neq b_n \). Fix a general \((X, P_1, \ldots, P_{n-1}) \in \mathcal{M}_{g,n-1} \). It is sufficient to prove the existence of \( P_n, Q_n \in X \setminus \{P_1, \ldots, P_{n-1}\} \) such that \( H(P_1, \ldots, P_n) \) is ordinary with \( a \in E(P_1, \ldots, P_n) \), \( H(P_1, \ldots, P_{n-1}, Q_n) \) is ordinary with \( b \in E(P_1, \ldots, P_{n-1}, Q_n) \) and \( P_n \neq Q_n \). Since \( \|a\| = \|b\| \) and \( a_n \neq b_n \), we have \( a_1 + \cdots + a_{n-1} = g - a_n \neq g - b_n \). Set \( R := \omega_X(-a_1P_1 + \cdots - a_{n-1}P_{n-1}) \) and \( L := \omega_X(-b_1P_1 + \cdots - b_{n-1}P_{n-1}) \). Since we are in characteristic zero and \( P_1, \ldots, P_{n-1} \) are general, \( h^1(R) = 1 \) and \( h^1(L) = 1 \), i.e. \( h^0(R) = a_n \) and \( h^0(L) = b_n \),
$h^0(L) = b_n$. Assume for the moment $a_n > 1$ and $b_n > 1$. By the Brill-Noether formula ([4, Theorem 15 (iv)]), $R$ (resp. $L$) has $a_n((a_n - 1)(g - 1) + g - 2 + n)$ (resp. $b_n((b_n - 1)(g - 1) + g - 2 + n)$) ramification points. Since this ramification points come from deformations of the ramification points of $\omega_X(-gP + a_iP)$ and $\omega_X(-gP + b_iP)$, these ramification points are simple ramification points. Since $a_n \neq b_n$ we may find ramification points $P_n$ of $R$ and $Q_n$ of $L$ with $Q_n \neq P_n$. Now assume that $\min \{a_n, b_n\} = 1$, say $b_n = 1$. In this case $|L|$ has a unique divisor of degree $g - 1$ and hence it is sufficient to use that (since $a_n > 1$) $R$ has at least $g$ ramification points.

(b) Now assume $\text{supp}(a) \subset \{1, \ldots, n\}$. With no loss of generality we may assume $\text{supp}(a) = \{1, \ldots, m\}$. Write $a = (a', 0, \ldots, 0)$ with $a' \in \mathbb{N}^m$. Take $P_1, \ldots, P_m$ such that $H(P_1, \ldots, P_m)$ is the only ordinary $m$-semigroup with $\{a'\} = E(P_1, \ldots, P_m)$. Since we are in characteristic zero, by [4, Theorem 15] it is sufficient to take as $P_{m+1}, \ldots, P_n$ and general $(P_{m+1}, \ldots, P_n) \in X^{n-m}$.

**Theorem 2.** Fix integers $g \geq 3$ and $n \geq 2$.

(a) If $n > g^3 - g^2$, then there is no ordinary $(X, P_1, \ldots, P_n)$ with $g e_i \in E(P_1, \ldots, P_n)$ for all $i$.

(b) If $2 \leq m \leq g^3 - g^2$, $n \geq m$, and $X$ is general, then there are $P_1, \ldots, P_n \in X$, $P_i \neq P_j$ for all $i \neq j$, such that $H(P_1, \ldots, P_n)$ is ordinary and $E(P_1, \ldots, P_n) = \{ge_i\}_{1 \leq i \leq m}$.

**Proof.** Part (a) follows from the Brill-Noether formula for the canonical line bundle, which says that $g^3 - g^2$ is the total weight of all Weierstrass points of a genus $g$ curve. Now assume that $X$ is general. All its Weierstrass points are ordinary and $g^3 - g^2$ is their number. Fix $n, m$ with $2 \leq m \leq g^3 - g^2$, $n \geq m$, and take as $P_1, \ldots, P_m$ any $m$ distinct Weierstrass points.

(a) Assume $n = m$. We have $h^0(O_X(tP_i)) = 1$ if $0 \leq t \leq g - 1$, $h^1(O_X(gP_i)) = 2$, $h^1(O_X(gP_i)) = 0$ and $h^1(O_X(tP_i)) = 0$ for all $t > g$. Therefore to prove that $H(P_1, \ldots, P_n)$ is ordinary and that $E(P_1, \ldots, P_n) = \{ge_i\}_{1 \leq i \leq n}$ it is sufficient to prove that $h^1(O_X(a_1P_1 + \cdots + a_nP_n)) = 0$ if either $a_1 + \cdots + a_n \geq g + 1$ or $a_1 + \cdots + a_n = g$ and $a_i < g$ for all $i$. Since $h^1(O_X(gP_i)) = 0$ for all $i$, it is sufficient to prove that $h^1(O_X(a_1P_1 + \cdots + a_nP_n)) = 0$ if $a_1 + \cdots + a_n = g$ and $a_i < g$ for all $i$. Assume the existence of $a = (a_1, \ldots, a_n)$ with $a_1 + \cdots + a_n = g$, $a_i < g$ for all $i$, and $h^1(O_X(a_1P_1 + \cdots + a_nP_n)) > 0$. Among these $a \in \mathbb{N}^n$ take one such that the integer $c := \sharp(\text{supp}(a))$ is minimal. We have $2 \leq c \leq g$. With no loss
of generality we may assume \( \supp(a) = \{1, \ldots, c\} \). Set \( L := \omega_X(-\sum_{i=1}^{c-1} a_i P_i) \). We have \( \deg(L) = g - 2 + a_n \).

(a1) Assume \( c = 2 \) and \( a_2 = 1 \). Since \( P_1 \) is an ordinary Weierstrass point, we have \( h^1(O_X((g-1)P_1)) = 1 \). Let \( D \) be the only element of \( |\omega_X(-(g-1)P_1)| \). Since \( h^1(O_X((g-1)P_1 + P_2)) > 0 \), \( P_2 \) is in the support of \( D \). Since the monodromy group of the Weierstrass points is the full symmetric group ([3]), we get that all the Weierstrass points of \( X \) different from \( P_1 \) are in the support of \( D \). Hence \( g - 1 = \deg(D) > g^2 - g^2 - 1 \), a contradiction.

(a2) Assume \( c = 2 \) and \( a_2 > 1 \). Since \( P_1 \) is ordinary, we have \( h^0(L) = a_2 \). Since \( h^1(O_X(a_1 P_1 + \cdots + a_n P_n)) > 0 \), \( P_2 \) is a ramification point of \( L \). Since the monodromy group of the Weierstrass points is the full symmetric group ([3]), all the Weierstrass points of \( X \) different from \( P_1 \) are ramification points of \( |L| \). The Brill-Segre formula for \( L \) gives that the ramification points of \( |L| \) are at most \( a_2 ((a_2 - 1)(g-1) + (g-2 + a_2)) \leq (g-1)((g-2)(g-1) + 2g-2) < g^3 - g^2 - 1 \), a contradiction.

(a3) Assume \( c > 2 \) and \( a_c = 1 \). Since \( c > 2 \), the minimality property for \( c \) shows that \( h^1(O_X(a_1 P_1 + \cdots + (a_{c-1} + a_c) P_{c-1})) = 0 \). Hence \( h^0(L) = 1 \). We get that \( P_c \) is in the base locus of \( L \). Since the monodromy group of the general Weierstrass points is the full symmetric group ([3]), we get that all Weierstrass points, except at most \( P_1, \ldots, P_{c-1} \), are in this base locus. We get a contradiction, because \( \deg(L) = g - 1 < g^3 - g^2 - (c-1) \), by our choice of the integer \( c \).

(a4) Assume \( c > 2 \) and \( a_c > 1 \). Since \( c > 2 \), the minimality property for \( c \) shows that \( h^1(O_X(a_1 P_1 + \cdots + (a_{c-1} + a_c) P_{c-1})) = 0 \). Hence \( h^0(L) = a_c \). We use again the Brill-Segre formula. The definition of \( c \) gives \( a_c \leq g+1-c \). Hence the number of ramification points of \( |L| \) is at most \( a_c ((a_c - 1)(g-1) + (g-2 + a_c)) \leq (g+1-c)((g-c)(g-1) + 2g-1-c) \). Since the monodromy group of the general Weierstrass points is the full symmetric group ([3]), all Weierstrass points of \( X \), except at most \( c-1 \), are ramification points of \( |L| \). Therefore it is sufficient to prove that for all \( c = 3, \ldots, g-1 \) we have \( (g+1-c)((g-c)(g-1) + 2g-1-c) \leq g^3 - g^2 - c \). Call \( u(g,c) \) the difference between the right hand side and the left hand side of the last inequality. Since \( u(g,c) \) in an increasing function of \( c \), it is sufficient to use that \( u(g,3) = g^3 - g^2 - 3 - (g-2)(g-3)(g-1) - (g-2)(2g-4) = g^3 - g^2 - 3 - g^3 + 6g^2 - g + 6 - 2g^2 + 8g - 8 > 0 \).

(b) Now assume \( n > m \). Take \( m \) distinct Weierstrass points \( P_1, \ldots, P_m \),
apply to them part (a) and then take a general \((P_{m+1}, \ldots, P_n) \in X^{n-m}\). Apply [4, Theorem 15].

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References


