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COMMON FIXED POINT THEOREMS FOR HYBRID PAIRS OF OCCASIONALLY WEAKLY COMPATIBLE MAPPINGS IN COMPLEX VALUED METRIC SPACES

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Abstract: In this paper we obtain some common fixed point theorems for hybrid pairs of single valued and multi-valued occasionally weakly compatible maps in complex valued metric space.

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Key Words: complex valued metric space, Common fixed point, occasionally weakly compatible mappings

1. Introduction

Azam et al. [3] introduced the concept of complex valued metric spaces and obtained sufficient conditions for the existence of common fixed points of a pair of contractive type mappings involving rational expressions. Subsequently many authors have studied the existence and uniqueness of the fixed points and common fixed points of self mapping in view of contrasting contractive conditions.

The study of fixed point theorems, involving four single-valued maps, began

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with the assumption that all of the maps are commuted. Sessa [12] weakened the condition of commutativity to that of pairwise weakly commuting. Jungck generalized the notion of weak commutativity to that of pairwise compatible [9] and then pairwise weakly compatible maps [10]. Jungck and Rhoades [11] introduced the concept of occasionally weakly compatible maps.

Abbas and Rhoades [2] generalized the concept of weak compatibility in the setting of single and multi-valued maps by introducing the notion of occasionally weakly compatible (owc).

In this paper we extended the result of Azam, Ahmed and Kumam [4] for hybrid pairs of occasionally weakly compatible (owc) mappings in complex valued metric space.

2. Preliminaries

Let C be the set of complex numbers and let $z_1, z_2 \in C$. Define a partial order $\leq$ on C as follows: $z_1 \leq z_2$ if and only if $Re(z_1) \leq Re(z_2), Im(z_1) \leq Im(z_2)$. It follows that $z_1 \leq z_2$ if one of the following conditions is satisfied:

(i) $Re(z_1) = Re(z_2), Im(z_1) < Im(z_2)$,
(ii) $Re(z_1) < Re(z_2), Im(z_1) = Im(z_2)$,
(iii) $Re(z_1) < Re(z_2), Im(z_1) < Im(z_2)$,
(iv) $Re(z_1) = Re(z_2), Im(z_1) = Im(z_2)$.

In particular, we will write $z_1 \lesssim z_2$ if one of (i), (ii) and (iii) is satisfied and we will write $z_1 < z_2$ if only (iii) is satisfied.

Definition 2.1. Let X be a non-empty set. Suppose that the mapping $d : X \times X \to C$ satisfies:

(i) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(iii) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a complex valued metric on X and (X, d) is called a complex valued metric space.

Definition 2.2. [4] Let (X, d) be a complex-valued metric space.

We denote the family of nonempty, closed and bounded subsets of a complex valued metric space by $CB(X)$.

From now on, we denote $s(z_1) = \{z_2 \in C : z_1 \preceq z_2\}$ for $z_1 \in C$, and $s(a, B) = \cup_{b \in B} s(d(a, b)) = \cup_{b \in B} \{z \in C : d(a, b) \preceq z\}$ for $a \in X$ and $B \in CB(X)$.

For $A, B \in CB(X)$, we denote

$$s(A, B) = \big(\cap_{a \in A} s(a, B) \big) \cap \big(\cap_{b \in B} s(b, A) \big).$$

Remark 2.3. [4] Let (X, d) be a complex-valued metric space. If $C = R$, then (X, d) is a metric space. Moreover, for $A, B \in CB(X), H(A, B) = \inf s(A, B)$ is the Hausdorff distance induced by d.

Definition 2.4. A point $x \in X$ is called a coincidence point (resp. fixed point) of $A : X \to X, B : X \to CB(X)$ if $Ax \in Bx$ (resp. $x = Ax \in Bx$).

Definition 2.5. Maps $f : X \to X$ and $T : X \to CB(X)$ are said to be weakly compatible if they commute at their coincidence points, that is $fx \in Tx$ for some $x \in X$ then $fTx = Tfx$.

Definition 2.6. Maps $f : X \to X$ and $T : X \to CB(X)$ are said to be occasionally weakly compatible (owc) if and only if there exist some point x in X such that $fx \in Tx$ and $fTx \subseteq Tfx$.

For example, if we take $X = [0, \infty)$ with usual metric then $f : X \to X$ and $T : X \to CB(X)$ defined by $f(x) = x^2$ and $T(x) = [0, \frac{1}{x}]$ are occasionally weakly compatible.

Example 2.7. let $X = [0, \infty)$ with usual metric. Define $f : X \to X, T : X \to CB(X)$ by

$$fx = \begin{cases} 0, & 0 \le x < 1 \\ x + 1, & 1 \le x < \infty \end{cases}$$

and

$$Tx = \begin{cases} \{x\}, & 0 \le x < 1 \\ \big[1, x + 2\big], & 1 \le x < \infty \end{cases}$$

It, can be easily verified that $x = 1$, is coincidence point of f and T, but f and T are not weakly compatible there. However the pair $\{f, T\}$ is occasionally weakly compatible, since the pair $\{f, T\}$ is weakly compatible at $x = 0$.

3. Main Results

Theorem 3.1. *Let (X, d) be a Complex valued metric space $f, g : X \to X$ and $F, G : X \to CB(X)$ be single valued and multi-valued maps respectively such that the pairs $\{f, F\}$ and $\{g, G\}$ are owc and satisfy inequality*

$$\alpha d(fx, gy) + \beta d(fx, Gy) + \gamma d(gy, Fx) \in s(Fx, Gy) \tag{3.1}$$

for all $x, y \in X$ for which $fx \neq gy$, where $\alpha, \beta, \gamma > 0$ and $(\alpha + \beta + \gamma) < 1$. Then f, g, F and G have a unique common fixed point.

Proof. Since the pairs $\{f, F\}$ and $\{g, G\}$ are owc, then there exists two points $x, y \in X$ such that $fx \in Fx, fFx \subseteq Ffx$ and $gy \in Gy, gGy \subseteq Ggy$.

First we prove that $fx = gy$. If not then by (3.1) we get

$$\alpha d(fx, gy) + \beta d(fx, Gy) + \gamma d(gy, Fx) \in s(Fx, Gy)$$

This implies that

$$\alpha d(fx, gy) + \beta d(fx, Gy) + \gamma d(gy, Fx) \in \left(\bigcap_{fx \in Fx} s(Fx, Gy) \right)$$
$$\alpha d(fx, gy) + \beta d(fx, Gy) + \gamma d(gy, Fx) \in s(Fx, Gy) = \bigcup_{gy \in Gy} s\big(d(fx, gy)\big)$$
$$\alpha d(fx, gy) + \beta d(fx, Gy) + \gamma d(gy, Fx) \in s\big(d(fx, gy)\big)$$

That is,

$$d(fx, gy) \leq \alpha d(fx, gy) + \beta d(fx, Gy) + \gamma d(gy, Fx)$$

Since $fx \in Fx$ and $gy \in Gy$, so we have

$$\begin{aligned} & d(Fx, Gy) \leq \alpha d(Fx, Gy) + \beta d(Fx, Gy) + \gamma d(Gy, Fx) \\ \text{or} \quad & d(Fx, Gy) \leq (\alpha + \beta + \gamma) d(Fx, Gy) \\ \text{as} \quad & (\alpha + \beta + \gamma) < 1, \ \text{ this implies that} \\ & d(Fx, Gy) < d(Fx, Gy), \end{aligned}$$

a contradiction, and hence $fx = gy$.

Next we claim that $x = fx$. If not then by (3.1) we get

$$\alpha d(fx, gfx) + \beta d(fx, Gfx) + \gamma d(gfx, Fx) \in s(Fx, Gfx)$$

This implies that

$$\begin{aligned}
&\alpha d(fx, gfx) + \beta d(fx, Gfx) + \gamma d(gfx, Fx) \in \left(\bigcap_{fx \in Fx} s(fx, Gfx) \right) \\
&\alpha d(fx, gfx) + \beta d(fx, Gfx) + \gamma d(gfx, Fx) \in s(fx, Gfx) \\
&= \bigcup_{gx \in Gx} s\big(d(fx, gfx)\big) \\
\text{or } \ &\alpha d(fx, gfx) + \beta d(fx, Gfx) + \gamma d(gfx, Fx) \in s\big(d(fx, gfx)\big)
\end{aligned}$$

That is,

$$d(fx, gfx) \le \alpha d(fx, gfx) + \beta d(fx, Gfx) + \gamma d(gfx, Fx)$$

Since $fx \in Fx$ and $gy \in Gy$, and $\{f, F\}$ and $\{g, G\}$ are owc so we have,

$$\begin{aligned}
&d(Fx, Gfx) \le \alpha d(Fx, Gfx) + \beta d(Fx, Gfx) + \gamma d(Gfx, Fx) \\
\text{or } \ &d(Fx, Gfx) \le (\alpha + \beta + \gamma) d(Fx, Gfx) \\
\text{as } \ &(\alpha + \beta + \gamma) < 1, \ \text{ this implies that} \\
&d(Fx, Gfx) < d(Fx, Gfx),
\end{aligned}$$

which is again a contradiction and the claim follows. Similarly we obtain $y = gy$. Thus f, g, F and G have a common fixed point. Uniqueness follows from (3.1). □

Theorem 3.2. *Let (X, d) be a Complex valued metric space $f, g : X \to X$ and $F, G : X \to CB(X)$ be single valued and multi-valued maps respectively such that the pairs $\{f, F\}$ and $\{g, G\}$ are owc and satisfy inequality*

$$k \max\big\{d(fx, gy), d(fx, Fx), d(fx, Gy), d(gy, Gy), d(gy, Fx)\big\} \in s(Fx, Gy) \tag{3.2}$$

for all $x, y \in X$ for which $fx \ne gy$ and $0 < k < 1$. Then f, g, F and G have a unique common fixed.

Proof. Since the pairs $\{f, F\}$ and $\{g, G\}$ are owc, then there exists two elements $x, y \in X$ such that $fx \in Fx, fFx \subseteq Ffx$ and $gy \in Gy, gGy \subseteq Ggy$.

First we prove that $fx = gy$. If not then by (3.2) we get

$$k \max\big\{d(fx, gy), d(fx, Fx), d(fx, Gy), d(gy, Gy), d(gy, Fx)\big\} \in s(Fx, Gy)$$

This implies that

$$\begin{aligned}
&k\max\big\{d(fx,gy),d(fx,Fx),d(fx,Gy),d(gy,Gy),d(gy,Fx)\big\}\\
&\in\left(\bigcap_{fx\in Fx}s(fx,Gy)\right)\\
&k\max\big\{d(fx,gy),d(fx,Fx),d(fx,Gy),d(gy,Gy),d(gy,Fx)\big\}\\
&\in s(fx,Gy)=\bigcup_{gy\in Gy}s\big(d(fx,gy)\big)\\
&k\max\big\{d(fx,gy),d(fx,Fx),d(fx,Gy),d(gy,Gy),d(gy,Fx)\big\}\in s\big(d(fx,gy)\big)
\end{aligned}$$

That is,

$$d(fx,gy)\le k\max\big\{d(fx,gy),d(fx,Fx),d(fx,Gy),d(gy,Gy),d(gy,Fx)\big\}$$

Since $fx\in Fx$ and $gy\in Gy$, so we have

$$\begin{aligned}
d(Fx,Gy)&\le kmax\big\{d(Fx,Gy),d(Fx,Fx),d(Fx,Gy),d(Gy,Gy),d(Gy,Fx)\big\}\\
&\le kd(Fx,Gy)
\end{aligned}$$

as $0<k<1$, this implies that
or $d(Fx,Gy)<d(Fx,Gy)$,
a contradiction, and hence $fx=gy$.

Next we claim that $x=fx$. If not then by (3.2) we get

$$k\max\big\{d(fx,gfx),d(fx,Fx),d(fx,Gfx),d(gfx,Gfx),d(gfx,Fx)\big\}\\
\in s(Fx,Gfx)$$

This implies that

$$\begin{aligned}
&k\max\big\{d(fx,gfx),d(fx,Fx),d(fx,Gfx),d(gfx,Gfx),d(gfx,Fx)\big\}\\
&\in s(fx,Gfx)\\
&\text{or}\\
&k\max\big\{d(fx,gfx),d(fx,Fx),d(fx,Gfx),d(gfx,Gfx),d(gfx,Fx)\big\}\\
&=\bigcup_{gx\in Gx}s\big(d(fx,gfx)\big)\\
&\text{or}\\
&k\max\big\{d(fx,gfx),d(fx,Fx),d(fx,Gfx),d(gfx,Gfx),d(gfx,Fx)\big\}
\end{aligned}$$

$$\in s\big(d(fx, gfx)\big)$$

That is

$$\begin{aligned}&d(fx, gfx)\\&\quad \leq k \max\left\{d(fx, gfx), d(fx, Fx), d(fx, Gfx), d(gfx, Gfx), d(gfx, Fx)\right\}.\end{aligned}$$

Since $fx \in Fx$ and $gy \in Gy$ and $\{f, F\}$ and $\{g, G\}$ are owc so we have,

$$\begin{aligned}d(Fx, Gfx) &\leq k \max\left\{d(Fx, Gfx), d(fx, Fx), d(Fx, Gfx),\right.\\&\qquad \left. d(gfx, Gfx), d(Gfx, Fx)\right\}\\&\leq kd(Fx, Gfx)\end{aligned}$$

as $0 < k < 1$, this implies that
or $d(Fx, Gfx) < d(Fx, Gfx)$,

which is again a contradiction and the claim follows. Similarly we obtain $y = gy$. Thus f, g, F and G have a common fixed point. Uniqueness follows from (3.2). □

Theorem 3.3. *Let (X, d) be a Complex valued metric space $f, g : X \to X$ and $F, G : X \to CB(X)$ be single valued and multi-valued maps respectively such that the pairs $\{f, F\}$ and $\{g, G\}$ are owc and satisfy inequality*

$$k \max\left\{d(fx, gy), d(fx, Fx), d(gy, Gy), \frac{d(fx, Gy) + d(gy, Fx)}{2}\right\} \in s(Fx, Gy) \quad (3.3)$$

for all $x, y \in X$ for which $fx \neq gy$ and $0 < k < 1$. Then f, g, F and G have a unique common fixed.

Proof. Clearly the result follows from Theorem 3.2. □

Theorem 3.4. *Let (X, d) be a Complex valued metric space $f, g : X \to X$ and $F, G : X \to CB(X)$ be single valued and multi-valued maps respectively such that the pairs $\{f, F\}$ and $\{g, G\}$ are owc and satisfy inequality*

$$k \max\left\{d(fx, gy), \frac{d(fx, Fx) + d(gy, Gy)}{2}, \frac{d(fx, Gy) + d(gy, Fx)}{2}\right\} \in s(Fx, Gy) \quad (3.4)$$

for all $x, y \in X$ for which $fx \neq gy$ and $0 < k < 1$. Then f, g, F and G have a unique common fixed.

Proof. Clearly the result follows from Theorem 3.2. □

Theorem 3.5. *Let (X,d) be a Complex valued metric space $f,g : X \to X$ and $F,G : X \to CB(X)$ be single valued and multi-valued maps respectively such that the pairs $\{f,F\}$ and $\{g,G\}$ are owc and satisfy inequality*

$$k \max\left\{d(fx,gy), \left(1-\frac{d(gy,Gy)+d(fx,Gy)}{d(fx,Fx)+d(gy,Fx)}\right)\right\} \in s(Fx,Gy) \tag{3.5}$$

for all $x,y \in X$ for which $fx \neq gy$ and $0 < k < 1$. Then f,g,F and G have a unique common fixed.

Proof. Since the pairs $\{f,F\}$ and $\{g,G\}$ are owc, then there exists two points $x,y \in X$ such that $fx \in Fx, fFx \in Ffx$ and $gy \in Gy, gGy \subseteq Ggy$.

First we prove that $fx = gy$. If not then by (3.5) we get

$$k \max\left\{d(fx,gy), \left(1-\frac{d(gy,Gy)+d(fx,Gy)}{d(fx,Fx)+d(gy,Fx)}\right)\right\} \in s(Fx,Gy)$$

This implies that

$$k \max\left\{d(fx,gy), \left(1-\frac{d(gy,Gy)+d(fx,Gy)}{d(fx,Fx)+d(gy,Fx)}\right)\right\} \in \left(\bigcap_{fx\in Fx} s(fx,Gy)\right)$$

$$k \max\left\{d(fx,gy), \left(1-\frac{d(gy,Gy)+d(fx,Gy)}{d(fx,Fx)+d(gy,Fx)}\right)\right\} \in s(fx,Gy)$$

$$= \left(\bigcup_{gy\in Gy} s\big(d(fx,Gy)\big)\right)$$

$$k \max\left\{d(fx,gy), \left(1-\frac{d(gy,Gy)+d(fx,Gy)}{d(fx,Fx)+d(gy,Fx)}\right)\right\} \in s\big(d(fx,gy)\big)$$

That is,

$$d(fx,gy) \leq k \max\left\{d(fx,gy), \left(1-\frac{d(gy,Gy)+d(fx,Gy)}{d(fx,Fx)+d(gy,Fx)}\right)\right\}$$

Since $fx \in Fx$ and $gy \in Gy$, so we have

$$d(Fx,Gy) \leq k \max\left\{d(Fx,Gy), \left(1-\frac{d(Gy,Gy)+d(Fx,Gy)}{d(fx,Fx)+d(Gy,Fx)}\right)\right\}$$

$$
\begin{aligned}
&= k\max\left\{d(Fx,Gy),\left(1-\frac{d(Fx,Gy)}{d(Gy,Fx)}\right)\right\}\\
&= k\max\left\{d(Fx,Gy),0\right\}\\
&\le kd(Fx,Gy)
\end{aligned}
$$

as $0<k<1$, this implies that

$$d(Fx,Gy) < d(Fx,Gy),$$

a contradiction, and hence $fx = gy$.

Next we claim that $x = fx$. If not then by (3.5) we get

$$k\max\left\{d(fx,gfx),\left(1-\frac{d(gfx,Gfx)+d(fx,Gfx)}{d(fx,Fx)+d(gfx,Fx)}\right)\right\}\in s(Fx,Gfx)$$

This implies that,

$$k\max\left\{d(fx,gfx),\left(1-\frac{d(gfx,Gfx)+d(fx,Gfx)}{d(fx,Fx)+d(gfx,Fx)}\right)\right\}\in\left(\bigcap_{fx\in Fx}s(fx,Gfx)\right)$$

$$k\max\left\{d(fx,gfx),\left(1-\frac{d(gfx,Gfx)+d(fx,Gfx)}{d(fx,Fx)+d(gfx,Fx)}\right)\right\}\in s(fx,Gfx)$$

$$=\left(\bigcup_{gx\in Gx}s\big(d(fx,gfx)\big)\right)$$

$$k\max\left\{d(fx,gfx),\left(1-\frac{d(gfx,Gfx)+d(fx,Gfx)}{d(fx,Fx)+d(gfx,Fx)}\right)\right\}\in s\big(d(fx,gfx)\big)$$

That is,

$$d(fx,gfx)\le k\max\left\{d(fx,gfx),\left(1-\frac{d(gfx,Gfx)+d(fx,Gfx)}{d(fx,Fx)+d(gfx,Fx)}\right)\right\}$$

Since $fx\in Fx$ and $gy\in Gy$ and $\{f,F\}$ and $\{g,G\}$ so we have,

$$
\begin{aligned}
d(Fx,Gfx) &\le k\max\left\{d(Fx,Gfx),\left(1-\frac{d(Gfx,Gfx)+d(Fx,Gfx)}{d(fx,Fx)+d(Gfx,Fx)}\right)\right\}\\
&= k\max\left\{d(Fx,Gfx),\left(1-\frac{d(Fx,Gfx)}{d(Gfx,Fx)}\right)\right\}
\end{aligned}
$$

$$= k \max\Big\{d(Fx, Gfx), 0\Big\}$$
$$\leq kd(Fx, Gfx)$$

as $0 < k < 1$, this implies that

$$d(Fx, Gfx) < d(Fx, Gfx),$$

which is again a contradiction and the claim follows. Similarly we obtain $y = gy$. Thus f, g, F and G have a common fixed point. Uniqueness follows from (3.5). □

Theorem 3.6. *Let (X, d) be a Complex valued metric space $f, g : X \to X$ and $F, G : X \to CB(X)$ be single valued and multi-valued maps respectively such that the pairs $\{f, F\}$ and $\{g, G\}$ are owc and satisfy inequality*

$$\alpha d(fx, gy)^p + (1-\alpha)\max\Big\{\Big(d(fx, Gy)\Big)^p, \Big(d(gy, Fx)\Big)^p, \Big(d(fx, Fx)\Big)^p, \Big(d(gy, Gy)\Big)^p\Big\} \in s(Fx, Gy)^p \quad (3.6)$$

for all $x, y \in X$ for which $fx \neq gy, p \geq 1$ and $\alpha \in (0, 1]$. Then f, g, F and G have a unique common fixed.

Proof. Clearly the result follows from Theorem 3.1. □

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