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SOME REGULAR EQUIVALENCE RELATION ON THE SEMIHYPERGROUP OF THE PARTIAL TRANSFORMATION SEMIGROUP ON A SET AND LOCAL SUBSEMIHYPERGROUPS WITH THAT REGULAR EQUIVALENCE RELATION

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Abstract: A *hyperoperation* $\circ$ on a nonempty set H is a function from $H \times H$ into $P^*(H)$ where $P^*(H)$ is the set of all nonempty subset of H and $(H, \circ)$ is call a *hypergroupoid*. A hypergroupoid $(H, \circ)$ is called a *semihypergroup* if the hyperoperation $\circ$ is associative. Thus, semihypergroups generalize semigroups. Moreover, if S is a semigroup; we can define a hyperoperation $\circ$ on S in order to make $(S, \circ)$ a semihypergroup. In 2013, R.I. Sararnrakskul defined a hyperoperation $\circ$ on the partial transformation semigroup $P(X)$ to make a semihypergroup. In this paper, we define a regular equivalence relation ρ on $(P(X), \circ)$ so that $P(X)/\rho$ is a semihypergroup and then we studies some subsemihypergroup of $P(X)/\rho$.

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1. Introduction

Hyperstructures theory was first initiated by F. Marty in 1934 at the 8th Congress of Scandinavian Mathematicians when he defined hypergroups as a generalization of group. Since then this theory has enjoyed a rapid development and many researchers have contributed to study of hypergroup with classical algebraic structure, modern algebra and its application. Elementary concepts and results on hypergroups can be found in [2]. In 2003, Corsini and Leoreanu presented numerous application of hyperstructures theory in their book [3]. These applications can be used in various areas such as geometry, topology, combinatorics, theory of binary relations, theory of fuzzy sets, probability theory, codes theory, automata theory especially the theories of group and semigroups.

First of all, we recall some basic definitions and examples of hypergroup theory from [2]. Let H be a nonempty set and $P^*(H)$ the set of all nonempty subsets of H. A *hyperoperation* on H is a function $\circ : H \times H \to P^*(H)$, the image of $(x, y) \in H \times H$ under $\circ$ is denoted by $x \circ y$ and called the *hyperproduct* of x and y, and $(H, \circ)$ is called a *hypergroupoid*. For nonempty subsets A and B of H, let $A \circ B = \bigcup_{a \in A, b \in B} a \circ b$, $A \circ b = A \circ \{b\}$ and $a \circ B = \{a\} \circ B$ where $a, b \in H$.

A hypergroupoid $(H, \circ)$ is called a *semihypergroup* if the hyperoperation $\circ$ is associative, that is, $(x \circ y) \circ z = x \circ (y \circ z)$ for all $x, y, z \in H$.

A semihypergroup $(H, \circ)$ is called a *hypergroup* if $x \circ H = H \circ x = H$ for all $x \in H$.

A hypergroupoid $(H, \circ)$ is called *commutative* if $x \circ y = y \circ x$ for all $x, y \in H$.

If $(H, \circ)$ is a hypergroupoid [semihypergroup], then for a nonempty subset K of H, K is called a *subhypergroupoid* [*subsemihypergroupoid*] if $K \circ K \subseteq K$, that is, $x \circ y \subseteq K$ for all $x, y \in K$.

One can see that semihypergroups are generalization of hypergroups. Besides, we can define a hyperoperation $\circ$ on a semigroup S which makes $(S, \circ)$ a semihypergroup.

The partial transformation semigroup, the full transformation semigroup, the 1-1 partial transformation semigroup or the symmetric inverse semigroup and the symmetric group on a nonempty set X are denoted $P(X)$, $T(X)$, $I(X)$ and $G(X)$, respectively.

Example 1.1. (see [10]) Let S be a semigroup and P be a nonempty subset of S and define $x \circ y = xPy (= \{xty \mid t \in P\})$ for all $x, y \in S$. Then $(S, \circ)$ is a semihypergroup. In particular, if S is a group, then $(S, \circ)$ is a hypergroup.

Then by Example 1.1, the semigroup $P(X)$ along with $I(X)$ is also a semihypergroup under the hyperoperation $\circ$, i.e.,

$$\alpha \circ \beta = \alpha I(X)\beta = \{\alpha\gamma\beta \mid \gamma \in I(X)\} \qquad \text{for all } \alpha, \beta \in P(X).$$

Our aim is to study another hyperoperation which is related to $(P(X), \circ)$ and regular equivalet relation ρ on $P(X)$ which make $P(X)/\rho$ is a semihypergroup. And we also study some subsemihypergroup on its.

Next, we recall some basic definitions and theorems of hypergroup theory from [2] which make help to investigate $P(X)/\rho$ is a semihypergroup.

Let ρ be an equivalence relation on a hypergroupoid $(H, \circ)$ and $A, B \subseteq H$, define $A\,\overline{\rho}\,B$ if and only if:

(i) $\forall\, a \in A, \exists\, b \in B$, $a\rho b$ and $\forall\, b \in B, \exists\, a \in A$, $a\rho b$ or

(ii) $\forall\, a \in A, \rho(a) \cap B \neq \emptyset$ and $\forall\, b \in B, \rho(b) \cap A \neq \emptyset$, where $\rho(x) = \{t \in H \mid t\rho x\} \subseteq H$.

An equivalence relation ρ is call *regular* if $x\rho y$ then $x \circ a \overline{\rho} y \circ a$ and $a \circ x \overline{\rho} a \circ y$ for all $x, y, a \in H$.

Theorem 1.2. *(see [2]) Let $(H, \circ)$ be a hypergroupoid ρ an equivalence on H. Define $\otimes$ on H/ρ by*

$$a\rho \otimes b\rho = \{x\rho \mid x \in a \circ b\}$$

for all $a, b \in H$ where $x\rho$ is the $\rho-$class of H containing x. Then:

(i) ρ is regular if and only if $\otimes$ is well-defined.

(ii) If $(H, \circ)$ is a semihypergroup and ρ is regular then $(H/\rho, \otimes)$ is a semihypergroup.

2. Main Result

First, we note that for any mapping α, α can be written as

$$\alpha = \begin{pmatrix} x\alpha^{-1} \\ x \end{pmatrix}_{x \in \operatorname{ran}\alpha}.$$

Proposition 2.1. Define a relation on $(P(X), \circ)$ by

$$\alpha\rho\beta \quad \text{if and only if} \quad |\operatorname{ran}\alpha| = |\operatorname{ran}\beta|$$

for all $\alpha, \beta \in P(X)$. Then ρ is an equivalence relation.

Lemma 2.2. *If $\alpha, \beta \in P(X)$ such that $\alpha\rho\beta$, then $\alpha I(X)\gamma\overline{\rho}\beta I(X)\gamma$, that is, $\alpha \circ \gamma\overline{\rho}\beta \circ \gamma$ for any $\gamma \in P(X)$.*

Proof. Let $\alpha, \beta \in P(X)$ and assume that $\alpha\rho\beta$. Then $|\operatorname{ran}\alpha| = |\operatorname{ran}\beta|$. So, there is an bijective mapping $\theta : \operatorname{ran}\alpha \to \operatorname{ran}\beta$. Let $f \in I(X)$. Since θ is a bijection from $\operatorname{ran}\alpha$ onto $\operatorname{ran}\beta$, $\operatorname{ran}\alpha = (\operatorname{ran}\beta)\theta^{-1} = \operatorname{ran}\beta\theta^{-1}$ and $\theta^{-1}f \in I(X)$. Therefore

$$\begin{aligned}
\operatorname{ran}(\alpha f\gamma) &= [\operatorname{ran}\alpha \cap \operatorname{dom}(f\gamma)](f\gamma) \\
&= [\operatorname{ran}\beta\theta^{-1} \cap \operatorname{dom}(f\gamma)](f\gamma) \\
&= \operatorname{ran}(\beta\theta^{-1} \cdot f\gamma) \\
&= \operatorname{ran}(\beta\theta^{-1}f\gamma).
\end{aligned}$$

Hence, $|\operatorname{ran}(\alpha f\gamma)| = |\operatorname{ran}(\beta\theta^{-1}f\gamma)|$, that is, $\alpha f\gamma\rho\beta(\theta^{-1}f)\gamma$.

Similarly, $\theta f \in I(X)$ and $\operatorname{ran}\beta = (\operatorname{ran}\alpha)\theta = \operatorname{ran}\alpha\theta$ Therefore

$$\begin{aligned}
\operatorname{ran}(\beta f\gamma) &= [\operatorname{ran}\beta \cap \operatorname{dom}(f\gamma)](f\gamma) \\
&= [\operatorname{ran}\alpha\theta \cap \operatorname{dom}(f\gamma)](f\gamma) \\
&= \operatorname{ran}(\alpha\theta \cdot f\gamma) \\
&= \operatorname{ran}(\alpha\theta f\gamma).
\end{aligned}$$

So, $|\operatorname{ran}(\beta f\gamma)| = |\operatorname{ran}(\alpha\theta f\gamma)|$, that is, $\alpha(\theta f)\rho\beta f\gamma$. □

Lemma 2.3. *Let $\alpha, \beta \in P(X)$ and $\gamma \in I(X)$ such that $\alpha\rho\beta$. If $f \in I(X)$ then there exists $h \in I(X)$ such that $|\operatorname{ran}\gamma f\alpha| = |\operatorname{ran}\gamma h\beta|$.*

Proof. There is an bijective mapping $\theta : \operatorname{ran}\alpha \to \operatorname{ran}\beta$ because of $\alpha\rho\beta$. Let $f \in I(X)$. Since θ is a bijection from $\operatorname{ran}\alpha$ onto $\operatorname{ran}\beta$ and $\operatorname{ran}\gamma f\alpha \subseteq \operatorname{ran}\alpha$, there exist $B \subseteq \operatorname{ran}\beta$ such that $\theta_{|\operatorname{ran}\gamma f\alpha} : \operatorname{ran}\gamma f\alpha \to B$ is a bijection. We define $\alpha^* \in I(X)$ by

for each $y \in \operatorname{ran}\gamma f\alpha$, choose $x_y \in y\alpha^{-1} \cap \operatorname{ran} f$, that is,

$$\alpha^* = \begin{pmatrix} x_y \\ y \end{pmatrix}_{\substack{y \in \operatorname{ran}\gamma f\alpha \\ x_y \in y\alpha^{-1} \cap \operatorname{ran} f}}.$$

So,

$$\operatorname{ran}\gamma f\alpha = \operatorname{ran}\gamma f\alpha^*. \tag{1}$$

Next, we define $\beta^* \in I(X)$ by
for each $y \in B$, choose $x_y \in y\beta^{-1}$, that is,

$$\beta^* = \begin{pmatrix} x_y \\ y \end{pmatrix}_{\substack{y\in B \\ x_y \in y\beta^{-1}}}.$$

Therefore $f\alpha^*\theta(\beta^*)^{-1} \in I(X)$ and $\operatorname{ran}\beta^* = \operatorname{ran}\beta$. Let $h = f\alpha^*\theta(\beta^*)^{-1}$.
Since $\operatorname{ran}\gamma f\alpha^* \subseteq \operatorname{ran}\alpha = \operatorname{dom}\theta$ and θ is injection,

$$|\operatorname{ran}\gamma f\alpha^*| = |(\operatorname{ran}\gamma f\alpha^*)\theta| = |\operatorname{ran}(\gamma f\alpha^*\theta)|. \tag{2}$$

Since $\operatorname{ran}\gamma f\alpha^* \subseteq \operatorname{ran}\alpha = \operatorname{dom}\theta$, $(\operatorname{ran}\gamma f\alpha^*)\theta = \operatorname{ran}(\gamma f\alpha^*\theta)$.
Similarly, $\operatorname{ran}(\gamma f\alpha^*\theta) = (\operatorname{ran}\gamma f\alpha^*)\theta \subseteq B = \operatorname{ran}\beta^* = \operatorname{dom}(\beta^*)^{-1}$ then

$$[\operatorname{ran}(\gamma f\alpha^*\theta)](\beta^*)^{-1} = \operatorname{ran}(\gamma f\alpha^*\theta(\beta^*)^{-1}). \tag{3}$$

Because $\operatorname{ran}(\gamma f\alpha^*\theta(\beta^*)^{-1}) \subseteq \operatorname{dom}\beta$ then

$$[\operatorname{ran}\left(\gamma f\alpha^*\theta(\beta^*)^{-1}\right)]\beta = \operatorname{ran}(\gamma f\alpha^*\theta(\beta^*)^{-1}\beta). \tag{4}$$

Since $\operatorname{ran}(\gamma f\alpha^*\theta) \subseteq B = \operatorname{dom}(\beta^*)^{-1}$ and $\beta_{|\operatorname{dom}\beta^*} = \beta^*$ is a bijection,

$$[\operatorname{ran}(\gamma f\alpha^*\theta)](\beta^*)^{-1}\beta = \operatorname{ran}(\gamma f\alpha^*\theta). \tag{5}$$

Therefore,

$$\begin{aligned} \operatorname{ran}(\gamma f\alpha^*\theta(\beta^*)^{-1}\beta) &= [\operatorname{ran}\left(\gamma f\alpha^*\theta(\beta^*)^{-1}\right)]\beta && \text{from (4)} \\ &= [(\operatorname{ran}(\gamma f\alpha^*\theta))](\beta^*)^{-1}\beta && \text{from (3)} \\ &= \operatorname{ran}(\gamma f\alpha^*\theta) && \text{from (5)}. \end{aligned}$$

Thus

$$\operatorname{ran}(\gamma f\alpha^*\theta(\beta^*)^{-1}\beta) = \operatorname{ran}(\gamma f\alpha^*\theta). \tag{6}$$

Consequently,

$$\begin{aligned} |\operatorname{ran}(\gamma h\beta)| &= |\operatorname{ran}(\gamma(f\alpha^*\theta(\beta^*)^{-1})\beta)| \\ &= |\operatorname{ran}(\gamma f\alpha^*\theta)| && \text{from (6)} \\ &= |\operatorname{ran}(\gamma f\alpha^*)| && \text{from (2)} \\ &= |\operatorname{ran}\gamma f\alpha| && \text{from (1)}. \end{aligned}$$

Hence the proof is complete. □

Lemma 2.4. *Let $\alpha, \beta \in P(X)$ and $\gamma \in I(X)$ such that $\alpha\rho\beta$. If $f \in I(X)$ then there exists $k \in I(X)$ such that $|\operatorname{ran}\gamma k\alpha| = |\operatorname{ran}\gamma f\beta|$.*

Proof. Let $\theta : \operatorname{ran}\alpha \to \operatorname{ran}\beta$ be a bijection. Since θ^{-1} is a bijection from $\operatorname{ran}\beta$ onto $\operatorname{ran}\alpha$ and $\operatorname{ran}\gamma f\beta \subseteq \operatorname{ran}\beta$, there exist $A \subseteq \operatorname{ran}\alpha$ such that $\theta^{-1}_{|\operatorname{ran}\gamma f\beta} : \operatorname{ran}\gamma f\beta \to A$ is a bijection. We define $\beta^* \in I(X)$ by

for each $y \in \operatorname{ran}\gamma f\beta$, choose $x_y \in y\beta^{-1} \cap \operatorname{ran} f$, that is,

$$\beta^* = \begin{pmatrix} x_y \\ y \end{pmatrix}_{\substack{y \in \operatorname{ran}\gamma f\beta \\ x_y \in y\beta^{-1} \cap \operatorname{ran} f}}.$$

Thus

$$\operatorname{ran}\gamma f\beta = \operatorname{ran}\gamma f\beta^*. \tag{7}$$

Next, we define $\alpha^* \in I(X)$ by

for each $y \in A$, choose $x_y \in y\alpha^{-1}$, that is,

$$\alpha^* = \begin{pmatrix} x_y \\ y \end{pmatrix}_{\substack{y \in A \\ x_y \in y\alpha^{-1}}}.$$

Therefore $f\beta^*\theta^{-1}(\alpha^*)^{-1} \in I(X)$ and $\operatorname{ran}\alpha^* = \operatorname{ran}\alpha$. Let $k = f\beta^*\theta^{-1}(\alpha^*)^{-1}$. It is enough to show that $|\operatorname{ran}\gamma k\alpha| = |\operatorname{ran}\gamma f\beta|$.

Since $\operatorname{ran}\gamma f\beta^* \subseteq \operatorname{ran}\beta = \operatorname{dom}\theta^{-1}$ and θ^{-1} is injection,

$$|\operatorname{ran}\gamma f\beta^*| = |(\operatorname{ran}\gamma f\beta^*)\theta^{-1}| = |\operatorname{ran}(\gamma f\beta^*\theta^{-1})|. \tag{8}$$

Since $\operatorname{ran}\gamma f\beta^* \subseteq \operatorname{ran}\beta = \operatorname{dom}\theta^{-1}$, $(\operatorname{ran}\gamma f\beta^*)\theta^{-1} = \operatorname{ran}(\gamma f\beta^*\theta^{-1})$. Similarly, $\operatorname{ran}(\gamma f\beta^*\theta^{-1}) = (\operatorname{ran}\gamma f\beta^*)\theta^{-1} \subseteq A = \operatorname{ran}\alpha^* = \operatorname{dom}(\alpha^*)^{-1}$ then

$$[\operatorname{ran}(\gamma f\beta^*\theta^{-1})](\alpha^*)^{-1} = \operatorname{ran}(\gamma f\beta^*\theta^{-1}(\alpha^*)^{-1}). \tag{9}$$

Because $\operatorname{ran}(\gamma f\beta^*\theta^{-1}(\alpha^*)^{-1}) \subseteq \operatorname{dom}\alpha$ then

$$[\operatorname{ran}\left(\gamma f\beta^*\theta^{-1}(\alpha^*)^{-1}\right)]\alpha = \operatorname{ran}(\gamma f\beta^*\theta^{-1}(\alpha^*)^{-1}\alpha). \tag{10}$$

Since $\operatorname{ran}(\gamma f\beta^*\theta^{-1}) \subseteq A = \operatorname{dom}(\alpha^*)^{-1}$ and $\alpha_{|\operatorname{dom}\alpha^*} = \alpha^*$ is a bijection,

$$[\operatorname{ran}(\gamma f\beta^*\theta^{-1})](\alpha^*)^{-1}\alpha = \operatorname{ran}(\gamma f\beta^*\theta^{-1}). \tag{11}$$

Therefore,

$$\operatorname{ran}(\gamma f\beta^*\theta^{-1}(\alpha^*)^{-1}\alpha) = [\operatorname{ran}\left(\gamma f\beta^*\theta^{-1}(\alpha^*)^{-1}\right)]\alpha \qquad \text{from (10)}$$

$$= [(\operatorname{ran}(\gamma f \beta^* \theta^{-1}))](\alpha^*)^{-1}\alpha \qquad \text{from (9)}$$
$$= \operatorname{ran}(\gamma f \beta^* \theta^{-1}) \qquad \text{from (11).}$$

Thus

$$\operatorname{ran}(\gamma f \beta^* \theta^{-1}(\alpha^*)^{-1}\alpha) = \operatorname{ran}(\gamma f \beta^* \theta^{-1}). \tag{12}$$

Consequently,

$$\begin{aligned} |\operatorname{ran}(\gamma k \alpha)| &= |\operatorname{ran}(\gamma(f \beta^* \theta^{-1}(\alpha^*)^{-1})\alpha)| \\ &= |\operatorname{ran}(\gamma f \beta^* \theta^{-1})| \qquad \text{from (12)} \\ &= |\operatorname{ran}(\gamma f \beta^*)| \qquad \text{from (8)} \\ &= |\operatorname{ran} \gamma f \beta| \qquad \text{from (7).} \end{aligned}$$

□

Lemma 2.5. *If $\alpha, \beta \in P(X)$ such that $\alpha\rho\beta$, then $\gamma I(X)\alpha\overline{\rho}\gamma I(X)\beta$, that is, $\gamma \circ \alpha\overline{\rho}\gamma \circ \beta$ for any $\gamma \in P(X)$.*

Proof. Let $\alpha, \beta, \gamma \in P(X)$ such that $\alpha\rho\beta$. Then $|\operatorname{ran}\alpha| = |\operatorname{ran}\beta|$. So, there is an bijective mapping $\theta : \operatorname{ran}\alpha \to \operatorname{ran}\beta$. From Lemma 2.3 and Lemma 2.4 there exists $h \in I(X)$ such that $|\operatorname{ran}\gamma f \alpha| = |\operatorname{ran}\gamma h \beta|$ and $k \in I(X)$ such that $|\operatorname{ran}\gamma k \alpha| = |\operatorname{ran}\gamma f \beta|$, respectively. Therefore $\gamma I(X)\alpha\overline{\rho}\gamma I(X)\beta$. □

Theorem 2.6. *Define the equivalence relation on $(P(X), \circ)$ by*

$$\alpha\rho\beta \qquad \textit{if and only if} \qquad |\operatorname{ran}\alpha| = |\operatorname{ran}\beta|$$

for all $\alpha, \beta \in P(X)$. Then ρ is regular.

Proof. It obtained directly from Lemma 2.2 and Lemma 2.5. □

Corollary 2.7. *Let ρ be an equivalence relation on $(P(X), \circ)$ defined by*

$$\alpha\rho\beta \qquad \textit{if and only if} \qquad |\operatorname{ran}\alpha| = |\operatorname{ran}\beta|$$

for all $\alpha, \beta \in P(X)$ and define $\otimes$ on $P(X)/\rho$ by

$$\alpha\rho \otimes \beta\rho = \{\gamma\rho \,|\, \gamma \in \alpha \circ \beta\}$$

where $\gamma\rho$ is the $\rho -$ class of $P(X)$ containing γ. Then $(P(X)/\rho, \otimes)$ is a semi-hypergroup. Moreover, $(G(X)/\rho, \otimes)$ is a subsemihypergroup of $(P(X)/\rho, \otimes)$.

Proof. Its follows from Theorem 1.2 and Theorem 2.6. □

In [4], [6], [7] and [8], the authors use the word a "local subsemigroup"of a semigroup S to mean a subsemigroup of S of the form eSe where e is an idempotent of S. In 2008, [9] the authors were motivated by this definition and defined a "local subset"and a "local subsemigroup"of a semigroup S in more general sense as follows : a *local subset* of a semigroup S is a subset of S of the form eAe where e is an idempotent of S and A is a subsemigroup of S. Note that a local subset of a semigroup S need not be a subsemigroup of S. Then they were interested in finding a necessary and sufficient condition for an idempotent e of S which guarantees that eAe becomes a subsemigroup of S for a given subsemigroup A of S. They called a local subset eAe of a semigroup S a *local subsemigroup* of S if eAe is a subsemigroup of S.

Denote by $E(S)$ the set of all idempotents of a semigroup S, that is,

$$E(S) = \{x \in S \mid x^2 = x\}.$$

The cardinality of a set X is denoted by $|X|$. The domain and range of a mapping α are denoted by $\operatorname{dom}\alpha$ and $\operatorname{ran}\alpha$, respectively, and the value of α at $x \in \operatorname{dom}\alpha$ is written by $x\alpha$. For $A \subseteq \operatorname{dom}\alpha$, $\alpha_{|A}$ denotes the restriction of α to A.

We also have that

$$\begin{aligned} E(P(X)) &= \{\alpha \in P(x) \mid \operatorname{ran}\alpha \subseteq \operatorname{dom}\alpha \text{ and } x\alpha = x \text{ for all } x \in \operatorname{ran}\alpha\} \\ &= \{\alpha \in P(x) \mid \operatorname{ran}\alpha \subseteq \operatorname{dom}\alpha \text{ and } \alpha_{|\operatorname{ran}\alpha} = 1_{\operatorname{ran}\alpha}\} \end{aligned}$$

([1], page 12). In [9],the authors provided a neccessary and sufficient condition of $\alpha \in E(T(X))$, where X is finite, so that $\alpha G(X)\alpha$ is a local subsemigroup of $T(X)$ as follows:

Theorem 2.8 ([9]). *Let X be a finite nonempty set and $\alpha \in E(T(X))$. Then $\alpha G(X)\alpha$ is a local subsemigroup of $T(X)$ if and only if either:*

(i) $\alpha = 1_X$, the identity mapping on X, or

(ii) for every $a \in \operatorname{ran}\alpha$, $|a\alpha^{-1}| \geq |\operatorname{ran}\alpha|$.

In the second case, $\alpha G(X)\alpha = \alpha T(\operatorname{ran}\alpha) \cong T(\operatorname{ran}\alpha)$.

From Example 1.1, if S is a semigroup and $\circ$ is a hyperoperation on S which makes $(S, \circ)$ a semihypergroup. For this reason in 2013, R.I. Sararnrakskul definea local subset and a local subsemihypergroup on $(S, \circ)$ in the similar way as follows : a *local subset* of a semihypergroup $(S, \circ)$ is a subset of S of the form $e \circ A \circ e$ where e is an idempotent of the semigroup S and A is a subsemihypergroup of S; moreover, if a local subset $e \circ A \circ e$ is a subsemihypergroup of S, then it is called a *local subsemihypergroup* of S.

Furthermore, she provided a neccessary and sufficient condition of $\alpha \in E(P(X))$, where X is finite, so that $\alpha \circ G(X) \circ \alpha$ is a local subsemihypergroup of $P(X)$ as follows: $\alpha \circ G(X) \circ \alpha$ is a local subsemihypergroup of $P(X)$ if and only if either:

(i) $\alpha = 1_A$, for some nonempty subset A of X, or

(ii) for every $a \in \operatorname{ran}\alpha, |a\alpha^{-1}| \geq |\operatorname{ran}\alpha|$.

If α satisfies (ii) then $\alpha \circ G(X) \circ \alpha = \alpha P(\operatorname{ran}\alpha) \cong P(\operatorname{ran}\alpha)$ under good isomorphism.

Lemma 2.9. *(see [10]) Let X be a nonempty set and $\alpha \in E(P(X)) \setminus \{0\}$. Then*

$$\alpha \circ G(X) \circ \alpha = \alpha I(X)\alpha.$$

Later on, we will find some subsemihypergroup on $P(X)/\rho$ which is related to Theorem 2.8 and we a call local subsemihypergroup of $P(X)/\rho$.

Corollary 2.10. *Let X be a nonempty set and $\alpha \in E(P(X)) \setminus \{0\}$. If $\alpha = 1_A$ for some nonempty subset A of X then*

$$\alpha \circ G(X) \circ \alpha = I(A).$$

Lemma 2.11. *(see [10]) Let X be a nonempty set and $\alpha \in E(P(X)) \setminus \{0\}$. Then $\alpha P(\operatorname{ran}\alpha)$ and $P(\operatorname{ran}\alpha)$ are subsemihypergroups of $P(X)$ and*

$$\alpha P(\operatorname{ran}\alpha) \cong P(\operatorname{ran}\alpha)$$

under good isomorphism of semihypergroup.

Lemma 2.12. *(see [10]) Let X be a nonempty set and $\alpha \in E(P(X)) \setminus \{0\}$. If $|a\alpha^{-1}| \geq |\operatorname{ran}\alpha|$ for every $a \in \operatorname{ran}\alpha$, then $\alpha I(X)\alpha = \alpha P(\operatorname{ran}\alpha)$.*

Theorem 2.13. *Let X be a nonempty set and $\alpha \in E(P(X)) \setminus \{0\}$. Then the local subset $\alpha\rho \otimes G(X)/\rho \otimes \alpha\rho$ of $P(X)/\rho$ is a local subsemihypergroup of $P(X)/\rho$ if:*

(i) $\alpha = 1_A$ *for some nonempty subset* A *of* X*, or*

(ii) $|a\alpha^{-1}| \geq |\operatorname{ran}\alpha|$ *for every* $a \in \operatorname{ran}\alpha$*.*

Proof. If α satisfies (i) then by Corollary 2.10, $\alpha \circ G(X) \circ \alpha$ is a subsemihypergroup of $P(X)$.

Assume that α satisfies (ii). Then by Lemma 2.9 and Lemma 2.10 we have $\alpha \circ G(X) \circ \alpha = \alpha I(X)\alpha = \alpha P(\operatorname{ran}\alpha)$ and then by Lemma 2.11 we have $\alpha P(\operatorname{ran}\alpha)$ is a subsemihypergroup of $P(X)$. Therefore $\alpha \circ G(X) \circ \alpha$ is a subsemihypergroup of $P(X)$.

Next, we let $\gamma, \gamma' \in G(X)$. Then

$$\begin{aligned}(\alpha\rho \otimes \gamma\rho) \otimes \alpha\rho &= \{x\rho \mid x \in \alpha \circ \gamma\} \otimes \alpha\rho \\ &= \bigcup_{x \in \alpha\circ\gamma} x\rho \otimes \alpha\rho \\ &= \bigcup_{x \in \alpha\circ\gamma} \{y\rho \mid y \in x \circ \alpha\} \\ &= \{y\rho \mid y \in \alpha \circ \gamma \circ \alpha\}.\end{aligned}$$

Similarly, $(\alpha\rho \otimes \gamma'\rho) \otimes \alpha\rho = \{y\rho \mid y \in \alpha \circ \gamma' \circ \alpha\}$. Therefore

$$\begin{aligned}(\alpha\rho \otimes \gamma\rho \otimes \alpha\rho) \otimes (\alpha\rho \otimes \gamma'\rho \otimes \alpha\rho) &= \{x\rho \mid x \in \alpha \circ \gamma \circ \alpha\} \otimes \{y\rho \mid y \in \alpha \circ \gamma' \circ \alpha\} \\ &= \bigcup_{\substack{x \in \alpha\circ\gamma\circ\alpha \\ y \in \alpha\circ\gamma'\circ\alpha}} x\rho \otimes y\rho \\ &= \bigcup_{\substack{x \in \alpha\circ\gamma\circ\alpha \\ y \in \alpha\circ\gamma'\circ\alpha}} \{z\rho \mid z \in x \circ y\} \\ &= \{z\rho \mid z \in (\alpha \circ \gamma \circ \alpha) \circ (\alpha \circ \gamma' \circ \alpha)\} \quad (13)\end{aligned}$$

Since $\alpha \circ G(X) \circ \alpha$ is a subsemihypergroup of $P(X)$, there exist $\beta \in G(X)$ such that

$$(\alpha \circ \gamma \circ \alpha) \circ (\alpha \circ \gamma' \circ \alpha) = \alpha \circ \beta \circ \alpha. \quad (14)$$

Consequently,

$$\begin{aligned}(\alpha\rho \otimes \gamma\rho \otimes \alpha\rho) \otimes (\alpha\rho \otimes \gamma'\rho \otimes \alpha\rho) &= \{z\rho \mid z \in (\alpha \circ \gamma \circ \alpha) \circ (\alpha \circ \gamma' \circ \alpha)\} \\ &\qquad \text{from (13)} \\ &= \{z\rho \mid z \in \alpha \circ \beta \circ \alpha\} \\ &\qquad \text{from (14)}\end{aligned}$$

$$= \alpha\rho \otimes \beta\rho \otimes \alpha\rho$$
$$\subseteq \alpha\rho \otimes G(X)/\rho \otimes \alpha\rho.$$

Hence $\alpha\rho \otimes G(X)/\rho \otimes \alpha\rho$ is a subsemihypergroup of $P(X)/\rho$. □

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