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WEAKLY T_F TYPE CONTRACTIVE MAPPINGS

Mehmet Kir[1 §], Hukmi Kiziltunc[2]
[1,2]Department of Mathematics
Ataturk University
25240, Erzurum, TURKEY

Abstract: In this paper, the concept of weakly T_F -contractive conditions are considered for the Banach, Kannan and Chatterjea fixed point theorems. It is shown that these mappings have a unique fixed point in a complete metric space.

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1. Introduction and Preliminaries

In 1922, Banach proved his famous theorem which ensures the existence and uniqueness of the fixed point.

A mapping $T : X \to X$, where (X, d) is a metric space, is said to be a contraction if there exists $k \in [0, 1)$ such that for all $x, y \in X$,

$$d(Tx, Ty) \leq kd(x, y). \tag{1.1}$$

If the metric space (X, d) is complete then the mapping satisfying (1.1) has a unique fixed point.

It is clear that the inequality (1.1) implies the continuity of T. A natural

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[§]Correspondence author

question is that whether one can find a contractive condition which will imply the existence of the fixed point but will not imply continuity of the mapppıng. Kannan [2] established the following result in which the question has been answered in the affirmative.

Theorem 1. *[2] If a mapping $T : X \to X$ where (X, d) is a complete metric space, satisfies the inequality*

$$d(Tx, Ty) \leq a[d(x, Tx) + d(y, Ty)] \tag{1.2}$$

where $a \in \left[0, \frac{1}{2}\right)$ and $x, y \in X$, then T has a unique fixed point.

A similar contractive condition has been introduced by Chatterjea [3] as following:

Theorem 2. *[3] If a mapping $T : X \to X$ where (X, d) is a complete metric space, satisfies the inequality*

$$d(Tx, Ty) \leq b[d(x, Ty) + d(y, Tx)] \tag{1.3}$$

such that $b \in \left[0, \frac{1}{2}\right)$ and $x, y \in X$, then T has a unique fixed point.

In 2010, Moradi and Beiranvand gave the following result [8]:

Theorem 3. *(T_F-Contraction Mapping Theorem) Let (X, d) be a complete metric space and $T, f : X \to X$ be self- mappings such that T is one-to-one and graph closed (or subsequentially convergent and continuous). Let f satisfying the inequality*

$$F(d(Tfx, Tfy)) \leq \alpha F(d(Tx, Ty)) \tag{1.4}$$

where $\alpha \in [0, 1)$ and $F : [0, \infty) \to [0, \infty)$ is nondecreasing continuous from the right and $F^{-1}(0) = \{0\}$. Then f has a unique fixed point in the complete metric space (X, d).

In this study our purpose is to introduce weakly T_F contractive conditions for Banach fixed point theorem, Kannan fixed point theorem and Chatterjea fixed point theorem.

Definition 1. [4] Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be sequentially convergent if we have, for every sequence $\{y_n\}$, if

$\{Ty_n\}$ converges then $\{y_n\}$ is also convergent. T is said to be subsequentially convergent if we have, for every sequence $\{y_n\}$, if $\{Ty_n\}$ converges then $\{y_n\}$ has a convergent subsequence.

2. Main Results

For the simplicity, we will use the following symbols.

1) We donote by $\mathcal{F}$ the set of all functions $F : [0, \infty) \to [0, \infty)$ which are continuous, monotone nondecreasing and $F(t) = 0$ if and only if $t = 0$.

2) We denote by Ψ the set of all functions $\psi : [0, \infty) \to [0, \infty)$ *is a continuous function such that* $\psi(t) = 0$ *if and only if* $t = 0$.

3) Also, we denote by $SSC(X)$ the set of all mappings $T : X \to X$ such that T is one-to-one , continuous and subsequentially convergent, by $SC(X)$ the set of all mappings $T : X \to X$ such that T is one-to-one , continuous and sequentially convergent.

Theorem 4. *(Weakly T_F Contractive Mapping Theorem) Let (X, d) be a complete metric space and $f : X \to X$ be a mapping. Let $T \in SSC$ and f satisfy the inequality*

$$F(d(Tfx, Tfy)) \leq F(d(Tx, Ty)) - \psi(d(Tx, Ty)) \tag{2.1}$$

where $F \in \mathcal{F}, \psi \in \Psi$. Then f has a unique fixed point in X. Also, if T is sequentially convergent then, for every $x_0 \in X$ the sequence of iterates $\{f^n x_0\}$ converges to the fixed point.

Proof. Let x_0 be an arbitrary point in X. We define the sequence $\{x_n\}$ by $x_{n+1} = fx_n = f^{n+1}x_0 \quad n = 1, 2, \cdots$. Using (2.1), we have

$$\begin{aligned} F(d(Tx_n, Tx_{n+1})) &= F(d(Tfx_{n-1}, Tfx_n)) \\ &\leq F(d(Tx_{n-1}, Tx_n)) - \psi(Tx_{n-1}, Tx_n) &(2.2) \\ &\leq F(d(Tx_{n-1}, Tx_n)). &(2.3) \end{aligned}$$

This implies that

$$d(Tx_n, Tx_{n+1}) \leq d(Tx_{n-1}, Tx_n).$$

It is clear $\{d(Tx_n, Tx_{n+1})\}$ is a monotone decreasing sequence, and consequently there exists an $r \geq 0$ such that

$$d(Tx_n, Tx_{n+1}) \to r \quad as\ n \to \infty.$$

Letting $n \to \infty$ in (2.2), we obtain that $F(r) \leq F(r) - \psi(r)$. This case implies that $r = 0$.

Now,we will prove that $\{Tx_n\}$ is a Cauchy sequence. Suppose that $\{Tx_n\}$ is not a Cahucy sequence. Then there exists an $\epsilon > 0$ and there exist subsequences $\{Tx_{m(k)}\}$, $\{Tx_{n(k)}\}$ of $\{Tx_n\}$ with $n(k) > m(k) > k$ such that

$$d\left(Tx_{m(k)}, Tx_{n(k)}\right) \geq \epsilon. \tag{2.4}$$

Further, corresponding to $m(k)$, we can choose $n(k)$ in such a way that it is the smallest integer with $n(k) > m(k)$ and

$$d\left(Tx_{m(k)}, Tx_{n(k)-1}\right) < \epsilon. \tag{2.5}$$

Also, using (2.4) we have

$$\begin{aligned} \epsilon &\leq d\left(Tx_{m(k)}, Tx_{n(k)}\right) \\ &\leq d\left(Tx_{m(k)}, Tx_{n(k)-1}\right) + d\left(Tx_{n(k)}, Tx_{n(k)-1}\right) \\ &< \epsilon + d\left(Tx_{n(k)}, Tx_{n(k)-1}\right). \end{aligned} \tag{2.6}$$

Letting $k \to \infty$ in (2.6), we have

$$\lim_{k\to\infty} d\left(Tx_{m(k)}, Tx_{n(k)}\right) = \epsilon. \tag{2.7}$$

Again, we have

$$d\left(Tx_{n(k)}, Tx_{m(k)}\right) \leq d\left(Tx_{n(k)}, Tx_{n(k)-1}\right) + d\left(Tx_{n(k)-1}, Tx_{m(k)-1}\right) + d\left(Tx_{m(k)-1}, Tx_{m(k)}\right) \tag{2.8}$$

$$\begin{aligned} &\leq d\left(Tx_{n(k)}, Tx_{n(k)-1}\right) \\ &\quad + [d\left(Tx_{n(k)-1}, Tx_{n(k)}\right) + d\left(Tx_{n(k)}, Tx_{m(k)}\right) \\ &\quad + d\left(Tx_{m(k)}, Tx_{m(k)-1}\right)] \\ &\quad + d\left(Tx_{m(k)-1}, Tx_{m(k)}\right) \end{aligned} \tag{2.9}$$

letting $k \to \infty$ in (2.8) and (2.9), we have

$$\epsilon \leq \lim_{k\to\infty} d\left(Tx_{n(k)-1}, Tx_{m(k)-1}\right) \leq \epsilon. \tag{2.10}$$

This implies that

$$\lim_{k\to\infty} d\left(Tx_{n(k)-1}, Tx_{m(k)-1}\right) = \epsilon. \tag{2.11}$$

Substituting (2.4) in (2.1)

$$\begin{aligned} F(\epsilon) &\leq F(d\left(Tx_{m(k)}, Tx_{n(k)}\right)) \\ &= F(d\left(Tfx_{m(k)-1}, Tfx_{n(k)-1}\right)) \quad (2.12)\\ &\leq F(d\left(Tx_{m(k)-1}, Tx_{n(k)-1}\right)) - \psi\left(d\left(Tx_{m(k)-1}, Tx_{n(k)-1}\right)\right) \quad (2.13) \end{aligned}$$

use the (2.7), (2.11) in both (2.12) and (2.13)

$$F(\epsilon) \leq F(\epsilon) - \psi\left(\epsilon\right). \quad (2.14)$$

This implies that $\epsilon = 0$. But this case is a contradiction. Thus $\{Tx_n\}$ is a Cauchy sequence in the complete metric space X. Hence, there is $v \in X$ such that

$$\lim_{n\to\infty} Tx_n = v. \quad (2.15)$$

Since T is subsequentially convergent, $\{x_n\}$ has a convergent subsequence. Thus there is an $u \in X$ such that

$$\lim_{k\to\infty} x_{n(k)} = u. \quad (2.16)$$

Since T is continous and $x_{n(k)} \to u$, we have

$$\lim_{n\to\infty} Tx_{n(k)} = Tu. \quad (2.17)$$

Since $\left\{Tx_{n(k)}\right\}$ is a subsequence of $\{Tx_n\}$, so that $Tu = v$. Now, we will show that $u \in X$ is a fixed point of f.

$$\begin{aligned} F(d\left(Tx_{n(k)}, Tfu\right)) &= F\left(d\left(Tfx_{n(k)-1}, Tfu\right)\right) \\ &\leq F(d\left(Tx_{n(k)-1}, Tu\right) - \psi\left(d\left(Tx_{n(k)-1}, Tu\right)\right) \quad (2.18) \end{aligned}$$

Letting $k \to \infty$ in (2.18) and using the (2.16), (2.17) we have

$$F\left(d\left(Tu, Tfu\right)\right) \leq F\left(0\right) - \psi\left(0\right) \quad (2.19)$$

this implies that $F\left(d\left(Tu, Tfu\right)\right) = 0$. As T is one-to-one, then $fu = u$. To prove the uniqueness of the fixed point, assume that $u' \in X$ is an other fixed points of f. Thus we have $fu' = u'$ and

$$\begin{aligned} F(d\left(Tu, Tu'\right)) &= F(d\left(Tfu, Tfu'\right)) \\ &\leq F(d\left(Tu, Tu'\right)) - \psi\left(d\left(Tu, Tu'\right)\right). \quad (2.20) \end{aligned}$$

Inequality (2.20) is a contradiction unless $\psi(d(Tu, Tu')) = 0$. This implies that $Tu = Tu'$. Since T is one-to-one $u = u'$, that is, the fixed point is unique.

Also, if T is sequentially convergent, by replacing $\{n\}$ with $\{n(k)\}$ we obtain that

$$\lim_{n\to\infty} x_n = u.$$

This implies that $\{x_n\}$ converges to the fixed point of f. □

Remark 1. In Theorem 4, if we take $\psi(t) = kF(t)$ where $t \in [0, \infty)$ and $k \in (0, 1]$, we obtain the above result that given by Moradi and Beiranvand [8].

If we take $Tx = x$, we obtain the following result given by Dutta and Choudhury [6].

Corollary 1. *Let (X, d) be a complete metric space and let $f : X \to X$ be a self-mapping satisfying the inequality*

$$\psi(d(fx, fy)) \leq \psi(d(x, y)) - \phi(d(x, y))$$

where ψ,$\phi : [0, \infty) \to [0, \infty)$ are both continuous and monotone nondecreasing functions with $\psi(t) = 0 = \phi(t)$ if and only if $t = 0$. Then f has a unique fixed point.

If we take $Tx = x$ and $F(s) = \int_0^s \varphi(t)\,dt$ with $\psi(t) = kF(t)$ where $t \in [0, \infty)$ and $k \in (0, 1)$, then we obtain the following result given by Branciari [11].

Corollary 2. *Let (X, d) be a complete metric spaces and let $f : X \to X$ be a mapping such that for each $x, y \in X$ and $c \in (0, 1)$,*

$$\int_0^{d(fx,fy)} \varphi(t)\,dt \leq c \int_0^{d(x,y)} \varphi(t)\,dt,$$

where $\varphi : [0, \infty) \to [0, \infty)$ is a Lebesgue-integrable mapping which summeble (i.e., with finite integral) on each compact subset of $[0, \infty)$, nonnegative, and such that for each $\epsilon > 0$, $\int_0^\epsilon \varphi(t)\,dt > 0$; then f has a unique fixed point $a \in X$ such that for each $x \in X$, $\lim_{n\to\infty} f^n x = a$.

Theorem 5. *(Weakly T_F Kannan Contractive Mapping Theorem) Let (X, d) be a complete metric space and $f : X \to X$ be mapping. Let $T \in SSC$ and f satisfies the inequality*

$$F(d(Tfx, Tfy)) \leq F\left(\frac{1}{2}\left[d(Tx, Tfx)) + d(Ty, Tfy)\right]\right) - \psi(d(Tx, Tfx), d(Ty, Tfy)) \quad (2.21)$$

where $F \in \mathcal{F}, \psi \in \Psi$. Then f has a unique fixed point. Also, if T is sequentially convergent then for $x_0 \in X$ the sequence of iterates $\{f^n x_0\}$ converges to this fixed point.

Proof. Let $x_0 \in X$ and $\{x_n\}$ be a sequence in X defined by $x_{n+1} = fx_n$, $n = 1, 2, \cdots$. Using the (2.21), we have

$$\begin{aligned} F(d(Tx_n, Tx_{n+1})) &= F(d(Tfx_{n-1}, Tfx_n)) \\ &\leq F(\frac{d(Tx_{n-1}, Tfx_{n-1})) + d(Tx_n, Tfx_n)}{2}) \\ &\quad -\psi(d(Tx_{n-1}, Tfx_{n-1}), d(Tx_n, Tfx_n)) \\ &= F(\frac{d(Tx_{n-1}, Tx_n) + d(Tx_n, Tx_{n+1})}{2}) \quad (2.22) \\ &\quad -\psi(d(Tx_{n-1}, Tx_n) + d(Tx_n, Tx_{n+1})) \quad (2.23) \\ &\leq F(\frac{d(Tx_{n-1}, Tx_n) + d(Tx_n, Tx_{n+1})}{2}). \quad (2.24) \end{aligned}$$

Note that F is nondecreasing continuous, then we have $d(Tx_n, Tx_{n+1}) \leq d(Tx_{n-1}, Tx_n)$. As the sequence $\{d(Tx_n, Tx_{n+1})\}$ is a monotone decreasing sequence of non-negative real numbers, thus

$$\lim_{n\to\infty} d(Tx_n, Tx_{n+1}) = 0. \quad (2.25)$$

By following the similar method in the proof of the Theorem 4, we obtain that $\{Tx_n\}$ is a Cauchy sequence in the complete metric space (X, d). Therefore there is an $u \in X$ such that $x_n \to u$ and $Tx_n \to v$ (as $n \to \infty$). Also, from the (2.21), we have

$$F(d(Tx_{n+1}, Tfu)) \leq F\left(\frac{d(Tx_n, Tfx_n) + d(Tu, Tfu)}{2}\right) - \psi(d(Tx_n, Tfx_n), d(Tu, Tfu)) \quad (2.26)$$

letting $n \to \infty$ in (2.26), we have $\psi(d(Tx_n, Tfx_n), d(Tu, Tfu)) \leq 0$. Thus we get $u \in X$ is a fixed point of f. It is easy to see uniqness of the fixed point. □

If we take $Fs = s$, then we obtain the following result, given by Moradi and Davood [7].

Corollary 3. *Let (X, d) be a complete metric space and $T, S : X \to X$ be mappings such that T is continuous, one to one and subsequentially convergent. If $\mu \in \left[0, \frac{1}{2}\right)$ and $x, y \in X$,*

$$d(TSx, TSy) \leq \mu\left[d(Tx, TSy) + d(Ty, TSx)\right]$$

then, S has a unique fixed point. Also, if T is sequentially convergent then for every $x_0 \in X$ the sequence of iterates $\{S^n x_0\}$ converges to the fixed point.t.

Theorem 6. *(Weakly T_F Chatterjea Contractive Mapping Theorem) Let (X, d) be a complete metric space and $f : X \to X$ be a mapping. Let $T \in SSC$ and f satisfies the inequality*

$$F(d(Tfx, Tfy)) \leq F\left(\frac{1}{2}\left[d(Tx, Tfy)) + d(Ty, Tfx)\right]\right) - \psi\left(d(Tx, Tfy), d(Ty, Tfx)\right) \quad (2.27)$$

where $F \in \digamma, \psi \in \Psi$. Then f has a unique fixed point. Also if T is sequentially convergent then for every $x_0 \in X$ the sequence of iterates $\{f^n x_0\}$ converges to this fixed point.

Proof. Let $x_0 \in X$ and $\{x_n\}$ be a sequence in X defined by $x_{n+1} = fx_n$, such that $n = 1, 2, \cdots$. From the (2.27), we have

$$\begin{aligned} F(d(Tx_n, Tx_{n+1})) &= F(d(Tfx_{n-1}, Tfx_n)) && (2.28) \\ &\leq F\left(\frac{1}{2}d(Tx_{n-1}, Tx_{n+1})\right) \\ &\quad -\psi\left(d(Tx_{n-1}, Tx_{n+1}), 0\right) && (2.29) \\ &\leq F\left(\frac{d(Tx_{n-1}, Tx_n) + d(Tx_n, Tx_{n+1})}{2}\right). && (2.30) \end{aligned}$$

Since F nondecreasing, we obtain that $d(Tx_n, Tx_{n+1}) \leq d(Tx_{n-1}, Tx_n)$. It is clear that $\{d(Tx_n, Tx_{n+1})\}$ is a monotone decreasing sequence of non-negative real numbers. Hence there is an $r \in \mathbb{R}$ such that

$$\lim_{n\to\infty} d(Tx_n, Tx_{n+1}) = r. \quad (2.31)$$

From the (2.28), we have

$$F(d(Tx_n, Tx_{n+1})) = F(d(Tfx_{n-1}, Tfx_n)) \quad (2.32)$$

$$\begin{aligned} &\leq F\left(\frac{1}{2}\left[d\left(Tx_{n-1}, Tfx_n\right)+d\left(Tx_n, Tfx_{n-1}\right)\right]\right) \\ &= F\left(\frac{1}{2}\left[d\left(Tx_{n-1}, Tx_{n+1}\right)\right]\right) \qquad (2.33) \\ &\leq F\left(\frac{1}{2}\left[d\left(Tx_{n-1}, Tx_n\right)+d\left(Tx_n, Tx_{n+1}\right)\right]\right). \qquad (2.34) \end{aligned}$$

Letting $n \to \infty$ respectively, in (2.32), (2.33), (2.34) we obtain

$$\lim_{n\to\infty} F\left(\frac{1}{2}\left[d\left(Tx_{n-1}, Tx_{n+1}\right)\right]\right) = F(r) \qquad (2.35)$$

Also, letting $n \to \infty$ in the (2.28) we obtain that $\psi(r, 0) = 0$. This implies that $r = 0$.

Following the similar process in the proof of the Theorem 4, we obtain that $\{Tx_n\}$ is a Cauchy sequence in the complete metric space (X, d) and there is an $u \in X$ such that $x_n \to u$ and $Tx_n \to v$ (as $n \to \infty$).

Also, from the 2.27, we have

$$\begin{aligned} F\left(d\left(Tx_{n+1}, Tfu\right)\right) &= F\left(d\left(Tfx_n, Tfu\right)\right) \\ &\leq F\left(\frac{1}{2}\left[d\left(Tx_n, Tfu\right)+d\left(Tu, Tfx_n\right)\right]\right) - \\ &\quad \psi\left(d\left(Tx_n, Tfu\right), d\left(Tu, Tfx_n\right)\right) \qquad (2.36) \end{aligned}$$

letting $k \to \infty$ in 2.36, it follows that $\psi(d(Tu, Tfu), d(Tu, Tfu)) \leq 0$. This implies that $Tu = Tfu$. Note that T is one-to-one so $fu = u$. Also, it is easy to see uniqness of the fixed point. □

If we take $Tx = x$ and $Fx = x$ then, we obtain the following result given by Choudhury [2].

Corollary 4. *Let (X, d) be a complete metric space and $f : X \to X$ be a mapping such that for each $x, y \in X$,*

$$d(fx, fy) \leq \frac{1}{2}\left[d(x, fy) + d(y, fx)\right] - \psi\left(d(x, fy), d(y, fx)\right)$$

where $\psi : [0, \infty) \times [0, \infty) \to [0, \infty)$ is a continuous function such that $\psi(x, y) = 0$ if and only if $x = y = 0$. Then f has a unique fixed point.

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References

[1] S. Banach, Sur les operations dans les ensembles abstraits et leur application aux equations integerales, *Fundamenta Mathematicae*, **3** (1922), 133-181.

[2] R. Kannan, Some results on fixed points, *Bull. Calcutta Math. Soc.*, **60** (1968), 71-76.

[3] S. Chatterjea, Fixed point theorems, *C.R. Acad. Bulgare Sci.*, **25** (1972), 727-730.

[4] B.E. Rhoades, Some theorems on weakly contractive maps, *Nonlinear Analysis: Theory, Methods and Applications*, **47**, No. 4 (2001), 2683-2693.

[5] B.S. Choudhury, Unique fixed point theorem for weakly $c-$contractive mappings, *Kathmandu Univ. J. Sci. Eng. and Tech.*, **5**, No. 1 (2009), 6-13.

[6] P.N. Dutta, B.S. Choudhury, A generalization of contraction principle in metric spces, *Fixed Point Theory Appl.* (2008), Article ID 406368.

[7] S. Moradi, A. Davood, New extension of Kannan fixed point theorem on complete metric and generalized metric spaces, *Int. Journal of Math. Analysis*, **5**, No. 47 (2011), 2313-1320.

[8] S. Moradi, A. Beiranvand, Fixed point of T_F-contractive single-valued mappings, *Iranian Journal of Mathematical Sciences and Informatics*, **5** (2010), 25-32.

[9] E. Karapınar, Fixed point theory for cyclic weaker $\phi-$contraction, *Appl. Math. Lett.*, **24**, No. 6 (2011), 882-825.

[10] M. Kir, H. Kiziltunc, T_F type contractive conditions for Kannan and Chatterjea fixed point theorems, *Advances in Fixed Point Theory*, **4**, No. 1 (2014), 140-148.

[11] A. Branciari, A fixed point theorem for mapping satisfying a general contractive condition of integral type, *Int. J.M. and M. Sciences*, **29**, No. 9 (2002), 531-536.