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GRÜSS TYPE INTEGRAL INEQUALITIES FOR GENERALIZED RIEMANN-LIOUVILLE FRACTIONAL INTEGRALS

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Abstract: In this article, we obtain generalizations for Grüss type integral inequality by using generalized Riemann-Liouville fractional integral.

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1. Introduction

If f and g are two continuous functions on $[a, b]$ satisfying $m \leq f(t) \leq M$ and $p \leq g(t) \leq P$ for all $t \in [a, b]$, $m, M, p, P \in \mathbb{R}$,

$$\left| \frac{1}{b-a} \int_a^b f(t)g(t)dt - \frac{1}{(b-a)^2} \int_a^b f(t)dt \int_a^b g(t)dt \right| \leq \frac{1}{4}(M-m)(P-p). \quad (1.1)$$

(1.1) inequality is well-known in literature as Grüss Inequality. It is defined as the integral inequality that establishes as a connection between the product of two functions and the product of the integrals [1].

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Grüss type inequalities are now vast and many extensions of the classical inequalities were intensively studied by many authors. Integral inequalities and applications have been addressed extensively by several researchers. For example, we refer the reader to $[4-10]$ and the references cited therein. Also, there are many extensions of Grüss type inequalities by using Riemann-Liouville fractional integrals.

2. Preliminaries

In this section, we will give some definitions, lemmas and theorems which we use later in this article.

Definition 1. [10, 11] *A function $f(t)$ is said to be in the $L_{p,k}[0,\infty)$ space if*

$$L_{p,k}[0,\infty) = \left\{ f : \|f\|_{L_{p,k[0,\infty)}} = \left(\int_0^\infty |f(t)|^p \, t^k dt \right)^{\frac{1}{p}} < \infty, \ 1 \leq p < \infty, \ k \geq 0 \right\}. \quad (2.1)$$

For $k = 0$,

$$L_p[0,\infty) = \left\{ f : \|f\|_{L_{p[0,\infty)}} = \left(\int_0^\infty |f(t)|^p \, dt \right)^{\frac{1}{p}} < \infty, \ 1 \leq p < \infty \right\}.$$

Definition 2. [12] *Let $f \in L_1[0,\infty)$.The Riemann- Liouville fractional integral of order $\alpha \geq 0$ is defined by*

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-x)^{\alpha-1} f(x) \, dx, \quad (2.2)$$

$$I^0 f(t) = f(t),$$

where Γ is the gamma function.

Definition 3. [10, 11] *Let $f \in L_{1,k}[0,\infty)$. The Generalized Riemann-Liouville fractional integral $I^{\alpha,k} f(x)$ of order $\alpha \geq 0$ and $k \geq 0$ is defined by*

$$I^{\alpha,k}f(x) = \frac{(k+1)^{1-\alpha}}{\Gamma(\alpha)} \int_0^x \left(x^{k+1} - t^{k+1}\right)^{\alpha-1} t^k f(t)\,dt, \tag{2.3}$$

$$I^{0,k}f(x) = f(x),$$

where Γ is the gamma function.

For the convenience of establishing our results, we give the semi-group property:

$$I^{\alpha,k}I^{\beta,k}f(t) = I^{\alpha+\beta,k}f(t), \quad k \geq 0,\ \alpha \geq 0,\ \beta \geq 0, \tag{2.4}$$

which implies the commutative property

$$I^{\alpha,k}I^{\beta,k}f(t) = I^{\beta,k}I^{\alpha,k}f(t). \tag{2.5}$$

From Definition 3, if $f(t) = t^{\gamma}$, then we have

$$I^{\alpha,k}t^{\gamma} = \frac{(k+1)^{-\alpha}\Gamma(\frac{\gamma}{k+1}+1)}{\Gamma(\alpha+\frac{\gamma}{k+1}+1)} t^{(k+1)\alpha+\gamma}, \quad k \geq 0,\ \alpha \geq 0,\ \beta \geq 0. \tag{2.6}$$

(2.3) – (2.6) results for Generalized Riemann-Liouville fractional integrals are reduced (2.2) fractional integral and its properties when $k = 0$.

Dahmani et al. [2] gave the following fractional integral inequalities for the integral (2.2) in Definition 2.

Theorem 1. *Let $f, g \in L[0,\infty)$ and satisfy the following conditions:*

$$m \leq f(t) \leq M, \quad p \leq g(t) \leq P, \quad t \in [0,\infty), \quad m, M, p, P \in \mathbb{R}.$$

Then for all $t > 0$, $\alpha > 0$, $\beta > 0$,

i) $$\left|\frac{t^{\alpha}}{\Gamma(\alpha+1)} I^{\alpha}(fg)(t) - I^{\alpha}f(t)I^{\alpha}g(t)\right| \leq \left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}\right)^2 (M-m)(P-p), \tag{2.7}$$

ii) $$\begin{aligned} &\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)} I^{\beta}(fg)(t) + \frac{t^{\beta}}{\Gamma(\beta+1)} I^{\alpha}(fg)(t) - I^{\alpha}f(t)I^{\beta}g(t) - I^{\beta}f(t)I^{\alpha}g(t)\right)^2 \\ &\leq \left\{\left(M\frac{t^{\alpha}}{\Gamma(\alpha+1)} - I^{\alpha}f(t)\right)\left(I^{\beta}f(t) - m\frac{t^{\beta}}{\Gamma(\beta+1)}\right)\right. \\ &\qquad \left. + \left(I^{\alpha}f(t) - m\frac{t^{\alpha}}{\Gamma(\alpha+1)}\right)\left(M\frac{t^{\beta}}{\Gamma(\beta+1)} - I^{\beta}f(t)\right)\right\} \\ &\qquad \times \left\{\left(P\frac{t^{\alpha}}{\Gamma(\alpha+1)} - I^{\alpha}g(t)\right)\left(I^{\beta}g(t) - p\frac{t^{\beta}}{\Gamma(\beta+1)}\right)\right. \\ &\qquad \left. + \left(I^{\alpha}g(t) - p\frac{t^{\alpha}}{\Gamma(\alpha+1)}\right)\left(P\frac{t^{\beta}}{\Gamma(\beta+1)} - I^{\beta}g(t)\right)\right\}. \end{aligned} \tag{2.8}$$

Jessada Tariboon et al in [3] proved the following fractional integral inequalities for functions which are bounded with integrable functions.

Theorem 2. *Let $f \in L[0,\infty)$. Suppose that:*

(H_1) There exist two integrable function φ_1, φ_2 on $[0,\infty)$ such that

$$\varphi_1(t) \leq f(t) \leq \varphi_2(t), \quad \forall t \in [0,\infty).$$

Then for $t > 0$, $\alpha, \beta > 0$, we have

$$I^\beta \varphi_1(t) I^\alpha f(t) + I^\alpha \varphi_2(t) I^\beta f(t) \geq I^\alpha \varphi_2(t) I^\beta \varphi_1(t) + I^\alpha f(t) I^\beta f(t). \tag{2.9}$$

Theorem 3. *Let f, $g \in L\ [0,\infty)$. Assume that:*

Condition (H_1) holds true, and moreover

(H_2)there exist two integrable function ψ_1, ψ_2 on $[0,\infty)$ such that

$$\psi_1(t) \leq g(t) \leq \psi_2(t), \quad \forall t \in [0,\infty).$$

Then for $t > 0$, $\alpha, \beta > 0$, the following inequalities hold:

$$\begin{aligned}
&(a)\ I^\beta \psi_1(t) I^\alpha f(t) + I^\alpha \varphi_2(t) I^\beta g(t) \geq I^\beta \psi_1(t) I^\alpha \varphi_2(t) + I^\alpha f(t) I^\beta g(t),\\
&(b)\ I^\beta \varphi_1(t) I^\alpha g(t) + I^\alpha \psi_2(t) I^\beta f(t) \geq I^\beta \varphi_1(t) I^\alpha \psi_2(t) + I^\beta f(t) I^\alpha g(t),\\
&(c)\ I^\alpha \varphi_2(t) I^\beta \psi_2(t) + I^\alpha f(t) I^\beta g(t) \geq I^\alpha \varphi_2(t) I^\beta g(t) + I^\beta \psi_2(t) I^\alpha f(t),\\
&(d)\ I^\alpha \varphi_1(t) I^\beta \psi_1(t) + I^\alpha f(t) I^\beta g(t) \geq I^\alpha \varphi_1(t) I^\beta g(t) + I^\beta \psi_1(t) I^\alpha f(t).
\end{aligned} \tag{2.10}$$

Lemma 1. *Let f, φ_1, $\varphi_2 \in L[0,\infty)$. Assume that the condition (H_1) hold. Then, for $t > 0$, $\alpha > 0$, we have*

$$\begin{aligned}
\frac{t^\alpha}{\Gamma(\alpha+1)} I^\alpha f^2(t) - (I^\alpha f(t))^2 &= \Big\{(I^\alpha \varphi_2(t) - I^\alpha f(t))(I^\alpha f(t) - I^\alpha \varphi_1(t))\\
&\quad - \frac{t^\alpha}{\Gamma(\alpha+1)} I^\alpha((\varphi_2(t) - f(t))(f(t) - \varphi_1(t))\Big\}\\
&\quad + \frac{t^\alpha}{\Gamma(\alpha+1)} I^\alpha \varphi_1(t) f(t) - I^\alpha \varphi_1(t) I^\alpha f(t)\\
&\quad + \frac{t^\alpha}{\Gamma(\alpha+1)} I^\alpha \varphi_2(t) f(t) - I^\alpha \varphi_2(t) I^\alpha f(t)\\
&\quad + I^\alpha \varphi_1(t) I^\alpha \varphi_2(t) - \frac{t^\alpha}{\Gamma(\alpha+1)} I^\alpha \varphi_1(t) \varphi_2(t).
\end{aligned} \tag{2.11}$$

Theorem 4. *Let $f, g, \varphi_1, \varphi_2, \psi_1, \psi_2 \in L[0,\infty)$, satisfy the conditions (H_1) and (H_2) on $[0,\infty)$. Then for all $t > 0$, $\alpha > 0$:*

$$\left|\frac{t^\alpha}{\Gamma(\alpha+1)} I^\alpha(fg)(t) - I^\alpha f(t) I^\alpha g(t)\right| \leq \sqrt{T(f,\varphi_1,\varphi_2)T(g,\psi_1,\psi_2)}, \tag{2.12}$$

where $T(u,v,w)$ is defined by

$$\begin{aligned} T(u,v,w) =& (I^\alpha w(t) - I^\alpha u(t))(I^\alpha u(t) - I^\alpha v(t)) \\ &+ \frac{t^\alpha}{\Gamma(\alpha+1)} I^\alpha v(t)u(t) - I^\alpha v(t) I^\alpha u(t) \\ &+ \frac{t^\alpha}{\Gamma(\alpha+1)} I^\alpha w(t)u(t) - I^\alpha w(t) I^\alpha u(t) \\ &+ I^\alpha v(t) I^\alpha w(t) - \frac{t^\alpha}{\Gamma(\alpha+1)} I^\alpha v(t)w(t). \end{aligned} \tag{2.13}$$

3. Main Results

In this section, using (2.3) and Generalized Riemann-Liouville fractional integral, we will obtain inequalities of Grüss type. Our first result is the following theorem.

Theorem 5. *Let $f \in L_{1,k}[0,\infty)$ and $k \geq 0$, $t > 0$, $\alpha, \beta > 0$. Suppose that there exist two integrable functions φ_1, φ_2 on $[0,\infty)$ such that*

$$\varphi_1(t) \leq f(t) \leq \varphi_2(t) \quad \forall t \in [0,\infty). \tag{3.1}$$

Then the following inequality

$$\begin{aligned} I^{\beta,k}\varphi_1(t) I^{\alpha,k} f(t) + I^{\alpha,k}\varphi_2(t) I^{\beta,k} f(t) \\ \geq I^{\alpha,k}\varphi_2(t) I^{\beta,k}\varphi_1(t) + I^{\alpha,k} f(t) I^{\beta,k} f(t) \end{aligned} \tag{3.2}$$

holds true.

Proof. From (3.1), for all $x \geq 0$, $y \geq 0$, we have

$$(\varphi_2(x) - f(x))(f(y) - \varphi_1(y)) \geq 0,$$

$$\varphi_2(x) f(y) + \varphi_1(y) f(x) \geq \varphi_1(y)\varphi_2(x) + f(x) f(y). \tag{3.3}$$

If we multiply both sides of (3.3) by

$$\frac{(k+1)^{1-\alpha}(t^{k+1}-x^{k+1})^{\alpha-1}x^k}{\Gamma(\alpha)},$$

and integrate with respect to x on $(0,t)$, we obtain

$$f(y)I^{\alpha,k}\varphi_2(t)+\varphi_1(y)I^{\alpha,k}f(t)\geq\varphi_1(y)I^{\alpha,k}\varphi_2(t)+f(y)I^{\alpha,k}f(t). \tag{3.4}$$

If we multiply both sides of (3.4) by

$$\frac{(k+1)^{1-\beta}(t^{k+1}-y^{k+1})^{\beta-1}y^k}{\Gamma(\beta)},$$

and integrate with respect to y on $(0,t)$, we get

$$I^{\beta,k}\varphi_1(t)I^{\alpha,k}f(t)+I^{\alpha,k}\varphi_2(t)I^{\beta,k}f(t) \\ \geq I^{\alpha,k}\varphi_2(t)I^{\beta,k}\varphi_1(t)+I^{\alpha,k}f(t)I^{\beta,k}f(t). \tag{3.5}$$

This proves the theorem. □

Corollary 1. *Let us put $k=0$ in Theorem 5. Then the inequality (2.9) in Theorem 2 holds.*

Corollary 2. *Let $f\in L_{1,k}[0,\infty)$. Suppose that $m\leq f(t)\leq M$, $\forall t\in[0,\infty)$ and $m,M\in\mathbb{R}$. Then for $k\geq 0$, $t>0$, $\alpha>0$, $\beta>0$, we have*

$$m\frac{(k+1)^{-\beta}t^{(k+1)\beta}}{\Gamma(\beta+1)}I^{\alpha,k}f(t)+M\frac{(k+1)^{-\alpha}t^{(k+1)\alpha}}{\Gamma(\alpha+1)}I^{\beta,k}f(t) \\ \geq Mm\frac{(k+1)^{-(\alpha+\beta)}t^{(k+1)(\alpha+\beta)}}{\Gamma(\beta+1)\Gamma(\alpha+1)}+I^{\alpha,k}f(t)I^{\beta,k}f(t). \tag{3.6}$$

If we take $k=0$ in (3.6) we obtain results in [3].

Theorem 6. *Let f and g be two integrable functions on $[0,\infty)$ and $k\geq 0$, $t>0$, $\alpha,\beta>0$. Suppose that (3.1) holds and moreover assume that there exist ψ_1 and ψ_2 integrable functions on $[0,\infty)$ such that*

$$\psi_1(t)\leq g(t)\leq\psi_2(t),\quad \forall t\in[0,\infty). \tag{3.7}$$

Then the following inequalities hold:

$$\begin{aligned}
&(a) \quad I^{\beta,k}\psi_1(t)I^{\alpha,k}f(t) + I^{\alpha,k}\varphi_2(t)I^{\beta,k}g(t)\\
&\qquad \geq I^{\beta,k}\psi_1(t)I^{\alpha,k}\varphi_2(t) + I^{\alpha,k}f(t)I^{\beta,k}g(t),\\
&(b) \quad I^{\beta,k}\varphi_1(t)I^{\alpha,k}g(t) + I^{\alpha,k}\psi_2(t)I^{\beta,k}f(t)\\
&\qquad \geq I^{\beta,k}\varphi_1(t)I^{\alpha,k}\psi_2(t) + I^{\beta,k}f(t)I^{\alpha,k}g(t),\\
&(c) \quad I^{\alpha,k}\varphi_2(t)I^{\beta,k}\psi_2(t) + I^{\alpha,k}f(t)I^{\beta,k}g(t)\\
&\qquad \geq I^{\alpha,k}\varphi_2(t)I^{\beta,k}g(t) + I^{\beta,k}\psi_2(t)I^{\alpha,k}f(t),\\
&(d) \quad I^{\alpha,k}\varphi_1(t)I^{\beta,k}\psi_1(t) + I^{\alpha,k}f(t)I^{\beta,k}g(t)\\
&\qquad \geq I^{\alpha,k}\varphi_1(t)I^{\beta,k}g(t) + I^{\beta,k}\psi_1(t)I^{\alpha,k}f(t).
\end{aligned} \tag{3.8}$$

Proof. From (3.1) and (3.7) for $\forall t \in [0, \infty)$, we have

$$(\varphi_2(x) - f(x))(g(y) - \psi_1(y)) \geq 0.$$

Then

$$\varphi_2(x)g(y) + \psi_1(y)f(x) \geq \psi_1(y)\varphi_2(x) + f(x)g(y). \tag{3.9}$$

If we multiply both sides of (3.9) by $\frac{(k+1)^{1-\alpha}(t^{k+1}-x^{k+1})^{\alpha-1}x^k}{\Gamma(\alpha)}$ and integrate with respect to x on $(0,t)$, we get

$$g(y)I^{\alpha,k}\varphi_2(t) + \psi_1(y)I^{\alpha,k}f(t) \geq \psi_1(y)I^{\alpha,k}\varphi_2(t) + g(y)I^{\alpha,k}f(t). \tag{3.10}$$

If we multiply both sides of (3.10) by $\frac{(k+1)^{1-\beta}(t^{k+1}-y^{k+1})^{\beta-1}y^k}{\Gamma(\beta)}$ and integrate with respect to y on $(0,t)$, we get

$$\begin{aligned}
&I^{\beta,k}\psi_1(t)I^{\alpha,k}f(t) + I^{\alpha,k}\varphi_2(t)I^{\beta,k}g(t)\\
&\qquad\qquad \geq I^{\alpha,k}\varphi_2(t)I^{\beta,k}\psi_1(t) + I^{\alpha,k}f(t)I^{\beta,k}g(t).
\end{aligned} \tag{3.11}$$

This proves the statement (a).

To prove (b)-(d), we use the following inequalities:

$$(b)\ (\psi_2(x) - g(x))(f(y) - \varphi_1(y)) \geq 0,$$

$$(c)\ (\varphi_2(x) - f(x))(g(y) - \psi_2(y)) \leq 0,$$

$$(d)\ (\varphi_1(x) - f(x))(g(y) - \psi_1(y)) \leq 0.$$

□

The following inequalities present a special case of Theorem 6.

Corollary 3. *Let $f, g \in L_{1,k}[0,\infty)$ and $k \geq 0$, $t > 0$, $\alpha > 0$, $\beta > 0$. Suppose that there exist real constants m, M, n, N, such that*

$$m \leq f(t) \leq M, \quad n \leq g(t) \leq N, \quad \forall t \in [0,\infty). \tag{3.12}$$

Then we have:

(a)

$$(a') \; n\frac{(k+1)^{-\beta}t^{(k+1)\beta}}{\Gamma(\beta+1)}I^{\alpha,k}f(t) + M\frac{(k+1)^{-\alpha}t^{(k+1)\alpha}}{\Gamma(\alpha+1)}I^{\beta,k}g(t) \geq nM\frac{(k+1)^{-(\alpha+\beta)}t^{(k+1)(\alpha+\beta)}}{\Gamma(\beta+1)\Gamma(\alpha+1)} + I^{\alpha,k}f(t)I^{\beta,k}g(t).$$

(b)

$$(b') \; m\frac{(k+1)^{-\beta}t^{(k+1)\beta}}{\Gamma(\beta+1)}I^{\alpha,k}g(t) + N\frac{(k+1)^{-\alpha}t^{(k+1)\alpha}}{\Gamma(\alpha+1)}I^{\beta,k}f(t) \geq mN\frac{(k+1)^{-(\alpha+\beta)}t^{(k+1)(\alpha+\beta)}}{\Gamma(\beta+1)\Gamma(\alpha+1)} + I^{\beta,k}f(t)I^{\alpha,k}g(t).$$

(c)

$$(c') \; NM\frac{(k+1)^{-(\alpha+\beta)}t^{(k+1)(\alpha+\beta)}}{\Gamma(\beta+1)\Gamma(\alpha+1)} + I^{\alpha,k}f(t)I^{\beta,k}g(t) \geq M\frac{(k+1)^{-\alpha}t^{(k+1)\alpha}}{\Gamma(\alpha+1)}I^{\beta,k}g(t) + N\frac{(k+1)^{-\beta}t^{(k+1)\beta}}{\Gamma(\beta+1)}I^{\alpha,k}f(t).$$

(d)

$$(d') \; mn\frac{(k+1)^{-(\alpha+\beta)}t^{(k+1)(\alpha+\beta)}}{\Gamma(\beta+1)\Gamma(\alpha+1)} + I^{\alpha,k}f(t)I^{\beta,k}g(t) \geq m\frac{(k+1)^{-\alpha}t^{(k+1)\alpha}}{\Gamma(\alpha+1)}I^{\beta,k}g(t) + n\frac{(k+1)^{-\beta}t^{(k+1)\beta}}{\Gamma(\beta+1)}I^{\alpha,k}f(t).$$

Lemma 2. *Let $f \in L_{1,k}[0,\infty)$ and assume φ_1, φ_2 be two integrable functions on $[0,\infty)$ and $k \geq 0$, $t > 0$, $\alpha > 0$. Suppose that the condition (3.1) holds. Then:*

$$\begin{aligned}
&\frac{(k+1)^{-\alpha}t^{(k+1)\alpha}}{\Gamma(\alpha+1)} I^{\alpha,k} f^2(t) - (I^{\alpha,k} f(t))^2 \\
=&(I^{\alpha,k}\varphi_2(t) - I^{\alpha,k}f(t))(I^{\alpha,k}f(t) - I^{\alpha,k}\varphi_1(t)) \\
&- \frac{(k+1)^{-\alpha}t^{(k+1)\alpha}}{\Gamma(\alpha+1)}(I^{\alpha,k}\varphi_2(t) - I^{\alpha,k}f(t))(I^{\alpha,k}f(t) - I^{\alpha,k}\varphi_1(t)) \\
&+ \frac{(k+1)^{-\alpha}t^{(k+1)\alpha}}{\Gamma(\alpha+1)} I^{\alpha,k}(\varphi_1(t)f(t)) - I^{\alpha,k}\varphi_1(t)I^{\alpha,k}f(t) \\
&+ \frac{(k+1)^{-\alpha}t^{(k+1)\alpha}}{\Gamma(\alpha+1)} I^{\alpha,k}(\varphi_2(t)f(t)) - I^{\alpha,k}\varphi_2(t)I^{\alpha,k}f(t) \\
&+ I^{\alpha,k}(\varphi_1(t)\varphi_2(t)) - \frac{(k+1)^{-\alpha}t^{(k+1)\alpha}}{\Gamma(\alpha+1)} I^{\alpha,k}(\varphi_1(t)\varphi_2(t)).
\end{aligned} \tag{3.13}$$

Proof. For any $x, y > 0$, and $k \geq 0$, we have

$$\begin{aligned}
&(\varphi_2(y) - f(y))(f(x) - \varphi_1(x)) + (\varphi_2(x) - f(x))(f(y) - \varphi_1(y)) \\
&\quad - (\varphi_2(x) - f(x))(f(x) - \varphi_1(x)) - (\varphi_2(y) - f(y))(f(y) - \varphi_1(y)) \\
=&f^2(x) + f^2(y) - 2f(x)f(y) + \varphi_2(y)f(x) + \varphi_1(x)f(y) \\
&\quad - \varphi_1(x)\varphi_2(y) + \varphi_2(x)f(y) + \varphi_1(y)f(x) - \varphi_1(y)\varphi_2(x) \\
&\quad - \varphi_2(x)f(x) + \varphi_1(x)\varphi_2(x) - \varphi_1(x)f(x) - \varphi_2(y)f(y) \\
&\quad + \varphi_1(y)\varphi_2(y) - \varphi_1(y)f(y).
\end{aligned} \tag{3.14}$$

If we multiply both sides of (3.14) by $\frac{(k+1)^{1-\alpha}(t^{k+1} - x^{k+1})^{\alpha-1}x^k}{\Gamma(\alpha)}$ and integrate with respect to x on $(0,t)$, we obtain

$$\begin{aligned}
&(\varphi_2(y) - f(y))(I^{\alpha,k}f(t) - I^{\alpha,k}\varphi_1(t)) + (I^{\alpha,k}\varphi_2(t) - I^{\alpha,k}f(t))(f(y) - \varphi_1(y)) \\
&- I^{\alpha,k}(\varphi_2(t) - f(t))(f(t) - \varphi_1(t)) \\
&- (\varphi_2(y) - f(y))(f(y) - \varphi_1(y))\frac{(k+1)^{-\alpha}t^{(k+1)\alpha}}{\Gamma(\alpha+1)} \\
=&I^{\alpha,k}f^2(t) + f^2(y)\frac{(k+1)^{-\alpha}t^{(k+1)\alpha}}{\Gamma(\alpha+1)} - 2f(y)I^{\alpha,k}f(t) + \varphi_2(y)I^{\alpha,k}f(t) \\
&+ f(y)I^{\alpha,k}\varphi_1(t)
\end{aligned}$$

$$\begin{aligned}
&- \varphi_2(y)I^{\alpha,k}\varphi_1(t) + f(y)I^{\alpha,k}\varphi_2(t) + \varphi_1(y)I^{\alpha,k}f(t) - \varphi_1(y)I^{\alpha,k}\varphi_2(t)\\
&- I^{\alpha,k}(\varphi_2(t)f(t)) + I^{\alpha,k}(\varphi_1(t)\varphi_2(t))\\
&- I^{\alpha,k}(\varphi_1(t)f(t)) - \varphi_2(y)f(y)\frac{(k+1)^{-\alpha}t^{(k+1)\alpha}}{\Gamma(\alpha+1)}\\
&+ \varphi_1(y)\varphi_2(y)\frac{(k+1)^{-\alpha}t^{(k+1)\alpha}}{\Gamma(\alpha+1)} - \varphi_1(y)f(y)\frac{(k+1)^{-\alpha}t^{(k+1)\alpha}}{\Gamma(\alpha+1)}.
\end{aligned} \tag{3.15}$$

If we multiply both sides of (3.15) by $\frac{(k+1)^{1-\alpha}(t^{k+1}-y^{k+1})^{\alpha-1}y^k}{\Gamma(\alpha)}$ and integrate with respect to y on $(0,t)$, we receive

$$\begin{aligned}
&(I^{\alpha,k}\varphi_2(t) - I^{\alpha,k}f(t))(I^{\alpha,k}f(t) - I^{\alpha,k}\varphi_1(t))\\
&+ (I^{\alpha,k}\varphi_2(t) - I^{\alpha,k}f(t))(I^{\alpha,k}f(t) - I^{\alpha,k}\varphi_1(t))\\
&- (I^{\alpha,k}\varphi_2(t) - I^{\alpha,k}f(t))(I^{\alpha,k}f(t) - I^{\alpha,k}\varphi_1(t))\frac{(k+1)^{-\alpha}t^{(k+1)\alpha}}{\Gamma(\alpha+1)}\\
&- (I^{\alpha,k}\varphi_2(t) - I^{\alpha,k}f(t))(I^{\alpha,k}f(t) - I^{\alpha,k}\varphi_1(t))\frac{(k+1)^{-\alpha}t^{(k+1)\alpha}}{\Gamma(\alpha+1)}\\
=&\frac{(k+1)^{-\alpha}t^{(k+1)\alpha}}{\Gamma(\alpha+1)}I^{\alpha,k}f^2(t) + \frac{(k+1)^{-\alpha}t^{(k+1)\alpha}}{\Gamma(\alpha+1)}I^{\alpha,k}f^2(t)\\
&- 2I^{\alpha,k}f(t)I^{\alpha,k}f(t) + I^{\alpha,k}\varphi_2(t)I^{\alpha,k}f(t) + I^{\alpha,k}\varphi_1(t)I^{\alpha,k}f(t)\\
&- I^{\alpha,k}\varphi_1(t)I^{\alpha,k}\varphi_2(t) + I^{\alpha,k}\varphi_2(t)I^{\alpha,k}f(t) + I^{\alpha,k}\varphi_1(t)I^{\alpha,k}f(t)\\
&- I^{\alpha,k}\varphi_1(t)I^{\alpha,k}\varphi_2(t) - \frac{(k+1)^{-\alpha}t^{(k+1)\alpha}}{\Gamma(\alpha+1)}I^{\alpha,k}(\varphi_2(t)f(t))\\
&+ \frac{(k+1)^{-\alpha}t^{(k+1)\alpha}}{\Gamma(\alpha+1)}I^{\alpha,k}(\varphi_1(t)\varphi_2(t))\\
&- \frac{(k+1)^{-\alpha}t^{(k+1)\alpha}}{\Gamma(\alpha+1)}I^{\alpha,k}(\varphi_1(t)f(t))\\
&- \frac{(k+1)^{-\alpha}t^{(k+1)\alpha}}{\Gamma(\alpha+1)}I^{\alpha,k}(\varphi_2(t)f(t))\\
&+ \frac{(k+1)^{-\alpha}t^{(k+1)\alpha}}{\Gamma(\alpha+1)}I^{\alpha,k}(\varphi_1(t)\varphi_2(t))\\
&- \frac{(k+1)^{-\alpha}t^{(k+1)\alpha}}{\Gamma(\alpha+1)}I^{\alpha,k}(\varphi_1(t)f(t)).
\end{aligned} \tag{3.16}$$

The lemma is proved. □

Corollary 4. *Let $k = 0$ in Lemma 2. Then we have inequality (2.11) in Lemma 1 holds true.*

If φ_1 and φ_2 are some constants, then the following result holds.

Corollary 5. *Let $f \in L_{1,k}[0, \infty)$. Suppose that $m \leq f(t) \leq M$, $\forall t \in [0, \infty)$ and $m, M \in \mathbb{R}$. Then for $k \geq 0$, $t > 0$, $\alpha > 0$, $\beta > 0$, we have*

$$\begin{aligned} &\frac{(k+1)^{-\alpha} t^{(k+1)\alpha}}{\Gamma(\alpha+1)} I^{\alpha,k} f^2(t) - (I^{\alpha,k} f(t))^2 \\ &\quad = (M \frac{(k+1)^{-\alpha} t^{(k+1)\alpha}}{\Gamma(\alpha+1)} - I^{\alpha,k} f(t))(I^{\alpha,k} f(t) - m \frac{(k+1)^{-\alpha} t^{(k+1)\alpha}}{\Gamma(\alpha+1)}) \\ &\qquad - \frac{(k+1)^{-\alpha} t^{(k+1)\alpha}}{\Gamma(\alpha+1)} I^{\alpha,k}((M - f(t))(f(t) - m). \end{aligned} \tag{3.17}$$

Theorem 7. *Let f and g be two integrable functions on $[0, \infty)$ and let $\varphi_1, \varphi_2, \psi_1$ and ψ_2 be four integrable functions on $[0, \infty)$ satisfying the conditions (3.1) and (3.7) on $[0, \infty)$. Then for all $t > 0$, $k \geq 0$, $\alpha > 0$:*

$$\begin{aligned} &\left| \frac{t^{\alpha(k+1)}(k+1)^{-\alpha}}{\Gamma(\alpha+1)} I^{\alpha,k}(f(t)g(t)) - I^{\alpha,k} f(t) I^{\alpha,k} g(t) \right| \\ &\qquad \leq \sqrt{T(f, \varphi_1, \varphi_2) T(g, \psi_1, \psi_2)} , \end{aligned} \tag{3.18}$$

where $T(u, v, w)$ is defined by the following equality

$$\begin{aligned} T(u, v, w) = &(I^{\alpha,k} w(t) - I^{\alpha,k} u(t))(I^{\alpha,k} u(t) - I^{\alpha,k} v(t)) \\ &+ \frac{(k+1)^{-\alpha} t^{\alpha(k+1)}}{\Gamma(\alpha+1)} I^{\alpha,k}(v(t)u(t)) - I^{\alpha,k} v(t) I^{\alpha,k} u(t) \\ &+ \frac{(k+1)^{-\alpha} t^{\alpha(k+1)}}{\Gamma(\alpha+1)} I^{\alpha,k}(w(t)u(t)) \\ &- I^{\alpha,k} w(t) I^{\alpha,k} u(t) + I^{\alpha,k} v(t) I^{\alpha,k} w(t) \\ &- \frac{(k+1)^{-\alpha} t^{\alpha(k+1)}}{\Gamma(\alpha+1)} I^{\alpha,k}(v(t)w(t)). \end{aligned} \tag{3.19}$$

Proof. Let f and g be two integrable functions defined on $[0, \infty)$ and satisfying (3.1) and (3.7). We define

$$H(x, y) = (f(x) - f(y))(g(x) - g(y)), \quad x, y \in (0, t), \ t > 0. \tag{3.20}$$

Multiplying both sides of (3.20) by

$$\frac{(k+1)^{2-2\alpha}(t^{k+1}-x^{k+1})^{\alpha-1}x^k(t^{k+1}-y^{k+1})^{\alpha-1}y^k}{2\Gamma^2(\alpha)},\ x,y\in(0,t)$$

and integrating the result with respect to x and y, from 0 to t, we obtain

$$\frac{(k+1)^{2-2\alpha}}{2\Gamma^2(\alpha)}\int_0^t\int_0^t(t^{k+1}-x^{k+1})^{\alpha-1}x^k(t^{k+1}-y^{k+1})^{\alpha-1}y^kH(x,y)dxdy$$
$$=\frac{(k+1)^{-\alpha}t^{\alpha(k+1)}}{\Gamma(\alpha+1)}I^{\alpha,k}(f(t)g(t))-I^{\alpha,k}f(t)I^{\alpha,k}g(t). \quad (3.21)$$

Applying the Cauchy-Schwarz inequality to (3.21), we have

$$\left(\begin{array}{c}\frac{(k+1)^{2-2\alpha}}{2\Gamma^2(\alpha)}\int_0^t\int_0^t(t^{k+1}-x^{k+1})^{\frac{\alpha-1}{2}}(t^{k+1}-x^{k+1})^{\frac{\alpha-1}{2}}(t^{k+1}-y^{k+1})^{\frac{\alpha-1}{2}}\\ \times(t^{k+1}-y^{k+1})^{\frac{\alpha-1}{2}}(f(x)-f(y))(g(x)-g(y))x^{\frac{k}{2}}x^{\frac{k}{2}}y^{\frac{k}{2}}y^{\frac{k}{2}}dxdy\end{array}\right)^2$$
$$\leq\frac{(k+1)^{2-2\alpha}}{2\Gamma^2(\alpha)}\int_0^t\int_0^t(t^{k+1}-x^{k+1})^{\alpha-1}(t^{k+1}-y^{k+1})^{\alpha-1}(f(x)-f(y))^2x^ky^kdxdy$$
$$\times\frac{(k+1)^{2-2\alpha}}{2\Gamma^2(\alpha)}\int_0^t\int_0^t(t^{k+1}-x^{k+1})^{\alpha-1}(t^{k+1}-y^{k+1})^{\alpha-1}(g(x)-g(y))^2x^ky^kdxdy. \quad (3.22)$$

From (3.21) and (3.22) we obtain

$$\left(\frac{(k+1)^{-\alpha}t^{\alpha(k+1)}}{\Gamma(\alpha+1)}I^{\alpha,k}(f(t)g(t))-I^{\alpha,k}f(t)I^{\alpha,k}g(t)\right)^2$$
$$\leq\frac{(k+1)^{-\alpha}t^{\alpha(k+1)}}{\Gamma(\alpha+1)}I^{\alpha,k}f^2(t)-\left(I^{\alpha,k}f(t)\right)^2$$
$$\times\frac{(k+1)^{-\alpha}t^{\alpha(k+1)}}{\Gamma(\alpha+1)}I^{\alpha,k}g^2(t)-\left(I^{\alpha,k}g(t)\right)^2. \quad (3.23)$$

Since $(\varphi_2(t)-f(t))(f(t)-\varphi_1(t))\geq 0$ and $(\psi_2(t)-g(t))(g(t)-\psi_1(y))\geq 0$, for $t\in[0,\infty)$, we have

$$\frac{(k+1)^{-\alpha}t^{\alpha(k+1)}}{\Gamma(\alpha+1)}I^{\alpha,k}\left[(\varphi_2(t)-f(t))(f(t)-\varphi_1(t))\right]\geq 0, \quad (3.24)$$

$$\frac{(k+1)^{-\alpha}t^{\alpha(k+1)}}{\Gamma(\alpha+1)} I^{\alpha,k}\left[(\psi_2(t)-g(t))(g(t)-\psi_1(y))\right] \geq 0. \tag{3.25}$$

Thus, using Lemma 2, we obtain

$$\begin{aligned}
&\frac{(k+1)^{-\alpha}t^{\alpha(k+1)}}{\Gamma(\alpha+1)} I^{\alpha,k} f^2(t) - (I^{\alpha,k} f(t))^2 \\
&\qquad \leq (I^{\alpha,k}\varphi_2(t) - I^{\alpha,k} f(t))(I^{\alpha,k} f(t) - I^{\alpha,k}\varphi_1(t)) \\
&\qquad + \frac{(k+1)^{-\alpha}t^{\alpha(k+1)}}{\Gamma(\alpha+1)} I^{\alpha,k}(\varphi_1(t) f(t)) - I^{\alpha,k}\varphi_1(t) I^{\alpha,k} f(t) \\
&\qquad + \frac{(k+1)^{-\alpha}t^{\alpha(k+1)}}{\Gamma(\alpha+1)} I^{\alpha,k}\varphi_2(t) - I^{\alpha,k}\varphi_2(t) I^{\alpha,k} f(t) \\
&\qquad + I^{\alpha,k}\varphi_1(t) I^{\alpha,k}\varphi_2(t) - \frac{(k+1)^{-\alpha}t^{\alpha(k+1)}}{\Gamma(\alpha+1)} I^{\alpha,k}(\varphi_1(t)\varphi_2(t)) \\
&\qquad = T(f,\varphi_1,\varphi_2),
\end{aligned} \tag{3.26}$$

$$\begin{aligned}
&\frac{(k+1)^{-\alpha}t^{\alpha(k+1)}}{\Gamma(\alpha+1)} I^{\alpha,k} g^2(t) - (I^{\alpha,k} g(t))^2 \\
&\qquad \leq (I^{\alpha,k}\psi_2(t) - I^{\alpha,k} g(t))(I^{\alpha,k} g(t) - I^{\alpha,k}\psi_1(t)) \\
&\qquad + \frac{(k+1)^{-\alpha}t^{\alpha(k+1)}}{\Gamma(\alpha+1)} I^{\alpha,k}(\psi_1(t) g(t)) - I^{\alpha,k}\psi_1(t) I^{\alpha,k} g(t) \\
&\qquad + \frac{(k+1)^{-\alpha}t^{\alpha(k+1)}}{\Gamma(\alpha+1)} I^{\alpha,k}\psi_2(t) - I^{\alpha,k}\psi_2(t) I^{\alpha,k} g(t) \\
&\qquad + I^{\alpha,k}\psi_1(t) I^{\alpha,k}\psi_2(t) - \frac{(k+1)^{-\alpha}t^{\alpha(k+1)}}{\Gamma(\alpha+1)} I^{\alpha,k}(\psi_1(t)\psi_2(t)) \\
&\qquad = T(g,\psi_1,\psi_2).
\end{aligned} \tag{3.27}$$

From (3.22), (3.26), and (3.27), we get (3.18). □

Remark 1. *If* $T(f,\varphi_1,\varphi_2) = T(f,m,M)$, $T(g,\psi_1,\psi_2) = T(g,p,P)$ *and* $m, M, p, P \in \mathbb{R}$, *then inequality* (3.18) *reduces to*

$$\begin{aligned}
&\left| \frac{(k+1)^{-\alpha}t^{\alpha(k+1)}}{\Gamma(\alpha+1)} I^{\alpha,k}(f(t)g(t)) - I^{\alpha,k} f(t) I^{\alpha,k} g(t) \right| \\
&\qquad \leq \left(\frac{(k+1)^{-\alpha}t^{\alpha(k+1)}}{2\Gamma(\alpha+1)} \right)^2 (M-m)(P-p).
\end{aligned} \tag{3.28}$$

Remark 2. *Let* $k = 0$ *and* $\alpha = 1$ *in* (3.28) *then we have Grüss inequality* (1.1) *on* $[0,t]$.

Corollary 6. *Let $k = 0$ in Theorem 7 and $T(f,\varphi_1,\varphi_2) = T(f,m,M)$, $T(g,\psi_1,\psi_2) = T(g,p,P)$ and $m,M,p,P \in \mathbb{R}$. Then we have inequality (2.7) in Theorem 1.*

Example 1. *Let f and g be two functions satisfying $t^p \leq f(t) \leq t^p + 1$ and $t^p - 1 \leq g(t) \leq t^p$ for $t \in [0,\infty)$. Then, for $p \geq 0$, $t > 0$, $\alpha > 0$, we have*

$$\left|\frac{(k+1)^{-\alpha}t^{\alpha(k+1)}}{\Gamma(\alpha+1)} I^{\alpha,k}(f(t)g(t)) - I^{\alpha,k}f(t)I^{\alpha,k}g(t)\right| \leq \sqrt{T(f,t^p,t^p+1)T(g,t^p-1,t^p)}.$$

Here:

$$\begin{aligned}
T(f,t^p,t^p+1) &= \left(\frac{(k+1)^{-\alpha}t^{\alpha(k+1)}}{\Gamma(\alpha+1)} + \frac{(k+1)^{-\alpha}t^{\alpha(k+1)+p}\Gamma(\frac{p}{k+1}+1)}{\Gamma(\alpha+\frac{p}{k+1}+1)} - I^{\alpha,k}f(t)\right) \\
&\times \left(I^{\alpha,k}f(t) - \frac{(k+1)^{-\alpha}t^{\alpha(k+1)+p}\Gamma(\frac{p}{k+1}+1)}{\Gamma(\alpha+\frac{p}{k+1}+1)}\right) + \frac{(k+1)^{-\alpha}t^{\alpha(k+1)}}{\Gamma(\alpha+1)} I^{\alpha,k}(f(t)t^p) \\
&- \frac{(k+1)^{-\alpha}t^{\alpha(k+1)+p}\Gamma(\frac{p}{k+1}+1)}{\Gamma(\alpha+\frac{p}{k+1}+1)} I^{\alpha,k}f(t) + \frac{(k+1)^{-\alpha}t^{\alpha(k+1)}}{\Gamma(\alpha+1)} I^{\alpha,k}((t^p+1)f(t)) \\
&- \left(\frac{(k+1)^{-\alpha}t^{\alpha(k+1)+p}\Gamma(\frac{p}{k+1}+1)}{\Gamma(\alpha+\frac{p}{k+1}+1)} + \frac{(k+1)^{-\alpha}t^{\alpha(k+1)}}{\Gamma(\alpha+1)}\right) I^{\alpha,k}f(t) \\
&+ \left(\frac{(k+1)^{-\alpha}t^{\alpha(k+1)+p}\Gamma(\frac{p}{k+1}+1)}{\Gamma(\alpha+\frac{p}{k+1}+1)}\right)\left(\frac{(k+1)^{-\alpha}t^{\alpha(k+1)+p}\Gamma(\frac{p}{k+1}+1)}{\Gamma(\alpha+\frac{p}{k+1}+1)} + \frac{(k+1)^{-\alpha}t^{\alpha(k+1)}}{\Gamma(\alpha+1)}\right) \\
&- \frac{(k+1)^{-\alpha}t^{\alpha(k+1)}}{\Gamma(\alpha+1)} \left(\frac{(k+1)^{-\alpha}t^{\alpha(k+1)+p}\Gamma(\frac{p}{k+1}+1)}{\Gamma(\alpha+\frac{p}{k+1}+1)} + \frac{(k+1)^{-\alpha}t^{\alpha(k+1)+2p}\Gamma(\frac{2p}{k+1}+1)}{\Gamma(\alpha+\frac{2p}{k+1}+1)}\right),
\end{aligned}$$

and

$$\begin{aligned}
T(g,t^p-1,t^p) &= \left(\frac{(k+1)^{-\alpha}t^{\alpha(k+1)+p}\Gamma(\frac{p}{k+1}+1)}{\Gamma(\alpha+\frac{p}{k+1}+1)} - I^{\alpha,k}g(t)\right) \\
&\times \left(I^{\alpha,k}g(t) - \frac{(k+1)^{-\alpha}t^{\alpha(k+1)+p}\Gamma(\frac{p}{k+1}+1)}{\Gamma(\alpha+\frac{p}{k+1}+1)} + \frac{(k+1)^{-\alpha}t^{\alpha(k+1)}}{\Gamma(\alpha+1)}\right) + \frac{(k+1)^{-\alpha}t^{\alpha(k+1)}}{\Gamma(\alpha+1)} I^{\alpha,k}(g(t)(t^p-1)) \\
&- \left(\frac{(k+1)^{-\alpha}t^{\alpha(k+1)+p}\Gamma(\frac{p}{k+1}+1)}{\Gamma(\alpha+\frac{p}{k+1}+1)} - \frac{(k+1)^{-\alpha}t^{\alpha(k+1)}}{\Gamma(\alpha+1)}\right) I^{\alpha,k}g(t) + \frac{(k+1)^{-\alpha}t^{\alpha(k+1)}}{\Gamma(\alpha+1)} I^{\alpha,k}(t^p g(t)) \\
&- \frac{(k+1)^{-\alpha}t^{\alpha(k+1)+p}\Gamma(\frac{p}{k+1}+1)}{\Gamma(\alpha+\frac{p}{k+1}+1)} I^{\alpha,k}g(t) + \left(\frac{(k+1)^{-\alpha}t^{\alpha(k+1)+p}\Gamma(\frac{p}{k+1}+1)}{\Gamma(\alpha+\frac{p}{k+1}+1)} - \frac{(k+1)^{-\alpha}t^{\alpha(k+1)}}{\Gamma(\alpha+1)}\right) \\
&\times \left(\frac{(k+1)^{-\alpha}t^{\alpha k+\alpha+p}\Gamma(\frac{p}{k+1}+1)}{\Gamma(\alpha+\frac{p}{k+1}+1)}\right) \\
&- \frac{(k+1)^{-\alpha}t^{\alpha(k+1)}}{\Gamma(\alpha+1)} \left(\frac{(k+1)^{-\alpha}t^{\alpha(k+1)+2p}\Gamma(\frac{2p}{k+1}+1)}{\Gamma(\alpha+\frac{2p}{k+1}+1)} - \frac{(k+1)^{-\alpha}t^{\alpha(k+1)+p}\Gamma(\frac{p}{k+1}+1)}{\Gamma(\alpha+\frac{p}{k+1}+1)}\right).
\end{aligned}$$

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