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SUBSPACE TRANSITIVITY OF TUPLES OF OPERATORS

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Abstract: In this paper, we introduce subspace hypercyclicity and transitivity of tuples of operators and we give some relations between these concepts and the subspace transitivity criterion for a tuple of operators.

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1. Introduction

By an n-tuple of operators we mean a finite sequence of length n of commuting continuous linear operators on a Banach space X.

Definition 1.1. Let $\mathcal{T} = (T_1, T_2, ..., T_n)$ be an n-tuple of operators acting on a separable infinite dimensional Banach space X over $\mathbb{C}$ and let M be a nonzero subspace of X. We will let

$$\mathcal{F} = \{{T_1}^{k_1}{T_2}^{k_2}...{T_n}^{k_n} : k_i \geq 0, i = 1, ..., n\}$$

be the semigroup generated by $\mathcal{T}$. For $x \in X$, the orbit of x under the tuple $\mathcal{T}$

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is the set $Orb(\mathcal{T}, x) = \{Sx : S \in \mathcal{F}\}$. A vector x is called a subspace-hypercyclic (or M-hypercyclic) vector for $\mathcal{T}$ if $Orb(\mathcal{T}, x) \cap M$ is dense in M and in this case the tuple $\mathcal{T}$ is called subspace-hypercyclic for M. The set of all M-hypercyclic vectors of $\mathcal{T}$ is denoted by $HC(\mathcal{T}, M)$. Also, for all $k \geq 2$, by $\mathcal{T}_d^{(k)}$ we will refer to the set of all k copies of an element of $\mathcal{F}$, i.e.

$$\mathcal{T}_d^{(k)} = \{S_1 \oplus ... \oplus S_k : S_1 = ... = S_k \in \mathcal{F}\}.$$

We say that $\mathcal{T}_d^{(k)}$ is subspace-hypercyclic, with respect to M, provided there exist $x_1, ..., x_k \in X$ such that $\{W(x_1 \oplus ... \oplus x_k) : W \in \mathcal{T}_d^{(k)}\} \cap M$ is dense in the k copies of M, $M \oplus ... \oplus M$.

Note that if $T_1, T_2, ..., T_n$ are commutative bounded linear operators on a Banach space X, and $\{m_j(i)\}_j$, is a sequence of natural numbers for $i = 1, ..., n$, then we say $\{{T_1}^{m_j(1)}{T_2}^{m_j(2)}...{T_n}^{m_j(n)} : j \geq 0\}$ is M-hypercyclic if there exists $x \in X$ such that $\{{T_1}^{m_j(1)}{T_2}^{m_j(2)}...{T_n}^{m_j(n)}x : j \geq 0\} \cap M$ is dense in M.

Definition 1.2. Suppose that $\mathcal{T} = (T_1, T_2, ..., T_n)$ is an n-tuple of operators acting on a separable infinite dimensional Banach space X over $\mathbb{C}$ and M is a nonzero subspace of X. We say that a tuple $\mathcal{T} = (T_1, T_2, ..., T_n)$ is called M-transitive with respect to a tuple of nonnegative integer sequences

$$(\{k_{j(1)}\}_j, \{k_{j(2)}\}_j, ..., \{k_{j(n)}\}_j),$$

if for every nonempty relatively open subsets U, V of X there exists $j_0 \in \mathbb{N}$ such that $T_1^{-k_{j_0(1)}} T_2^{-k_{j_0(2)}} ... T_n^{-k_{j_0(n)}}(U) \cap V$ contains a relatively open nonempty subset of M. Also, we say that an n-tuple $\mathcal{T}$ is M-transitive if it is M-transitive with respect an n-tuple of nonnegative integer sequences.

Suprisingly, there are something that does not happen for single operators. For example, hypercyclic tuples can arise in finite dimensional, and there are operators that have somewhere dense orbits that are not everywhere dense (see [1]). Also, we note that there are subspace-hypercyclic operators that are not hypercyclic (see [2]).

2. Main Results

In this section, we investigate subspace-transitivity for tuples of operators.

Theorem 2.1. *Suppose that $\mathcal{T} = (T_1, T_2, ..., T_n)$ is an n-tuple of operators acting on a separable infinite dimensional Banach space X over $\mathbb{C}$ and*

M is a nonzero subspace of X. Then $\mathcal{T}$ is M-transitive if and only if for any nonempty sets $U \subset M$ and $V \subset M$, both relatively open, there exists a tuple $(k_1, k_2, ..., k_n)$ of integers such that $T_1^{-k_1} T_2^{-k_2} ... T_n^{-k_n}(U) \cap V$ is nonempty and $T_1^{-k_1} T_2^{-k_2} ... T_n^{-k_n} M \subset M$.

Proof. Let $\mathcal{T}$ be M-transitive with respect to M and let U and V be nonempty relatively open subsets of M. Hence there exists a tuple $(k_1, k_2, ..., k_n)$ of integers such that $T_1^{-k_1} T_2^{-k_2} ... T_n^{-k_n}(U) \cap V$ contains a relatively open nonempty set W. To show that $T_1^{-k_1} T_2^{-k_2} ... T_n^{-k_n} M \subset M$, let $x \in M$ and note that

$$T_1^{k_1} T_2^{k_2} ... T_n^{k_n} W \subset U \cap T_1^{k_1} T_2^{k_2} ... T_n^{k_n} V \subset U \subset M.$$

If $x_0 \in W$, then there exists $r > 0$ small enough such that $x_0 + rx \in W$, since W is relatively open. Thus

$$T_1^{k_1} T_2^{k_2} ... T_n^{k_n}(x_0 + rx) = T_1^{k_1} T_2^{k_2} ... T_n^{k_n} x_0 + r T_1^{k_1} T_2^{k_2} ... T_n^{k_n} x \in M,$$

which implies that $T_1^{k_1} T_2^{k_2} ... T_n^{k_n} x \in M$. Thus $T_1^{k_1} T_2^{k_2} ... T_n^{k_n} M \subset M$. Conversely, note that $T_1^{k_1} T_2^{k_2} ... T_n^{k_n} : M \to M$ is continuous and so

$$T_1^{-k_1} T_2^{-k_2} ... T_n^{-k_n}(U) \cap V$$

is relatively open and nonempty subset of M. This completes the proof. □

Lemma 2.2. *Suppose that $\mathcal{T} = (T_1, T_2, ..., T_n)$ is an n-tuple of operators acting on a separable infinite dimensional Banach space X over $\mathbb{C}$ and M is a nonzero subspace of X. Suppose that $\mathcal{T}$ is subspace-transitive with respect to M. Then M contains a dense G_δ set.*

Proof. Let $\{B_n : n \in \mathbb{N}\}$ be a countable open basis for the relative topology of M. In Theorem 2.1, put $U = B_i$ and $V = B_j$, then there exists $K_{i,j}^m \in \mathbb{N} \cup \{0\}$ for $m = 1, ..., n$ satisfying that $T_1^{-k_{i,j}^1} T_2^{-k_{i,j}^2} ... T_n^{-k_{i,j}^n}(B_i) \cap B_j$ is relatively open. Hence, the sets

$$G_i = \bigcup_j T_1^{-k_{i,j}^1} T_2^{-k_{i,j}^2} ... T_n^{-k_{i,j}^n}(B_i) \cap B_j$$

are relatively open. Also, each G_i is dense since it intersects each relatively open set in M. Hence, $\bigcap_i G_i$ is also dense and the proof is complete. □

Corollary 2.3. *Under the hypothesis and the notations of Lemma 2.2, the set*

$$\bigcap_i \bigcup \{T_1^{-k_1} T_2^{-k_2} ... T_n^{-k_n}(B_i) \cap M : \ K_1, ..., k_n \geq 0\}$$

is a dense subset of M.

Proof. Since
$$\bigcap_i \bigcup_j T_1^{-k_{i,j}^1} T_2^{-k_{i,j}^2} ... T_n^{-k_{i,j}^n}(B_i) \cap B_j$$
is a subset of
$$\bigcap_i \bigcup \{T_1^{-k_1} T_2^{-k_2} ... T_n^{-k_n}(B_i) \cap M :\ K_1, ..., k_n \geq 0\},$$
it is clear. □

Theorem 2.4. *Suppose that $\mathcal{T} = (T_1, T_2, ..., T_n)$ is an n-tuple of operators acting on a separable infinite dimensional Banach space X over $\mathbb{C}$ and M is a nonzero subspace of X. If $\mathcal{T}$ is M-transitive, then $\mathcal{T}$ is M-hypercyclic.*

Proof. Let $\{B_n : n \in \mathbb{N}\}$ be a countable open basis for the relative topology of M. Note that
$$x \in \bigcap_i \bigcup \{T_1^{-k_1} T_2^{-k_2} ... T_n^{-k_n}(B_i) \cap M :\ K_1, ..., k_n \geq 0\}$$
if and only if for any integer $j \geq 1$, there exists a tuple $(k_1, k_2, ..., k_n)$ of integers such that $T_1^{k_1} T_2^{k_2} ... T_n^{k_n} x \in B_j$. This occurs if and only if $Orb(\mathcal{T}, x) \cap M$ is dense in M or equivalently, if $x \in HC(\mathcal{T}, M)$. Hence
$$HC(\mathcal{T}, M) = \bigcap_i \bigcup \{T_1^{-k_1} T_2^{-k_2} ... T_n^{-k_n}(B_i) \cap M :\ K_1, ..., k_n \geq 0\}.$$
Now by Corollary 2.3, the proof is complete. □

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