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ON AN INEQUALITY RELATED TO HILBERT'S WITH LAPLACE TRANSFORM

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Abstract: Considering different parameters, we obtain new Hilbert-type inequalities involving the Laplace transform. Then we extract from our results some special cases.

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1. Introduction

The well known classical discrete Hilbert-type inequality takes the following form [1]:
If $a_n, b_n \geq 0, 0 < \sum_{n=1}^{\infty} a_n^2 < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^2 < \infty$, then we have

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{m+n} \leq \frac{\pi}{\sin(\pi/p)} \left(\sum_{m=1}^{\infty} a_m^p\right)^{1/p} \left(\sum_{n=1}^{\infty} b_n^q\right)^{1/q}. \tag{1}$$

Inequality (1) has the following integral analogous:

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$$\int_0^\infty \int_0^\infty \frac{f(x)\,g(y)}{x+y} dx\,dy \le \frac{\pi}{\sin(\pi/p)} \left(\int_0^\infty f^p(x) dx \right)^{1/p} \left(\int_0^\infty g^q(x) dx \right)^{1/q}, \qquad (2)$$

unless $f(x) \equiv 0$ or $g(x) \equiv 0$, where $p > 1$, $q = p/(p-1)$. The constant $\pi\, cosec(\pi/p)$, in (1) and (2), is the best possible see [1].

Inequalities (1) and (2), which have many generalizations, see for example [2], [3] and references therein, with their improvements have played fundamental roles in the development of many mathematical branches see for instance [2], [4] and references therein.

Recently, some mathematicians have become interested in what so called half-discrete Hilbert-type inequalities see for instance [5] and [6].
In the left-hand sides of this kind of inequalities we have a "mixture" of summations and integrals with a combination between discrete and continuous variables.

For example in [5] we find the following inequality:
If $0 < \alpha$, $0 < r \le 1$, $\frac{1}{p} + \frac{1}{q} = 1$ $(p \ne 0, 1)$, $\lambda_1 > 0$, $p\lambda_2\alpha + (1-p)\lambda_2 r \le 1$, $\lambda = \lambda_1 + \lambda_2$, $a_n \ge 0$, and $f(x) \ge 0$ is a real measurable function in $(0, \infty)$, then for $p > 1$:

$$\sum_{n=1}^\infty a_n \int_0^\infty \frac{f(x)}{(x^\alpha + n^r)^\lambda} dx \le \left(\frac{1}{\alpha} \beta(\lambda_1, \lambda_2) \right)^{\frac{1}{q}} \left(\frac{1}{r} \beta(\xi, \zeta) \right)^{\frac{1}{p}} \left(\sum_{n=1}^\infty n^{q(1-\lambda_2\alpha)-1} a_n^q \right)^{\frac{1}{q}}$$
$$\times \left(\int_0^\infty x^{p\lambda_2\alpha(\frac{\alpha}{r}-1)+p(1-\lambda_1\alpha)-1} f^p(x) dx \right)^{\frac{1}{p}}, \qquad (3)$$

where $\xi = \lambda_1 - p\lambda_2(\frac{\alpha}{r} - 1)$ and $\zeta = \lambda_2 + p\lambda_2(\frac{\alpha}{r} - 1)$.

In (3), if $r = \alpha = 1$, $p = q = 2$, and $\lambda_1 = \lambda_2 = \frac{1}{2}$, we obtain the following inequality (which has been proved in [7])

$$\sum_{n=1}^\infty a_n \int_0^\infty \frac{f(x)}{x+n} dx < \pi \left(\sum_{n=1}^\infty a_n^2 \int_0^\infty f^2(x) \right)^{1/2}, \qquad (4)$$

where the constant π is the best possible.

In this paper we consider a further generalization of the notion of having a "mixture" of summations and integrals on the left-hand side of the inequality. The upper bound of the inequality under investigation involves Laplace transforms for the functions in the left-hand side of the inequality.

In the following section we state the main result of this paper of which many special cases can be obtained.

Before proving the main theorem of this paper, Theorem 2.1, let us state and prove the following lemma:

Lemma 1.1. (see [8]) *For $n \in \mathcal{N}$, if $\alpha > -\frac{1}{n}$, $f(x)$ and $g(x)$ are non-negative measurable functions on the interval $(0,\infty)$, then*

$$\int_0^\infty \int_0^\infty \frac{f(x)\,g(y)}{(x+y)^{\alpha n+1}} dx\,dy = \frac{1}{\Gamma(\alpha n+1)} \int_0^\infty s^{\alpha n}\, F(s)\, G(s)\, ds, \tag{5}$$

where $F(s)$ and $G(s)$ are Laplace transforms of $f(x)$ and $g(x)$ respectively, and $\Gamma(z)$ is the gamma function.

Proof. Using the Fubini's theorem, we have

$$\begin{aligned}
\int_0^\infty s^{\alpha n}\, F(s)\, G(s)\, ds &= \int_0^\infty s^{\alpha n}\, ds \int_0^\infty e^{-sx}\, f(x)\, ds \int_0^\infty e^{-sy}\, g(y)\, ds \\
&= \int_0^\infty f(x)\, dx \int_0^\infty g(y)\, dy \int_0^\infty s^{\alpha n}\, e^{-s(x+y)}\, ds \\
&= \int_0^\infty f(x)\, dx \int_0^\infty g(y)\, dy \int_0^\infty \frac{t^{\alpha n}\, e^{-t}}{(x+y)^{\alpha n+1}}\, dt \\
&= \Gamma(\alpha n+1) \int_0^\infty \int_0^\infty \frac{f(x)\, g(y)}{(x+y)^{\alpha n+1}}\, dx\, dy.
\end{aligned}$$

This completes the proof. □

2. Main Results

In this section we state and discuss our main theorem together with its special cases.

Theorem 2.1. *Suppose that $\alpha > -\frac{1}{n}$, $\frac{1}{p} + \frac{1}{q} = 1$, $a_n \geq 0$, and $f(x)$, $g(x) \geq 0$ are real measurable functions on $(0,\infty)$. Then for $p > 1$, the* *following inequalities hold:*

$$J := \left(\sum_{n=1}^{\infty} \frac{n^{p\alpha-1}}{(\Gamma(\alpha n+1))^p} \left[\int_0^{\infty} s^{\alpha n}\, F(s)\, G(s)\, ds \right]^p \right)^{\frac{1}{p}} \tag{6}$$

$$\leq \left(\sum_{n=1}^{\infty} \frac{n^{p\alpha-1}}{(\Gamma(\alpha n+1))^p} \left(\int_0^{\infty} s^{p(\frac{\alpha n}{2}+\frac{1-\alpha}{q})} F^p(s) ds \right) \left(\int_0^{\infty} s^{\frac{\alpha n q}{2}+\alpha-1} G^q(s) ds \right)^{\frac{p}{q}} \right)^{\frac{1}{p}},$$

$$I := \sum_{n=1}^{\infty} a_n \int_0^{\infty}\int_0^{\infty} \frac{f(x)\, g(y)}{(x+y)^{\alpha n+1}} dx\, dy$$

$$\leq \left(\sum_{n=1}^{\infty} \frac{n^{p\alpha-1}}{(\Gamma(\alpha n+1))^p} \left(\int_0^{\infty} s^{p(\frac{\alpha n}{2}+\frac{1-\alpha}{q})} F^p(s) ds \right) \left(\int_0^{\infty} s^{\frac{\alpha n q}{2}+\alpha-1} G^q(s) ds \right)^{\frac{p}{q}} \right)^{\frac{1}{p}}$$

$$\times \left(\sum_{n=1}^{\infty} n^{q(1-\alpha)-1} a_n^q \right)^{\frac{1}{q}}, \tag{7}$$

where $F(s)$ and $G(s)$ are Laplace transforms for the functions $f(x)$ and $g(x)$ respectively.

Proof. Using Hölder's inequality produces

$$\left[\int_0^{\infty} s^{\alpha n}\, F(s)\, G(s)\, ds \right]^p = \left[\int_0^{\infty} s^{\frac{\alpha n}{2}} \frac{s^{(1-\alpha)/q}\Gamma(\alpha n+1)}{n^{(1-\alpha)/p}} F(s)\, s^{\frac{\alpha n}{2}} \frac{n^{(1-\alpha)/p}}{\Gamma(\alpha n+1) s^{(1-\alpha)/q}} G(s)\, ds \right]^p \tag{8}$$

$$\leq \int_0^{\infty} s^{p\left(\frac{\alpha n}{2}+\frac{1-\alpha}{q}\right)} \frac{(\Gamma(\alpha n+1))^p}{n^{(1-\alpha)}} F^p(s)\, ds$$

$$\times \left[\int_0^{\infty} s^{\frac{\alpha n q}{2}-(1-\alpha)} \frac{n^{q(1-\alpha)/p}}{(\Gamma(\alpha n+1))^q} G^q(s)\, ds \right]^{\frac{p}{q}}$$

$$= \left[\int_0^\infty s^{p\left(\frac{\alpha n}{2}+\frac{1-\alpha}{q}\right)} F^p(s)\, ds \right] \left[\int_0^\infty s^{\frac{\alpha n q}{2}+\alpha-1} G^q(s)\, ds \right]^{\frac{p}{q}} . \tag{9}$$

Using Lebesgue term-by-term integration theorem (see [9]) and (9), then inequality (6) can be written as follows

$$J \le \left(\sum_{n=1}^{\infty} \frac{n^{p\alpha-1}}{(\Gamma(\alpha n+1))^p} \int_0^\infty s^{p\left(\frac{\alpha n}{2}+\frac{1-\alpha}{q}\right)} F^p(s)\, ds \times \left[\int_0^\infty s^{\frac{\alpha n q}{2}+\alpha-1} G^q(s)\, ds \right]^{\frac{p}{q}} \right)^{\frac{1}{p}} .$$

This completes the proof of (6). To prove (7), applying Lemma 1.1 then by Hölder's inequality and inequality (6) we obtain

$$\begin{aligned}
I :=& \sum_{n=1}^{\infty} a_n \int_0^\infty \int_0^\infty \frac{f(x)\, g(y)}{(x+y)^{\alpha n+1}} dx\, dy \\
=& \sum_{n=1}^{\infty} \left(n^{\frac{1}{p}-\alpha} a_n \right) \left(\frac{n^{\alpha-\frac{1}{p}}}{\Gamma(\alpha n+1)} \int_0^\infty s^{\alpha n} F(s)\, G(s)\, ds \right) \\
\le& \left[\sum_{n=1}^{\infty} \frac{n^{\left(\alpha-\frac{1}{p}\right)p}}{(\Gamma(\alpha n+1))^p} \left(\int_0^\infty s^{\alpha n} F(s)\, G(s)\, ds \right)^p \right]^{\frac{1}{p}} \left[\sum_{n=1}^{\infty} n^{\left(\frac{1}{p}-\alpha\right)q} a_n^q \right]^{\frac{1}{q}} \\
=& J \left[\sum_{n=1}^{\infty} n^{\left(\frac{1}{p}-\alpha\right)q} a_n^q \right]^{\frac{1}{q}} \\
\le& \left(\sum_{n=1}^{\infty} \frac{n^{p\alpha-1}}{(\Gamma(\alpha n+1))^p} \left(\int_0^\infty s^{p\left(\frac{\alpha n}{2}+\frac{1-\alpha}{q}\right)} F^p(s)\, ds \right) \left(\int_0^\infty s^{\frac{\alpha n q}{2}+\alpha-1} G^q(s)\, ds \right)^{\frac{p}{q}} \right)^{\frac{1}{p}} \\
& \times \left(\sum_{n=1}^{\infty} n^{q(1-\alpha)-1} a_n^q \right)^{\frac{1}{q}} .
\end{aligned}$$

This completes the proof. □

Some special cases of Theorem 2.1 are as follows:

Case 1. If $\alpha = \frac{1}{p}$, then inequality (7) takes the form:

$$\sum_{n=1}^{\infty} a_n \int_0^{\infty}\int_0^{\infty} \frac{f(x)\, g(y)}{(x+y)^{\frac{n}{p}+1}} dx\, dy \leq$$
$$\left(\sum_{n=1}^{\infty} \frac{1}{\left(\frac{n}{p}\Gamma(\frac{n}{p})\right)^p} \int_0^{\infty} s^{\frac{n}{2}+\frac{p}{q^2}} F^p(s)\, ds \left(\int_0^{\infty} s^{\frac{nq-2p}{2pq}} G^q(s)\, ds \right)^{\frac{p}{q}} \right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} a_n^q \right)^{\frac{1}{q}}. \tag{10}$$

Using the elementary inequality (see [[10], p. 30])

$$x^{\frac{1}{p}}\, y^{\frac{1}{q}} \leq \frac{x}{p} + \frac{y}{q}$$

where $x > 0\, y > 0$ and $\frac{1}{p} + \frac{1}{q} = 1$ with $p > 1$, then inequality (10) gives the estimate

$$\sum_{n=1}^{\infty} a_n \int_0^{\infty}\int_0^{\infty} \frac{f(x)\, g(y)}{(x+y)^{\frac{n}{p}+1}} dx\, dy$$
$$\leq \frac{1}{p} \left(\sum_{n=1}^{\infty} \frac{1}{\left(\frac{n}{p}\Gamma(\frac{n}{p})\right)^p} \int_0^{\infty} s^{\frac{n}{2}+\frac{p}{q^2}} F^p(s)\, ds \left(\int_0^{\infty} s^{\frac{nq-2p}{2pq}} G^q(s)\, ds \right)^{\frac{p}{q}} \right)$$
$$+ \frac{1}{q} \left(\sum_{n=1}^{\infty} a_n^q \right).$$

However, when $p = q = 2$ inequality (10) becomes

$$\sum_{n=1}^{\infty} a_n \int_0^{\infty}\int_0^{\infty} \frac{f(x)\, g(y)}{(x+y)^{\frac{n}{2}+1}} dx\, dy$$
$$\leq \left(\sum_{n=1}^{\infty} \frac{1}{\left(\frac{n}{2}\Gamma(\frac{n}{2})\right)^2} \int_0^{\infty} s^{\frac{n}{2}+\frac{1}{2}} F^2(s)\, ds \int_0^{\infty} s^{\frac{n}{4}-\frac{1}{2}} G^2(s)\, ds \right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} a_n^2 \right)^{\frac{1}{2}}.$$

Case 2. If $\alpha = 1$ and $p = q = 2$, then inequality (7) yields:

$$\sum_{n=1}^{\infty} a_n \int_0^{\infty}\int_0^{\infty} \frac{f(x)\,g(y)}{(x+y)^{n+1}} dx\,dy$$
$$\leq \left(\sum_{n=1}^{\infty} \frac{1}{n\,(\Gamma(n))^2} \int_0^{\infty} s^n\,F^2(s)\,ds \int_0^{\infty} s^n\,G^2(s)\,ds\right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} \frac{a_n^2}{n}\right)^{\frac{1}{2}}.$$

Case 3. If $\alpha = 0$ and $p = q = 2$, then inequality (7) produces:

$$\sum_{n=1}^{\infty} a_n \int_0^{\infty}\int_0^{\infty} \frac{f(x)\,g(y)}{x+y} dx\,dy$$
$$\leq \left(\sum_{n=1}^{\infty} \frac{1}{n}\right)^{\frac{1}{2}} \left(\int_0^{\infty} s\,F^2(s)\,ds\right)^{\frac{1}{2}} \left(\int_0^{\infty} \frac{1}{s}\,G^2(s)\,ds\right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} n a_n^2\right)^{\frac{1}{2}}.$$

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