STEADY-STATE BEHAVIOR OF AN M/M/1 QUEUE
IN RANDOM ENVIRONMENT SUBJECT TO SYSTEM
FAILURES AND REPAIRS

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Abstract: In this paper, we consider an M/M/1 queueing model with disasters in random environment. This model contains a repair state, a checking state along with a set of operational units. When a disaster occurs the system is transferred to the failure state(state o) and then to the checking state(state 1) for the verification process. After the verification process, the system moves to any one of the operational units. The steady-state behavior of the underlying queueing model along with the average queue size is analyzed. Numerical illustrations are provided to study the effect of parameters on the average queue size.

Key Words: M/M/1 queueing model, disasters, generating functions, steady-state equations, average queue size

1. Introduction

Queueing models are used to analyze many real time situations. Depending upon the real time environment various types of queueing models are studied by different authors. When the real situation involves complexity, authors
analyzed the system as queueing models along with constraints like disaster, server vocation, retrials etc...

One of the major areas of interest in the queueing models is queueing system in random environment with disasters. When a system is working, a disaster may occur either due to an outside source or a cause from within the system. Due to this disaster the customers get annihilated and the system is inactive until the arrival of a new customer. In computer networks, inactivation of the processors by transmission of virus is not uncommon and the system can be restored to normalcy only after taking appropriate corrective measures. (see Chao [2], Chen and Renshaw [3]). A queueing system with this constraint is of much interest and is studied by many authors. Such models are useful for the performance evaluation of communications and computer networks which are characterized by time-varying arrival, service and failure rates. To incorporate the effects of external environmental factors into a stochastic model a queue with a random environment is considered. The random environment process may take a number of varied forms such as a discrete- or continuous-time Markov chain, a random walk, a semi-Markov process, or a Brownian motion, etc. If the random environment is Markovian, the primary stochastic process to which it is attached is said to be Markov-modulated. Kella and Whitt [5] considered an infinite-capacity storage model with two-state random environment and characterized its steady-state behavior. Kakubava [4] formulated a queueing system under batch service in an M/G/1 System and assumed that the batch size depends on the state of the system and also on the state of some random environment whereas Korotaev and Spivak [6] considered a queueing system in a semi-Markovian random environment.

Krishnamoorthy and Arivudainambi [7] considered an M/M/1 queueing system with catastrophes and found the transient solution of the system concerned. Krishna et al. [8] analyzed discouraged arrivals queue with catastrophes and derived explicit expressions of the transient solution along with the moments. Sudhesh [10] studied a queue with system disasters and customer impatience and derived its system-size probabilities using continued fraction technology. Yechiali [11] considered an M/M/c queue with system disasters and impatience customers and derived various quality of service measures such as mean sojourn time of a served customer, proportion of customers served and rate of lost customers due to disasters. Some more articles on queues with disasters can be found in Altman and Yechiali [1] and Yi et al. [12].

The long run behavior of any system with disaster or the steady-state is very important. Many researchers have studied the system behavior in the steady-state. Paz and Yechiali [9] analyzed a M/M/1 queue in a multi-phase
random environment, where the system occasionally suffers a disastrous failure, causing all present jobs to be lost. They considered a repair phase, where the system gets repaired and found the steady-state behavior and some performance measures of the system concerned.

Instead of having one repair state if we have a checking state to check the overall condition of the system after getting repaired the efficiency of the model will get improved. Hence in our paper we have considered an $M/M/1$ multi-phase queueing model in random environment with disasters. This model contains $n+1$ operational units including a repair state followed by a checking state. Whenever a disaster occurs the system moves to the repair state and after getting repaired it moves to the checking state with probability one. From the checking state it moves to any one of the remaining units. For the above model steady-state behavior is analyzed.

The organization of the paper is as follows: In Section 2, a detailed description is given for the proposed model. Section 3 deals with the balance equation of the model under consideration. An expression for the steady-state probability of having $m$ customers in state $i$, $m \geq 0$ is obtained. The above probabilities are analyzed using generating functions in Section 4. In Section 5 some performance measures are discussed.

2. Model Description

In this paper, we consider an $M/M/1$ queueing model with $n+1$ units operating in a random environment. These $n+1$ operating units form a continuous time Markov chain with units $i = 0, 1, 2, 3, ..., n$ and the corresponding Transition Probability Matrix $[q]$ as given below

\[
\begin{pmatrix}
0 & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & q_2 & q_3 & \cdots & q_n \\
1 & 0 & 0 & 0 & \cdots & 0 \\
1 & 0 & 0 & 0 & \cdots & 0 \\
1 & 0 & 0 & 0 & \cdots & 0 \\
1 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

The units $i = 2$ to $n$ are operating units whereas the unit $i = 0$ is the failure unit and $i = 1$ is the checking unit. When the system is in the unit $i$, $i = 2$ to $n$ it acts as an $M/M/1$ queue with arrival rate $\lambda_i \geq 0$ and service rate $\mu_i \geq 0$. The sojourn time for unit $i$ is the exponential random variable with mean $\frac{1}{\gamma_i}$, $i = 2, 3, 4, ..., n$. 
Occasionally a failure occurs when the system is in unit \(i \geq 2\). At that instant the system is transferred to the unit \(i = 0\) and then to \(i = 1\). The time spent by a system at the failure unit \(i = 0\) is an exponentially distributed random variable with mean \(\frac{1}{\gamma_0}\). The time taken for the verification process at \(i = 1\) is also an exponentially distributed random variable with mean \(\frac{1}{\gamma_1}\).

After the completion of the verification process the system is transferred to any one of the operating units \(i \geq 2\) with probability \(q_i\) and hence \(\sum_{i=2}^{n} q_i = 1\). The arrival rate for each of the failure units \(i = 0\) and \(i = 1\) are \(\lambda_0 \geq 0\) and \(\lambda_1 \geq 0\) respectively. As there is no service in these units \(\mu_0 = \mu_1 = 0\). In each active unit \(i = 2\) to \(n\), the system stays until a breakdown occurs which sends it to unit 0 and then to unit 1.

The above system can be represented by the stochastic process \(\{U(t), N(t)\}\) where \(U(t)\) denotes the unit in which the system operates at time \(t\) and \(N(t)\) denotes the number of customers present at time \(t\). The steady state probabilities of the system being in state \(i\) with \(m\) customers is denoted by \(p_{im}\). That is

\[
p_{im} = \lim_{t \to \infty} \{P(U(t) = i, N(t) = m)\},
\]

\(\forall t \geq 0, \ 0 \leq i \leq n, \ m = 0, 1, 2, \ldots\) \quad (2.1)

### 3. Steady-State Equations

Based on the above assumptions the underlying steady-state balance equations for the system under consideration are as follows. For \(m = 0\):

\[
(\lambda_0 + \gamma_0) p_{00} - \sum_{i=2}^{n} \gamma_i \sum_{m=0}^{\infty} p_{im} = 0, \quad \text{for } i = 0,
\]

\[
(\lambda_1 + \gamma_1) p_{10} - \sum_{i=2}^{n} \gamma_i \sum_{m=0}^{\infty} p_{im} = 0, \quad \text{for } i = 1,
\]

and for \(m \geq 1\)

\[
(\lambda_0 + \gamma_0) p_{0m} = \lambda_0 p_{0m-1}, \quad \text{for } i = 0
\]

\[
(\lambda_1 + \gamma_1) p_{1m} = \lambda_1 p_{1m-1}, \quad \text{for } i = 1
\]
For $i = 2, 3, 4, \ldots n$, the balance equations are

$$(\lambda_i + \gamma_i)p_{i0} = \mu_i p_{i1} + \gamma_1 q_i p_{10}, \quad \text{for } m = 0,$$

and

$$(\lambda_i + \mu_i + \gamma_i)p_{im} = \lambda_i p_{i(m-1)} + \mu_i p_{i(m+1)} + \gamma_1 q_i p_{1m}, \quad \text{for } m \geq 1$$

From (3.3), we get

$$p_{0m} = \left( \frac{\lambda_0}{\lambda_0 + \gamma_0} \right)^m p_{00}$$

which yields

$$\sum_{m=0}^{\infty} p_{0m} = \left( \frac{\lambda_0 + \gamma_0}{\gamma_0} \right) p_{00}$$

Proceeding as before (3.4) gives

$$\sum_{m=0}^{\infty} p_{1m} = \left( \frac{\lambda_1 + \gamma_1}{\gamma_1} \right) p_{10}$$

Let $d_j, j = 0, 1, 2, \ldots$ denote the limiting probabilities of the underlying Markov Chain $Q$. That is

$$d_j = \lim_{t \rightarrow \infty} P(U(t) = j)$$

satisfy

$$\sum_{j=0}^{n} d_j = 1 \quad \text{and} \quad d_0 = d_1 = \sum_{j=2}^{n} d_j, \quad (3.10)$$

where $d_j = d_0 q_j = d_1 q_j, \quad \text{for } j \geq 2$.

Thus from (3.10) we have

$d_0 = d_1 = \frac{1}{3}$ also $d_j = \frac{q_j}{3}, j \geq 2$. In other words, $d_0 = d_1 = \frac{1}{3}$ shows that the Markov Chain constantly alternates between the units $i = 0, i = 1$ and one of the other units $i \geq 2$. Therefore the system visits the three units $0, 1$ and $i \geq 2$, $\frac{1}{3}$ of the times. Hence

$$\sum_{m=0}^{\infty} p_{im} = \frac{d_i}{\gamma_i} = \left( \frac{1}{\gamma_0} + \frac{1}{\gamma_1} \right) + \sum_{k=2}^{n} \frac{q_k}{\gamma_k}, \quad 2 \leq i \leq n,$$

$$(3.11)$$
and more explicitly
\[ \sum_{m=0}^{\infty} p_{0m} = \frac{d_0}{\gamma_0} \sum_{k=0}^{n} \frac{d_k}{\gamma_k} = \frac{1}{\gamma_0} \cdot \frac{\gamma_0 + \gamma_1}{\gamma_0 \gamma_1} \sum_{k=2}^{n} \frac{q_k}{\gamma_k} \]

Assuming \( \beta = \frac{\gamma_0 + \gamma_1}{\gamma_0 \gamma_1} + \sum_{k=2}^{n} \frac{q_k}{\gamma_k} \) we get
\[ \sum_{m=0}^{\infty} p_{0m} = \frac{1}{\beta \gamma_0} \]

from (3.11) it follows
\[ \sum_{m=0}^{\infty} p_{im} = \frac{q_i}{\beta \gamma_i} \]

which eventually yields
\[ \sum_{m=0}^{\infty} p_{im} = \frac{q_i \sum_{m=0}^{\infty} p_{0m} \gamma_0}{\gamma_i} \]

Continuing as before for \( i = 1 \) we obtain
\[ \sum_{m=0}^{\infty} p_{im} = \frac{q_i \sum_{m=0}^{\infty} p_{1m} \gamma_1}{\gamma_i} \]

Using (3.7) and (3.12) \( p_{00} \) can be expressed in terms of \( \beta \) as
\[ p_{00} = \frac{1}{\beta (\lambda_0 + \gamma_0)} \]

similarly
\[ p_{10} = \frac{1}{\beta (\lambda_1 + \gamma_1)} \]
4. Generating Functions

Define

\[ G_i(z) = \sum_{m=0}^{\infty} p_{im} z^m, \quad i = 0, 1, 2, \ldots, n, \quad 0 \leq z \leq 1. \]  \hspace{1cm} (4.1)

Setting \( i = 0 \) in (4.1), we get

\[ G_0(z) = \frac{\lambda_0 + \gamma_0}{\lambda_0(1 - z) + \gamma_0} p_{00} \]  \hspace{1cm} (4.2)

using (3.1), (4.2) modifies to

\[ (\lambda_0(1 - z) + \gamma_0)G_0(z) = \sum_{i=2}^{n} \sum_{m=0}^{\infty} \gamma_i p_{im}. \]  \hspace{1cm} (4.3)

Setting \( z = 1 \) and \( i = 0 \) in (4.1) and using (4.3) we get

\[ \gamma_0 \sum_{m=0}^{\infty} p_{0m} = \sum_{i=2}^{n} \gamma_i \sum_{m=0}^{\infty} p_{im}. \]  \hspace{1cm} (4.4)

For \( i = 1 \) the equations corresponding to (4.2),(4.3) and (4.4) are obtained as

\[ G_1(z) = \frac{\lambda_1 + \gamma_1}{\lambda_1(1 - z) + \gamma_1} p_{10} \]  \hspace{1cm} (4.5)

\[ (\lambda_1(1 - z) + \gamma_1)G_1(z) = \sum_{i=2}^{n} \sum_{m=0}^{\infty} \gamma_i p_{im}. \]  \hspace{1cm} (4.6)

and

\[ \gamma_1 \sum_{m=0}^{\infty} p_{1m} = \sum_{i=2}^{n} \gamma_i \sum_{m=0}^{\infty} p_{im}. \]  \hspace{1cm} (4.7)

Note that (4.4) and (4.7) coincides implying \( \gamma_0 \sum_{m=0}^{\infty} p_{0m} = \gamma_1 \sum_{m=0}^{\infty} p_{1m}. \)

Arranging the terms of (3.6) and using (3.5) and after some algebraic manipulations we get

\[ G_i(z)[\lambda_i z(1 - z) + \mu_i(z - 1) + z\gamma_i] - \gamma_1 q_i zG_1(z) = p_{i0} \mu_i(z - 1), \quad i \geq 2 \]  \hspace{1cm} (4.8)
Now, $G_0(z)$ and $G_1(z)$ can be determined from (4.3) and (4.6), each $G_i(z)$, $i \geq 2$ can be found from (4.8) if $p_{i0}$ is known.

Define

$$f_0(z) = \lambda_0(1 - z) + \gamma_0,$$

$$f_1(z) = \lambda_1(1 - z) + \gamma_1$$

and

$$f_i(z) = \lambda_i z(1 - z) + \mu_i(z - 1) + \gamma_i z, \quad i \geq 2$$

The quadratic polynomial $f_i(z), i \geq 2$, each have two real roots. Let $z_i$ denote the only real positive root of $f_i(z)$ in the interval $(0,1)$. Then we have

$$z_i = \frac{(\lambda_i + \mu_i + \gamma_i) - \sqrt{(\lambda_i + \mu_i + \gamma_i)^2 - 4\lambda_i \mu_i}}{2\lambda_i} \quad (4.9)$$

In fact, $z_i$ represents the Laplace-Stieltjes Transform, evaluated at a point $\gamma_i$, of the busy period in an $M/M/1$ queue with arrival rate $\lambda_i$ and service rate $\mu_i$. Substitution of (4.9) in (4.8) gives

$$p_{i0} = \frac{\gamma_1 q_i z_i G_1(z_i)}{\mu_i(1 - z_i)}.$$  

Using (4.6) $p_{i0}$ modifies to

$$p_{i0} = \frac{\gamma_1^2 \sum_{m=0}^{\infty} p_{1m} q_i z_i}{(\lambda_1(1 - z_i) + \gamma_i)\mu_1(1 - z_i)}, \quad 2 \leq i \leq n.$$  

Now, each $G_i(z)$ can be completely determined using $p_{i0}$

5. **Average Queue Size**

Define

$$G'_i(1) = E[L_i] = \sum_{m=1}^{\infty} mp_{im}, \quad i = 0, 1, 2, \ldots, n.$$  

Taking derivatives of (4.3) and (4.6) at $z = 1$, and using (4.2) and (4.5), we obtain

$$G'_0(1) = E[L_0] = \frac{\lambda_0}{\gamma_0}(\frac{\lambda_0 + \gamma_0}{\gamma_0})p_{00} \quad (5.1)$$
STEADY-STATE BEHAVIOR OF AN M/M/1 QUEUE...

Figure (1): $E(L_0)$ versus $\gamma_0$ for $p_{00} = .667$

$$G'_1(1) = E[L_1] = \frac{\lambda_1}{\gamma_1} \left( \frac{\lambda_1 + \gamma_1}{\gamma_1} p_{10} \right)$$  \hspace{1cm} (5.2)

Now, $E[L_i]$ can be determined by differentiating (4.8), setting $z = 1$ and using (4.1) as

$$E[L_i] = \frac{1}{\gamma_i} [\mu_i p_{i0} + \frac{q_i}{\beta} \left( \frac{\lambda_1}{\gamma_1} + \frac{\lambda_i}{\gamma_i} - \frac{\mu_i}{\gamma_i} \right)]$$  \hspace{1cm} (5.3)

The total number of customers cleared from the system is given by

$$E[L] = \sum_{i=0}^{n} E[L_i]$$

$$= E[L_0] + E[L_1] + \sum_{i=2}^{n} \frac{1}{\gamma_i} [\mu_i p_{i0} + \gamma_i \sum_{m=0}^{\infty} p_{im} (\frac{\lambda_1}{\gamma_1} + \frac{\lambda_i}{\gamma_i} - \frac{\mu_i}{\gamma_i})]$$
6. Numerical Illustrations

In this section, the effect of the system parameters on the mean queue size are discussed numerically. In Figures 1 and 2 the mean queue size of the unit 0 and unit 1 are plotted against \( \gamma_0 \) and \( \gamma_1 \) for varying values of \( \lambda_0 \) and \( \lambda_1 \) respectively. Both the graphs show a steep decrease for increasing values of \( \gamma_0 \) and \( \gamma_1 \) and reaches stability. Figure 3 depicts the graph of the mean queue size of the unit 2 verses \( \gamma_2 \). The behavior of the graph is similar to \( E[L_0] \) and \( E[L_1] \) and it reaches stability for slightly higher values of \( \gamma_2 \). Similar graphs can be drawn for the average queue size of the other units. Figure 4 compares the average queue size of the units \( i = 1 \) and \( i = 2 \) verses the parameter \( \gamma_1 \). The average queue size of the unit 2 is greater when compared to the average queue size of the unit 1. \( E[L_1] \) and \( E[L_2] \) reaches stability more or less at the same period.

7. Conclusions

In this paper, we have discussed an \( M/M/1 \) queueing model under random environment with disasters. This is modeled with a failure state(0) and a checking
The steady-state probabilities are analyzed for all the units, using generating functions. Some performance measures are explained with numerical illustrations. The future work is to extend this model for a failure state with customers in the orbit.

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References


Figure (4): Mean queue size versus $\gamma_1$ for $\lambda_1 = 0.04, \lambda_2 = .025, \mu_2 = .03, \gamma_2 = 2, p_{10} = .667$ and $p_{20} = .335$


