International Journal of Pure and Applied Mathematics
Volume 101 No. 2 2015, 267-279
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v101i2.11
ijpam.eu

STEADY-STATE BEHAVIOR OF AN $M/M/1$ QUEUE IN RANDOM ENVIRONMENT SUBJECT TO SYSTEM FAILURES AND REPAIRS

S. Sophia[1 §], B. Praba[2]
[1,2]Department of Mathematics
SSN College of Engineering
SSN Nagar, 603 110, TamilNadu, INDIA

Abstract: In this paper, we consider an $M/M/1$ queueing model with disasters in random environment. This model contains a repair state, a checking state along with a set of operational units. When a disaster occurs the system is transferred to the failure state(state o) and then to the checking state(state 1) for the verification process. After the verification process, the system moves to any one of the operational units. The steady-state behavior of the underlying queueing model along with the average queue size is analyzed. Numerical illustrations are provided to study the effect of parameters on the average queue size.

Key Words: $M/M/1$ queueing model, disasters, generating functions, steady-state equations, average queue size

1. Introduction

Queueing models are used to analyze many real time situations. Depending upon the real time environment various types of queueing models are studied by different authors. When the real situation involves complexity, authors

Received: November 19, 2014

[§]Correspondence author

analyzed the system as queueing models along with constraints like disaster, server vocation, retrials etc...

One of the major area of interest in the queueing models is queueing system in random environment with disasters. When a system is working, a disaster may occur either due to an outside source or a cause from within the system. Due to this disaster the customers get annihilated and the system is inactive until the arrival of a new customer. In computer networks, inactivation of the processors by transmission of virus is not uncommon and the system can be restored to normalcy only after taking appropriate corrective measures. (see Chao [2], Chen and Renshaw [3]). A queueing system with this constraint is of much interest and is studied by many authors. Such models are useful for the performance evaluation of communications and computer networks which are characterized by time-varying arrival, service and failure rates. To incorporate the effects of external environmental factors into a stochastic model a queue with a random environment is considered. The random environment process may take a number of varied forms such as a discrete- or continuous-time Markov chain, a random walk, a semi-Markov process, or a Brownian motion, etc. If the random environment is Markovian, the primary stochastic process to which it is attached is said to be Markov-modulated. Kella and Whitt [5] considered an infinite-capacity storage model with two-state random environment and characterized its steady-state behavior. Kakubava [4] formulated a queueing system under batch service in an M/G/1 System and assumed that the batch size depends on the state of the system and also on the state of some random environment whereas Korotaev and Spivak [6] considered a queueing system in a semi-Markovian random environment.

Krishna kumar and Arivudainambi [7] considerd an $M/M/1$ queueing system with catastrophes and found the transient solution of the system concerned. Krisnna kumar et al. [8] analyzed discouraged arrivals queue with catastrophes and derived explicit expressions of the transient solution along with the moments. Sudhesh [10] studied a queue with system disasters and customer impatience and derived its system-size probabilities using continued fraction technology. Yechiali [11] considered an $M/M/c$ queue with system disasters and impatience customers and derived various quality of service measures such as mean sojourn time of a served customer, proportion of customers served and rate of lost customers due to disasters. Some more articles on queues with disasters can be found in Altman and Yechiali [1] and Yi et al. [12].

The long run behavior of any system with disaster or the steady-state is very important. Many researchers have studied the system behavior in the steady-state. Paz and Yechiali [9] analyzed a $M/M/1$ queue in a multi-phase random environment, where the system occasionally suffers a disastrous failure, causing all present jobs to be lost. They considered a repair phase, where the system gets repaired and found the steady-state behavior and some performance measures of the system concerned.

Instead of having one repair state if we have a checking state to check the overall condition of the system after getting repaired the efficiency of the model will get improved. Hence in our paper we have considered an $M/M/1$ multi-phase queueing model in random environment with disasters. This model contains $n+1$ operational units including a repair state followed by a checking state. Whenever a disaster occurs the system moves to the repair state and after getting repaired it moves to the checking state with probability one. From the checking state it moves to any one of the remaining units. For the above model steady-state behavior is analyzed.

The organization of the paper is as follows: In Section 2, a detailed description is given for the proposed model. Section 3 deals with the balance equation of the model under consideration. An expression for the steady-state probability of having m customers in state i, $m \geq 0$ is obtained. The above probabilities are analyzed using generating functions in Section 4. In Section 5 some performance measures are discussed.

2. Model Description

In this paper, we consider an $M/M/1$ queueing model with $n+1$ units operating in a random environment. These $n+1$ operating units form a continuous time Markov chain with units $i = 0, 1, 2, 3, ...n$ and the corresponding Transition Probability Matrix $[q]$ as given below

$$\begin{pmatrix} 0 & 1 & 0 & 0 & ... & 0 \\ 0 & 0 & q_2 & q_3 & \cdots & q_n \\ 1 & 0 & 0 & 0 & ... & 0 \\ 1 & 0 & 0 & 0 & ... & 0 \\ 1 & 0 & 0 & 0 & ... & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The units $i = 2$ to n are operating units whereas the unit $i = 0$ is the failure unit and $i = 1$ is the checking unit. When the system is in the unit i, $i = 2$ to n it acts as an $M/M/1$ queue with arrival rate $\lambda_i \geq 0$ and service rate $\mu_i \geq 0$. The sojourn time for unit i is the exponential random variable with mean $\frac{1}{\gamma_i}$, $i = 2, 3, 4, ..., n$.

Occasionally a failure occurs when the system is in unit $i \geq 2$. At that instant the system is transferred to the unit $i = 0$ and then to $i = 1$. The time spent by a system at the failure unit $i = 0$ is an exponentially distributed random variable with mean $\frac{1}{\gamma_0}$. The time taken for the verification process at $i = 1$ is also an exponentially distributed random variable with mean $\frac{1}{\gamma_1}$.

After the completion of the verification process the system is transferred to any one of the operating units $i \geq 2$ with probability q_i and hence $\sum_{i=2}^{n} q_i = 1$. The arrival rate for each of the failure units $i = 0$ and $i = 1$ are $\lambda_0 \geq 0$ and $\lambda_1 \geq 0$ respectively. As there is no service in these units $\mu_0 = \mu_1 = 0$. In each active unit $i = 2$ to n, the system stays until a breakdown occurs which sends it to unit 0 and then to unit 1.

The above system can be represented by the stochastic process $\{U(t), N(t)\}$ where $U(t)$ denotes the unit in which the system operates at time t and $N(t)$ denotes the number of customers present at time t. The steady state probabilities of the system being in state i with m customers is denoted by p_{im}. That is

$$p_{im} = lim_{t\to\infty}\{P(U(t) = i, N(t) = m)\},$$
$$\forall\ t \geq 0,\ 0 \leq i \leq n,\ m = 0, 1, 2, ... \quad (2.1)$$

3. Steady-State Equations

Based on the above assumptions the underlying steady-state balance equations for the system under consideration are as follows. For $m = 0$:

$$(\lambda_0 + \gamma_0)p_{00} - \sum_{i=2}^{n} \gamma_i \sum_{m=0}^{\infty} p_{im} = 0, \quad \text{for} \quad i = 0, \qquad (3.1)$$

$$(\lambda_1 + \gamma_1)p_{10} - \sum_{i=2}^{n} \gamma_i \sum_{m=0}^{\infty} p_{im} = 0, \quad \text{for} \quad i = 1, \qquad (3.2)$$

and for $m \geq 1$

$$(\lambda_0 + \gamma_0)p_{0m} = \lambda_0 p_{0m-1}, \quad \text{for} \quad i = 0 \qquad (3.3)$$

$$(\lambda_1 + \gamma_1)p_{1m} = \lambda_1 p_{1m-1}, \quad \text{for} \quad i = 1 \qquad (3.4)$$

For $i = 2, 3, 4, ...n$, the balance equations are

$$(\lambda_i + \gamma_i)p_{i0} = \mu_i\ p_{i1} + \gamma_1 q_i p_{10}, \quad \text{for } \ m = 0, \tag{3.5}$$

and

$$(\lambda_i + \mu_i + \gamma_i)p_{im} = \lambda_i p_{im-1} + \mu_i p_{im+1} + \gamma_1 q_i p_{1m}, \quad \text{for } \ m \geq 1 \tag{3.6}$$

From (3.3), we get

$$p_{0m} = (\frac{\lambda_0}{\lambda_0 + \gamma_0})^m\ p_{00} \tag{3.7}$$

which yields

$$\sum_{m=0}^{\infty} p_{0m} = (\frac{\lambda_0 + \gamma_0}{\gamma_0})\ p_{00} \tag{3.8}$$

Proceeding as before (3.4) gives

$$\sum_{m=0}^{\infty} p_{1m} = (\frac{\lambda_1 + \gamma_1}{\gamma_1})\ p_{10} \tag{3.9}$$

Let $d_j, j = 0, 1, 2, ...$ denote the limiting probabilities of the underlying Markov Chain Q. That is

$$d_j = lim_{t \longrightarrow \infty} P(U(t) = j)$$

satisfy

$$\sum_{j=0}^{n} d_j = 1 \ \text{ and } \ d_0 = d_1 = \sum_{j=2}^{n} d_j, \tag{3.10}$$

where $d_j = d_0 q_j = d_1 q_j, \quad$ for $\ j \geq 2$.

Thus from (3.10) we have
$d_0 = d_1 = \frac{1}{3}$ also $d_j = \frac{q_j}{3}, j \geq 2$. In other words, $d_0 = d_1 = \frac{1}{3}$ shows that the Markov Chain constantly alternates between the units $i = 0, i = 1$ and one of the other units $i \geq 2$. Therefore the system visits the three units $0, 1$ and $i \geq 2$, $\frac{1}{3}$ of the times. Hence

$$\sum_{m=0}^{\infty} p_{im} = \frac{\frac{d_i}{\gamma_i}}{\sum_{k=0}^{n} \frac{d_k}{\gamma_k}} = \frac{\frac{q_i}{\gamma_i}}{(\frac{1}{\gamma_0} + \frac{1}{\gamma_1}) + \sum_{k=2}^{n} \frac{q_k}{\gamma_k}}, \quad 2 \leq i \leq n, \tag{3.11}$$

and more explicitly

$$\sum_{m=0}^{\infty} p_{0m} = \frac{\frac{d_0}{\gamma_0}}{\sum_{k=0}^{n} \frac{d_k}{\gamma_k}} = \frac{\frac{1}{\gamma_0}}{\frac{\gamma_0+\gamma_1}{\gamma_0\gamma_1} + \sum_{k=2}^{n} \frac{q_k}{\gamma_k}}$$

Assuming $\beta = \frac{\gamma_0+\gamma_1}{\gamma_0\gamma_1} + \sum_{k=2}^{n} \frac{q_k}{\gamma_k}$ we get

$$\sum_{m=0}^{\infty} p_{0m} = \frac{1}{\beta\gamma_0} \tag{3.12}$$

from (3.11) it follows

$$\sum_{m=0}^{\infty} p_{im} = \frac{q_i}{\beta\gamma_i}$$

which eventually yields

$$\sum_{m=0}^{\infty} p_{im} = \frac{q_i \sum_{m=0}^{\infty} p_{0m}\gamma_0}{\gamma_i}$$

Continuing as before for $i = 1$ we obtain

$$\sum_{m=0}^{\infty} p_{im} = \frac{q_i \sum_{m=0}^{\infty} p_{1m}\gamma_1}{\gamma_i}$$

Using (3.7) and (3.12) p_{00} can be expressed in terms of β as

$$p_{00} = \frac{1}{\beta(\lambda_0 + \gamma_0)}$$

similarly

$$p_{10} = \frac{1}{\beta(\lambda_1 + \gamma_1)}$$

4. Generating Functions

Define

$$G_i(z) = \sum_{m=0}^{\infty} p_{im} z^m, \quad i = 0, 1, 2, ...n, \ 0 \leq z \leq 1. \tag{4.1}$$

Setting $i = 0$ in (4.1), we get

$$G_0(z) = \frac{\lambda_0 + \gamma_0}{\lambda_0(1-z) + \gamma_0} p_{00} \tag{4.2}$$

using (3.1), (4.2) modifies to

$$(\lambda_0(1-z) + \gamma_0)G_0(z) = \sum_{i=2}^{n} \gamma_i \sum_{m=0}^{\infty} p_{im}. \tag{4.3}$$

Setting $z = 1$ and $i = 0$ in (4.1) and using (4.3) we get

$$\gamma_0 \sum_{m=0}^{\infty} p_{0m} = \sum_{i=2}^{n} \gamma_i \sum_{i=0}^{\infty} p_{im}. \tag{4.4}$$

For $i = 1$ the equations corresponding to (4.2),(4.3) and (4.4) are obtained as

$$G_1(z) = \frac{\lambda_1 + \gamma_1}{\lambda_1(1-z) + \gamma_1} p_{10} \tag{4.5}$$

$$(\lambda_1(1-z) + \gamma_1)G_1(z) = \sum_{i=2}^{n} \gamma_i \sum_{m=0}^{\infty} p_{im} \tag{4.6}$$

and

$$\gamma_1 \sum_{m=0}^{\infty} p_{1m} = \sum_{i=2}^{n} \gamma_i \sum_{i=0}^{\infty} p_{im} \tag{4.7}$$

Note that (4.4) and (4.7) coincides implying $\gamma_0 \sum_{m=0}^{\infty} p_{0m} = \gamma_1 \sum_{m=0}^{\infty} p_{1m}$.

Arranging the terms of (3.6) and using (3.5) and after some algebraic manipulations we get

$$G_i(z)[\lambda_i z(1-z) + \mu_i(z-1) + z\gamma_i] - \gamma_1 q_i z G_1(z) = p_{i0}\mu_i(z-1), i \geq 2 \tag{4.8}$$

Now, $G_0(z)$ and $G_1(z)$ can be determined from (4.3) and (4.6), each $G_i(z)$, $i \geq 2$ can be found from (4.8) if p_{i0} is known.

Define

$$f_0(z) = \lambda_0(1-z) + \gamma_0,$$

$$f_1(z) = \lambda_1(1-z) + \gamma_1$$

and

$$f_i(z) = \lambda_i z(1-z) + \mu_i(z-1) + \gamma_i z, i \geq 2$$

The quadratic polynomial $f_i(z), i \geq 2$, each have two real roots. Let z_i denote the only real positive root of $f_i(z)$ in the interval (0,1). Then we have

$$z_i = \frac{(\lambda_i + \mu_i + \gamma_i) - \sqrt{(\lambda_i + \mu_i + \gamma_i)^2 - 4\lambda_i\mu_i}}{2\lambda_i} \tag{4.9}$$

In fact, z_i represents the Laplace-Stieltjes Transform, evaluated at a point γ_i, of the busy period in an $M/M/1$ queue with arrival rate λ_i and service rate μ_i. Substitution of (4.9) in (4.8) gives

$$p_{i0} = \frac{\gamma_1 q_i z_i G_1(z_i)}{\mu_i(1-z_i)}.$$

Using (4.6) p_{i0} modifies to

$$p_{i0} = \frac{\gamma_1^2 \sum_{m=0}^{\infty} p_{1m} q_i z_i}{(\lambda_1(1-z_i) + \gamma_i)\mu_1(1-z_i)}, 2 \leq i \leq n.$$

Now, each $G_i(z)$ can be completely determined using p_{i0}

5. Average Queue Size

Define

$$G_i'(1) = E[L_i] = \sum_{m=1}^{\infty} m p_{im}, \quad i = 0, 1, 2, ..., n.$$

Taking derivatives of (4.3) and (4.6) at $z = 1$, and using (4.2) and (4.5), we obtain

$$G_0'(1) = E[L_0] = \frac{\lambda_0}{\gamma_0}\left(\frac{\lambda_0 + \gamma_0}{\gamma_0}\right)p_{00} \tag{5.1}$$

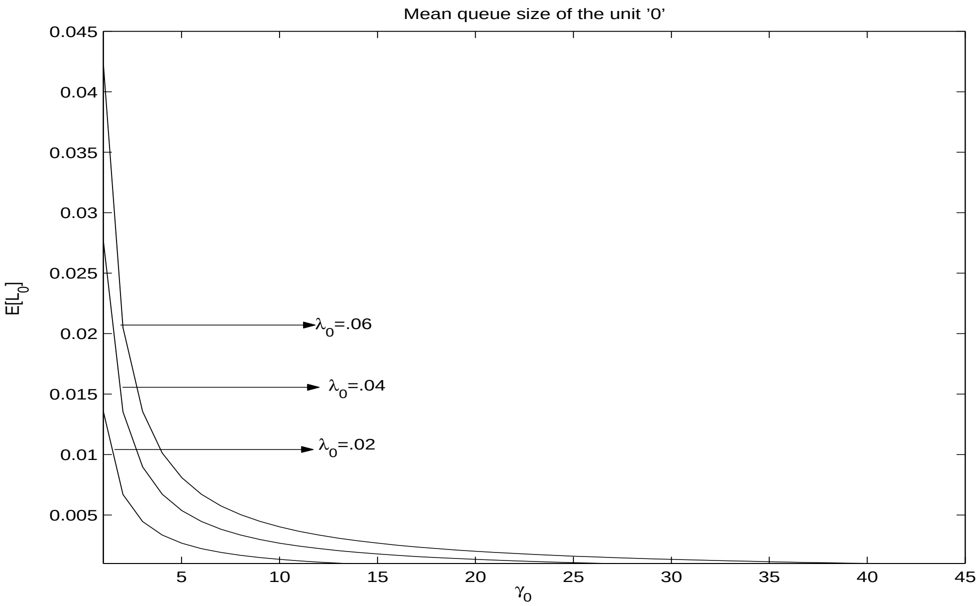

Figure (1): $E(L_0)$ versus γ_0 for $p_{00} = .667$

$$G_1'(1) = E[L_1] = \frac{\lambda_1}{\gamma_1}(\frac{\lambda_1 + \gamma_1}{\gamma_1})p_{10} \tag{5.2}$$

Now, $E[L_i]$ can be determined by differentiating (4.8), setting $z = 1$ and using (4.1) as

$$\begin{aligned} E[L_i] &= \frac{1}{\gamma_i}[\mu_i p_{i0} + \frac{q_i}{\beta}(\frac{\lambda_1}{\gamma_1} + \frac{\lambda_i}{\gamma_i} - \frac{\mu_i}{\gamma_i})] \qquad (5.3) \\ &= \frac{1}{\gamma_i}[\mu_i p_{i0} + \gamma_i \sum_{m=0}^{\infty} p_{im}(\frac{\lambda_1}{\gamma_1} + \frac{\lambda_i}{\gamma_i} - \frac{\mu_i}{\gamma_i})] \end{aligned}$$

The total number of customers cleared from the system is given by

$$\begin{aligned} E[L] &= \sum_{i=0}^{n} E[L_i] \\ &= E[L_0] + E[L_1] + \sum_{i=2}^{n} \frac{1}{\gamma_i}[\mu_i p_{i0} + \gamma_i \sum_{m=0}^{\infty} p_{im}(\frac{\lambda_1}{\gamma_1} + \frac{\lambda_i}{\gamma_i} - \frac{\mu_i}{\gamma_i})] \end{aligned}$$

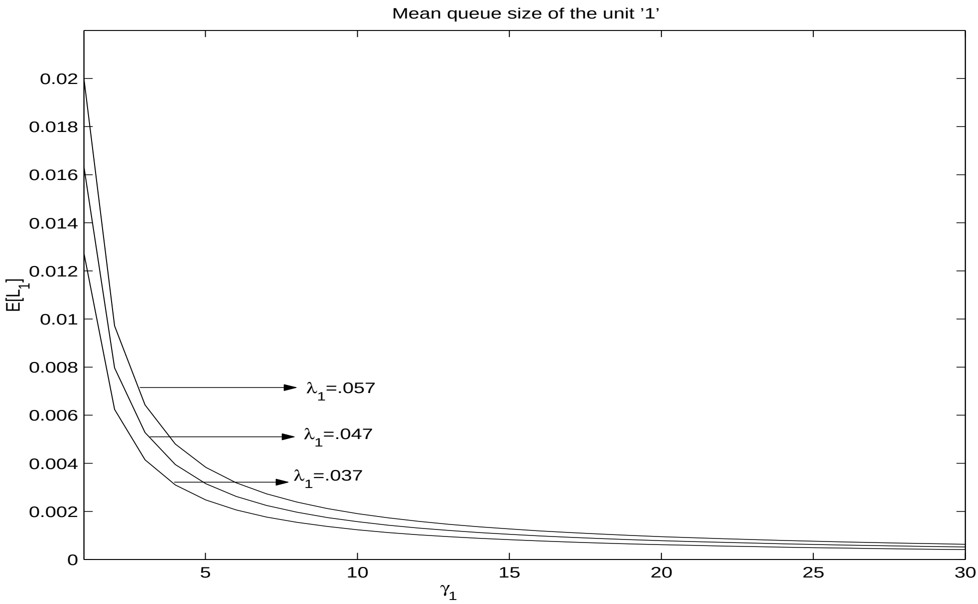

Figure (2):$E(L_1)$ versus γ_1 for $p_{10} = .333$

6. Numerical Illustrations

In this section, the effect of the system parameters on the mean queue size are discussed numerically. In Figures 1 and 2 the mean queue size of the unit 0 and unit 1 are plotted against γ_0 and γ_1 for varying values of λ_0 and λ_1 respectively. Both the graphes show a steep decrease for increasing values of γ_0 and γ_1 and reaches stability. Figure 3 depicts the graph of the mean queue size of the unit 2 verses γ_2. The behavior of the graph is similar to $E[L_0]$ and $E[L_1]$ and it reaches stability for slightly higher values of γ_2. Similar graphes can be drawn for the average queue size of the other units. Figure 4 compares the average queue size of the units $i = 1$ and $i = 2$ verses the parameter γ_1. The average queue size of the unit 2 is greater when compared to the average queue size of the unit 1. $E[L_1]$ and $E[L_2]$ reaches stability more or less at the same period.

7. Conclusions

In this paper, we have discussed an $M/M/1$ queueing model under random environment with disasters. This is modeled with a failure state(0) and a checking

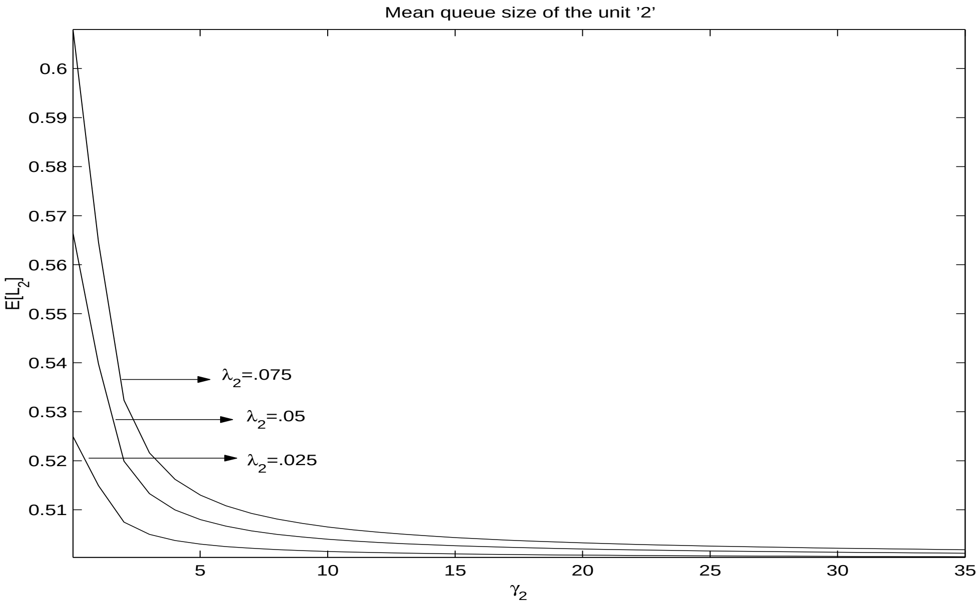

Figure (3): $E(L_2)$ versus γ_2 for $\lambda_1 = .04, \gamma_1 = .08, \mu_2 = .03$ and $p_{20} = .667$

state(1). The steady-state probabilities are analyzed for all the units, using generating functions. Some performance measures are explained with numerical illustrations. The future work is to extend this model for a failure state with customers in the orbit.

Acknowledgments

The authors sincerely thank the management and the principal, SSN Institutions for their constant support towards the successful completion of this work.

References

[1] E. Altma and U. Yechiali, Analysis of customers impatience in queues with server vacations, *Queueing Systems*, **52**, No. 4 (2006), 261-279.

[2] X. Chao, A queueing network model with catastrophes and product form solution, *Operations Research Letters*, **18**, No. 2 (1995), 75-79.

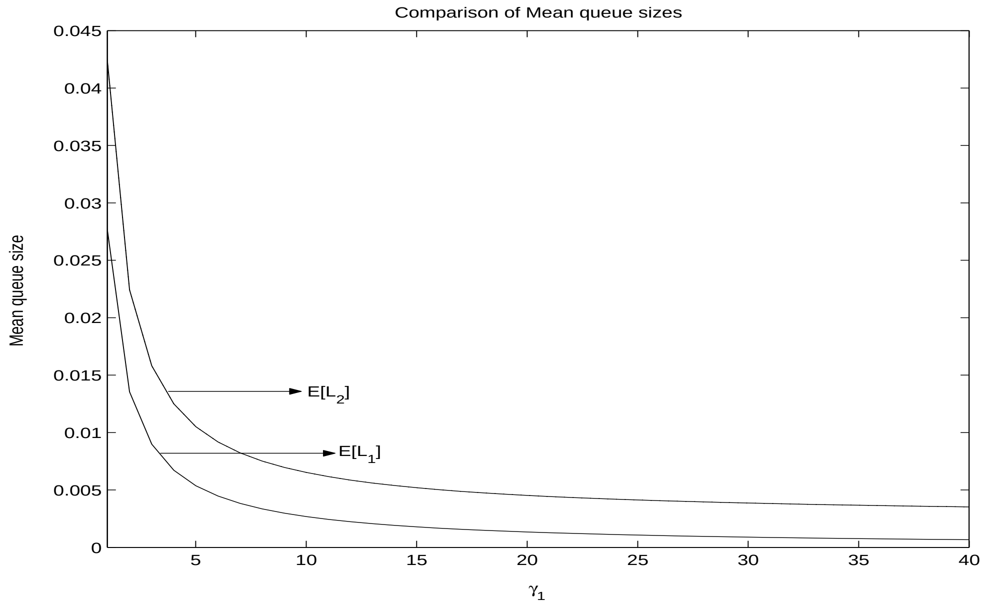

Figure (4): Mean queue size versus γ_1 for
$\lambda_1 = 0.04, \lambda_2 = .025, \mu_2 = .03, \gamma_2 = 2, p_{10} = .667$ and $p_{20} = .335$

[3] A. Chen and E. Rensha, The M/M/1 queue with mass exodeus and mass arrivals when empty, *Journal of Applied Probability*, **34** (1997), 192-207.

[4] R. V. Kakubava, Analysis of queues under batch service in an $M/G/1$ system in a random environment, *Automation and Remote Control*, **62**, No. 5 (2001), 782-788.

[5] O. Kella and W. Whitt, A storage model with a two-stage random environment, *Operations Research*, **40**, No. 3 (1992), S257-S262.

[6] I. A. Korotaev, and L. R. Spivak, Queueing systems in a semi-Markovian random environment, *Automation and Remote Control*, **53**, No. 7 (1992), 1028-1033.

[7] B. Krishna Kumar, and D. Arivudainambi, Transient solution of an M/M/1 queue with catastrophes, *Computers and Mathematics with Applications*, **40**, No. 10 (2000), 1233-1240.

[8] B. Krihna kumar, A. Vijayakumar and S.Sophia, Transient analysis for

state-dependent queues with catastrophes, *Stochastic Analysis and Applications*, **26**, No. 6 (2008), 1201-1217.

[9] N. Paz, and U.Yechiali, An M/M/1 queue in random environment with disasters, *Asia-Pacific Journal of Operational Research*, **31**, No. 3 (2013), 1-12.

[10] R. Sudhesh, Transient analysis of a queue with system disasters and customer impatience, *Queueing Systems*, **66**, No. 1 (2010), 95-105.

[11] U. Yechiali, Queues with system disasters and impatient customers when system is down, *Queueing Systems*, **56**, No. 3 (2007), 195-202.

[12] X.W. Yi, J.D Kim, D.W.Choi, K.C.Chae, The Geo/G/1 queue with disasters and multiple working vacations, *Stochastic Models*, **23**, No. 4 (2007), 537-549.