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EQUIVALENT CONDITIONS OF SUBSPACE TRANSITIVITY CRITERION FOR TUPLES OF OPERATORS

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Abstract: In this paper we investigate the conditions under which a tuple of operator holds in the subspace-transitivity criterion.

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1. Introduction

By an n-tuple of operators we mean a finite sequence of length n of commuting continuous linear operators on a Banach space X.

Definition 1.1. Let $\mathcal{T} = (T_1, T_2, ..., T_n)$ be an n-tuple of operators acting on an infinite dimensional Banach space X. We will let $\mathcal{F} = \{{T_1}^{k_1}{T_2}^{k_2}...{T_n}^{k_n} : k_i \in \mathbb{Z}_+; \ \ i = 1, ..., n\}$ be the semigroup generated by $\mathcal{T}$. For $x \in X$, the orbit of x under the tuple $\mathcal{T}$ is the set $Orb(\mathcal{T}, x) = \{Sx : S \in \mathcal{F}\}$. A vector x is called a M-hypercyclic vector for $\mathcal{T}$ if $Orb(\mathcal{T}, x) \cap M$ is dense in M and in this case the tuple $\mathcal{T}$ is called M-hypercyclic. The set of all M-hypercyclic vectors of $\mathcal{T}$ is denoted by $HC(\mathcal{T}, M)$. Also, by $\mathcal{T}_d^{(2)}$ we will refer to the set of all k copies of an element of $\mathcal{F}$, i.e. $\mathcal{T}_d^{(2)} = \{S_1 \oplus S_2 : S_1 = S_2 \in \mathcal{F}\}$.

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Definition 1.2. Suppose that $\mathcal{T} = (T_1, T_2, ..., T_n)$ is an n-tuple of operators acting on a separable infinite dimensional Banach space X over $\mathbb{C}$ and M is a nonzero subspace of X. We say that a tuple $\mathcal{T} = (T_1, T_2, ..., T_n)$ is called M-transitive with respect to a tuple of nonnegative integer sequences

$$(\{k_{j(1)}\}_j, \{k_{j(2)}\}_j, ..., \{k_{j(n)}\}_j),$$

if for every nonempty relatively open subsets U, V of X there exists $j_0 \in \mathbb{N}$ such that $T_1^{-k_{j_0(1)}} T_2^{-k_{j_0(2)}} ... T_n^{-k_{j_0(n)}}(U) \cap V$ contains a relatively open nonempty subset of M. Also, we say that an n-tuple $\mathcal{T}$ is M-transitive if it is M-transitive with respect an n-tuple of nonnegative integer sequences.

Definition 1.3. Suppose that $\mathcal{T} = (T_1, T_2, ..., T_n)$ is an n-tuple of operators acting on a separable infinite dimensional Banach space X over $\mathbb{C}$ and M is a nonzero subspace of X. We say that $\mathcal{T}_d^{(2)}$ is M-transitive if for every nonempty relatively open subsets U_1, V_1, U_2, V_2 of M, there exists $m_i, \in \mathbb{N}$, $i = 1, ..., n$, such that $T_1^{-m_1} T_2^{-m_2} ... T_n^{-m_n}(U_i) \cap V_i$ contains a relatively open nonempty subset of M for $i = 1, 2$.

Suprisingly, there are something that does not happen for single operators. For example, hypercyclic tuples can arise in finite dimensional, and there are operators that have somewhere dense orbits that are not everywhere dense. Also, we note that there are subspace-hypercyclic operators that are not hypercyclic. For some topics we refer to [1]-[3].

2. Main Result

In this section we characterize the equivalent conditions for a tuple of operator satisfying the subspace-transitivity criterion.

Theorem 2.1. (Subspace-Transitivity Criterion for Tuples) *Let $\mathcal{T} = (T_1, T_2, ..., T_n)$ be a tuple of continuous operators acting on a separable infinite dimensional Banach space X. Suppose that there exist two dense subsets Y and Z in M, and strictly increasing sequences of positive integers $\{m_{j(i)}\}_j$ for $i = 1, ..., n$ such that:*

1. $T_1^{m_{j(1)}} ... T_n^{m_{j(n)}} y \to 0$ for all $y \in Y$ as $j \to 0$,

2. For every $z \in Z$, there exists a sequence $\{x_j\}_j$ in M such that $x_j \to 0$ and $T_1^{m_{j(1)}} ... T_n^{m_{j(n)}} x_j \to z$,

3. M is invariant subspace for $T_1^{m_{j(1)}} ... T_n^{m_{j(n)}}$ for all j.
Then $\mathcal{T}$ is subspace-transitive with respect to M.

Theorem 2.2. *Let $\mathcal{T} = (T_1, T_2, ..., T_n)$ be a tuple of operators acting on a separable infinite dimensional Banach space X and M be closed subspace of X. Then the followings are equivalent:*

(i) $\mathcal{T}$ satisfies the subspace-transitivity criterion with respect to M.

(ii) $\mathcal{T}_d^{(2)}$ is M-transitive.

Proof. (i) $\rightarrow$ (ii): Let Y, Z, $(\{m_{j(1)}\}_j, \{m_{j(2)}\}_j, ..., \{m_{j(n)}\}_j)$ be as given in the M-transitivity criterion. Notice that M-transitivity criterion will also be satisfied by any tuple of subsequences

$$(\{m_{j_k(1)}\}_k, \{m_{j_k(2)}\})_k, ..., \{m_{j_k(n)}\}_k)$$

of $(\{m_{j(1)}\}_j, \{m_{j(2)}\}_j, ..., \{m_{j(n)}\}_j)$. Now for $i = 1, 2$, let (U_i, V_i) be a pair of nonempty relatively open subsets of M. We want to show that there are arbitrary large positive integers $m_1, ..., m_n$ satisfying $T_1^{-m_1} T_2^{-m_2} ... T_n^{-m_n}(U_i) \cap V_i$ contains a nonempty relatively open subset of M for $i = 1, 2$. Since

$$\{T_1{}^{m_{j(1)}} T_2{}^{m_{j(2)}} ... T_n{}^{m_{j(n)}} : j \geq 1\}$$

is M-transitive, there exists a tuple of subsequences

$$(\{m_{j_k(1)}\}_k, \{m_{j_k(2)}\})_k, ..., \{m_{j_k(n)}\}_k)$$

of $(\{m_{j(1)}\}_j, \{m_{j(2)}\}_j, ..., \{m_{j(n)}\}_j)$ with $T_1^{-m_{j_k(1)}} T_2^{-m_{j_k(2)}} ... T_n^{-m_{j_k(n)}}(U_1) \cap V_1$ containing a nonempty relatively open subset of M for all $j \geq 1$. But as we stated earlier, $\mathcal{T}$ is also M-transitive with respect to

$$(\{m_{j(1)}\}_j, \{m_{j(2)}\}_j, ..., \{m_{j(n)}\}_j),$$

thus for $i = 1, ..., n$, there exist $m_i \in \{m_{j_k(i)}\}$ arbitrary large so that

$$T_1^{-m_1} T_2^{-m_2} ... T_n^{-m_n}(U_2) \cap V_2$$

contains a nonempty relatively open subset of M. Hence

$$T_1^{-m_1} T_2^{-m_2} ... T_n^{-m_n}(U_i) \cap V_i$$

contains a nonempty relatively open subset of M for $i = 1, 2$ and so $\mathcal{T}_d^{(2)}$ is M-transitive.

(ii) $\rightarrow$ (i): Suppose that $\mathcal{T}_d^{(2)}$ is M-transitive, hence it is M-hypercyclic. Let $x \oplus y$ be a M-hypercyclic vector for $\mathcal{T}_d^{(2)}$. In particular, x and y are M-hypercyclic vectors for $\mathcal{T}$. Note that for all tuple of nonnegative integers $(m_1, m_2, ..., m_n)$, the vector $T_1^{m_1}T_2^{m_2}...T_n^{m_n}y$ is M-hypercyclic vector for $\mathcal{T}$. This implies that for all nonempty relatively open subsets G, U of M, there is $g \in G$ and $u \in U$ such that (g, u) is a M-hypercyclic vector for $\mathcal{T}_d^{(2)}$. Fix such g and put $U_k = B(0, 1/k) \cap M$ for all $k \geq 1$. Proceeding by induction we find $u_j \in U_j$ for all $j \in \mathbb{N}$, and increasing sequences $\{m_{j(i)}\}_j$ $(i = 1, ..., n)$ of natural numbers satisfying ${T_1}^{m_{j(1)}}{T_2}^{m_{j(2)}}...{T_n}^{m_{j(n)}}g \in U_j$ and ${T_1}^{m_{j(1)}}{T_2}^{m_{j(2)}}...{T_n}^{m_{j(n)}}u_j \in g + U_j$ for all $j \in \mathbb{N}$. This is possible since:

1. $Orb(\mathcal{T}_d^{(2)}, g \oplus u_j) \cap M \oplus M$ is dense in $M \oplus M$.
2. $g \oplus u_j \in HC(\mathcal{T}^{(2)}, M \oplus M)$.
3. $0 \oplus g \in M \oplus M$,
4. $U_j \oplus g + U_j$ contains $0 \oplus g$.

Now let $Y = Z = Orb(\mathcal{T}, g)$ which is dense in X. Then we have that ${T_1}^{m_{j(1)}}{T_2}^{m_{j(2)}}...{T_n}^{m_{j(n)}}g \to 0$ and so ${T_1}^{m_{j(1)}}{T_2}^{m_{j(2)}}...{T_n}^{m_{j(n)}}u_j \to g$, and $u_j \to 0$. For $z = T_1^{m_1}T_2^{m_2}...T_n^{m_n}g$, define $x_j = T_1^{m_1}T_2^{m_2}...T_n^{m_n}u_j$. Then $x_j \to 0$ since $u_j \to 0$. Also, note that

$${T_1}^{m_j(1)}{T_2}^{m_j(2)}...{T_n}^{m_j(n)}x_j = {T_1}^{m_1}{T_2}^{m_2}...{T_n}^{m_n}{T_1}^{m_j(1)}{T_2}^{m_j(2)}...{T_n}^{m_j(n)}u_j$$

which tends to z. Hence $\mathcal{T}$ satisfies the subspace-transitivity criterion with respect to M and the proof is complete. □

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