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DOWNSETS AND GENERALIZED DESCRIPTIVE SET THEORY

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Abstract: Downsets over a regular cardinal κ arise from Π^1_1 normal form. For inaccessible κ downsets correspond to the trees of generalized descriptive set theory. Basic properties of downsets, trees, and the correspondence are considered. A new characterization of weak compactness is given. Some facts about Σ^1_1 WPS's are proved.

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1. Introduction

In [2] the question is left open whether a $\mathcal{U}_{\Sigma^1_1}$-Mahlo cardinal is weakly compact. This would be of great interest if true, as it would provide evidence that weakly compact cardinals can be built-up by collecting the universe. Once the "weakly compact barrier" has been broken, justifying larger indescribable cardinals (e.g. totally indescribable) would presumably be routine. The conjecture is plausible, in view of the fact that there is no obvious Π^1_1 sentence enforcing the $\mathcal{U}_{\Sigma^1_1}$-Mahlo cardinals.

A relevant question, of considerable interest in its own right, is as follows. Suppose there is a Σ^1_1 WPS (well preorder on a subset; see [2]) of order type α. Is there a uniform Σ^1_1 WPS of order type β where $\beta \geq \alpha$? A positive answer to this question would be evidence in favor of the first question having a positive

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answer.

An attempt to devise methods to deal with these and related questions was begun in [3], where Π^1_1 normal form over a regular cardinal was defined. The success of Π^1_1 normal form in descriptive set theory suggests that it might be useful over a regular cardinal. Descriptive set theory over a regular cardinal has been considered already in [5], [6], [7], [8], [11], [12], [13], [16], [17], [18], [21], and [9], [10], [19], [23] for $\kappa = \aleph_1$.

Just as in the case of ω, Π^1_1 normal form gives rise to a combinatorial notion, analogous to trees of finite sequences, which is called a downset in [3]. Greater understanding of these should be attained; some considerations on this problem will be given here. The reader is assumed to be familiar with [3].

Throughout this paper, κ will denote a regular uncountable cardinal. In some cases, κ will be required to be (strongly) inaccessible. In this case $|V_\kappa| = \kappa = \beth_\kappa$, any set has cardinality less than κ, and any class (in particular the class $\mathcal{A}_g$ defined below) has cardinality at most κ.

2. Basic Definitions

In a partial order $\leq$ on a set S a downset (or downward closed or lower set) is a subset D such that if $x \in D$, $w \in S$, and $w \leq x$ then $w \in D$. Algebraic properties which hold for downsets include the following.

- Any subset of S is a contained in a smallest downset, its "downward closure".
- The downsets form a complete lattice under union and intersection.

For a regular cardinal κ, consider the following two sets:

- $\mathcal{A}_g = \{f : x \mapsto V_\kappa : x \in V_\kappa\}$
- $\mathcal{A} = \{f : \alpha \mapsto \kappa : \alpha \in \kappa\}$

These sets are partially ordered by $\subseteq$. Let DS denote $\{D \subseteq \mathcal{A}_g : D$ is a downset$\}$, and let Tr denote $\{D \subseteq \mathcal{A} : D$ is a downset$\}$. Either the notation DS(D) or $D \in$ DS may be used.

Let $\mathcal{N}_g$ denote $(V_\kappa)^{V_\kappa}$. For $D \in$ DS, a branch of D is a function $F \in \mathcal{N}_g$ such that $F \restriction x \in D$ for all $x \in V_\kappa$. Let $[D]$ denote the set of branches of D. If $[D] \neq \emptyset$ say that D is branched; otherwise D is unbranched. Let UDS denote the unbranched downsets of DS.

Let $\mathcal{N}$ denote κ^κ. A branch in a downset $D \in$ Tr is a function $F \in \mathcal{N}$ such that $F \restriction \alpha \in D$ for all $\alpha < \kappa$. $[D]$ denotes the set of branches, and UTr the unbranched downsets.

As seen in [3], $\mathcal{N}_g$ is useful in considering Π^1_1 normal form in a regular cardinal, since Skolem functions map V_κ to V_κ. $\mathcal{N}$ is a more common object of study, and is often called generalized Baire space, ordinary Baire space being the case $\kappa = \omega$. Various facts for each case will be shown to hold. In the case of $\mathcal{N}$, many of these have been given in the references cited above, in particular [7]; explicit citation will generally be omitted.

In the literature on generalized Baire space, the requirement that $\kappa^{<\kappa} = \kappa$ is often imposed. This breaks in to the following cases:

- For a successor cardinal κ^+, the requirement holds iff $2^\kappa = \kappa^+$.
- If κ is weakly inaccessible but not inaccessible, and $2^\lambda > \kappa$ for some $\lambda < \kappa$, the requirement does not hold.
- If κ is weakly inaccessible but not inaccessible, and $2^\lambda \leq \kappa$ for all $\lambda < \kappa$, the requirement holds.
- If κ is inaccessible the requirement holds.

The requirement follows from GCH.

Next, some facts are given which are not used thereafter, but might be of interest.

Theorem 1. *If $D_1 \in DS$, $D_2 \in UDS$, and $D_1 \subseteq D_2$ then $D_1 \in UDS$. The analogous statement holds for Tr.*

Proof. Immediate. □

Suppose $\eta < \kappa$. Let $\mathcal{A}_g^{\|\eta}$ denote those $f \in \mathcal{A}_g$ where $f(x) \in (V^\kappa)^\eta$. For $f \in \mathcal{A}_g^{\|\eta}$ and $\xi < \eta$ let $f_{\|\xi}$ be the function $f_{\|\xi}(x) = f(x)(\xi)$. Suppose D_ξ for $\xi < \eta$ are downsets. Let $\|_{\xi<\eta} D_\xi$ be $\{f \in \mathcal{A}_g^{\|\eta} : f_{\|\xi} \in D_\xi \text{ for all } \xi < \eta\}$. It is readily seen that $\|_{\xi<\eta} D_\xi$ is a downset.

Theorem 2. *In the case of $\mathcal{N}_g$, suppose $\eta < \kappa$ and D_ξ for $\xi < \eta$ is a downset.*

a. *$\cup_{\xi<\eta} D_\xi$ is unbranched iff every D_ξ is unbranched.*

b. *$\|_{\xi<\eta}\, D_\xi$ is unbranched iff some D_ξ is unbranched.*

Proof. For part a, one direction follows by theorem 1. For the other direction, suppose $F \in \mathcal{N}_g$. Let $x_\xi \in V_\kappa$ be such that $F \restriction x_\xi \notin D_\xi$. Let $x = \cup_{\xi<\eta} x_\xi$; then $x \in V_\kappa$. If $F \restriction x \in \cup_\xi D_\xi$ then $F \restriction x \in D_\xi$ for some ξ, so $F \restriction x_\xi \in D_\xi$, a contradiction. Thus, $F \restriction x \notin \cup_\xi D_\xi$. Since F was arbitrary $\cup_\xi D_\xi$ is unbranched.

For part b, suppose $D_\xi \subseteq \mathcal{A}_g$ is unbranched. Given F, if $F \notin \mathcal{N}_g^{\|\eta}$ then for some x $F \restriction x \notin \mathcal{A}_g^{\|\eta}$. Otherwise, $F =\|_\xi\, F_\xi$ for some F_ξ's. By hypothesis for some ξ and x $F_\xi \restriction x \notin D_\xi$. By definition $F \restriction x \notin\|_\xi D_\xi$. Since F was arbitrary $\|_\xi D_\xi$ is unbranched. For the converse, suppose F_ξ is a branch of D_ξ; then $\|_\xi F_\xi$ is a branch of $\|_\xi D_\xi$. □

Part a holds for $\mathcal{N}$, as a variation of the argument shows. For part b, there is no analogous definition of $\|_{\xi<\eta} D_\xi$ unless $\kappa^{|\eta|} = \kappa$. If this does hold (e.g., if $\eta < \omega$) then a bijection between κ and κ^η can be used to define $\|_{\xi<\eta} D_\xi$, and part b proved.

Another construction which generalizes is that of "trees on products" (Section 2.C of [15]). For $\vec{x}, \vec{y} \in \mathcal{A}_g^k$ let $\vec{x} \subseteq_{\mathrm{c}} \vec{y}$ denote "componentwise inclusion", $x_i \subseteq y_i$ for all i. Let $\mathcal{A}_g^{k,\mathrm{e}}$ denote $\{\vec{f} \in \mathcal{A}_g^k : \mathrm{Dom}(f_1) = \cdots = \mathrm{Dom}(f_k)\}$. This may be ordered by $\subseteq_{\mathrm{c}}$. The following definitions are also made: For $\vec{F} \in \mathcal{N}_g^k$ $\vec{f} \subseteq {}_{\mathrm{c}}\vec{F}$ if for all i $f_i \subseteq F_i$. $U_{\vec{f}} = \{\vec{F} : \vec{f} \subseteq_{\mathrm{c}} \vec{F}\}$; $\{U_{\vec{f}} : \vec{\in}\mathcal{A}_g^{k,\mathrm{e}}\}$ is readily seen to be a base for the topology on $\mathcal{N}_g^k$ (see below). $\mathrm{Dom}(\vec{f})$ denotes the common value $\mathrm{Dom}(f_i)$. A downset is defined as usual. A branch in a downset D is a vector $\vec{F} \in \mathcal{N}_g^k$ such that for all x the componentwise restriction of F to x is in D. The definitions of $\mathcal{A}^{k,\mathrm{e}}$, etc., are a straightforward variation.

3. Topology

A topology can be defined on $\mathcal{N}_g$ or $\mathcal{N}$, which as will be seen in the next section is related to the downsets. For $f \in \mathcal{A}_g$ let $U_f = \{F \in \mathcal{N}_g : f \subseteq F\}$. Standard arguments show that $\{U_f\}$ is a base for a topology on $\mathcal{N}_g$. Indeed, say that $f, g \in \mathcal{A}_g$ are compatible if $f \restriction (x \cap y) = g \restriction (x \cap y)$ where $x = \mathrm{Dom}(f)$ and $y = \mathrm{Dom}(g)$. $U_f \cap U_g$ is nonempty iff f and g are compatible, and in this case, $U_f \cap U_g = U_{f \cup g}$. Also by standard arguments, U_f is clopen; indeed, $U_f^c = \cup_g U_g$ where g ranges over the functions with $\mathrm{Dom}(g) = \mathrm{Dom}(f)$ other than f. The topology with base $\{U_f\}$ is totally disconnected.

For $f \in \mathcal{A}$ U_f is defined analogously, and the facts stated in the previous paragraph hold also for the topological space $\mathcal{N}$. A further fact is that f and g are compatible iff they are comparable.

If κ is inaccessible then $|V_\kappa| = \kappa$ and it follows that $\mathcal{N}_g$ and $\mathcal{N}$ are homeomorphic. Indeed, suppose $E : \kappa \mapsto V_\kappa$ is a bijection; E induces a bijection $\hat{E} : \mathcal{N} \mapsto \mathcal{N}_g$, where $\hat{E}(F) = EFE^{-1}$. An injection from $\mathcal{A}$ to $\mathcal{A}_g$ is also induced; using $\hat{E}$ to denote this as well, if f has domain α then $\hat{E}(f)$ has domain $E[\alpha]$ and $\hat{E}(f) = EfE^{-1}$.

$\hat{E} : \mathcal{N} \mapsto \mathcal{N}_g$ is a homeomorphism: For a basic open subset $U_f \subseteq \mathcal{N}$, $\hat{E}[U_f]$ is the basic open subset $U_{\hat{E}(f)}$. For a basic open subset $U_f \subseteq \mathcal{N}_g$, let $h = E^{-1}(f)$

and $\alpha = \sup \mathrm{Dom}(h)$; then $\hat{E}^{-1}[U_f] = \cup\{U_g : \mathrm{Dom}(g) = \alpha, h \subseteq g\}$.

It may also be verified that if $\mathcal{N}_g$ and $\mathcal{N}$ are homeomorphic then κ is inaccessible. As already noted, in this paper, basic definitions and theorems will be given "in general", that is, for both $\mathcal{N}_g$ and $\mathcal{N}$ for any regular cardinal κ. If κ is inaccessible one version usually follows from the other by virtue of a homeomorphism of the form $\hat{E}$.

Lemma 3. *Suppose $S \subseteq \mathcal{A}_g$. $\cap_{f\in S} U_f \neq \emptyset$ iff S is compatible (i.e., every pair is compatible). Suppose S is compatible and $|S| < \kappa$, and let $g = \cup S$. Then $\cap_{f\in S} U_f = U_g$. The analogous statements hold in $\mathcal{N}$.*

Proof. If S is not compatible then there is an incompatible pair f, g and $U_f \cap U_g = \emptyset$. If S is compatible then $\cup S$ is single valued, whence there is some $F \in \mathcal{N}_g$ extending it, and $F \in \cap_{f\in S} U_f$, and the first claim is proved. For the second claim, since κ is regular and $|S| < \kappa$, $g \in V_\kappa$. For any $f \in S$ $f \subseteq g$, whence $U_g \subseteq U_f$; thus, $U_g \subseteq \cap_{f\in S} U_f$. Let $x_f = \mathrm{Dom}(f)$ for $f \in S$ and $y = \mathrm{Dom}(g)$; then $y = \cup_{f\in S} x_f$. Suppose $F \in \cap_{f\in S} U_f$. Then $F \restriction x_f = f$. It follows that $F \restriction y = g$, and so $F \in U_g$. The argument for $\mathcal{N}$ is a straightforward variation. □

Theorem 4. *The intersection of $\eta < \kappa$ open subsets of $\mathcal{N}_g$ is open, the union of η closed subsets is closed, and the Boolean algebra of clopen subsets is κ-complete. The analogous statements hold in $\mathcal{N}$.*

Proof. Suppose $W = \cap_{\xi<\eta} W_\xi$ where W_ξ is open, say $W_\xi = \cup_{\zeta\in S_\xi} U_{\xi\zeta}$ where $U_{\xi\zeta}$ is basic clopen. Let $T = \{t : \eta \mapsto \cup_{\xi<\eta} S_\xi : t(\xi) \in S_\xi\}$. Then $W = \cap_{\xi<\eta} \cup_{\zeta\in S_\xi} U_{\xi\zeta} = \cup_{t\in T} \cap_{\xi<\eta} U_{\xi,t(\xi)}$. By lemma 3 W is open. This proves the first claim, and the remaining claims follow. The argument for $\mathcal{N}$ is a straightforward variation. □

In $\mathcal{N}_g$ the number of basic clopen sets is at most $\beth_\kappa$ (since $\mathcal{A}_g \subseteq V_\kappa$ since by hypothesis κ is regular), and it is easy to see that the number equals $\beth_\kappa$. If $\lambda < \kappa$ then there are at least $2^{\beth_\lambda}$ clopen sets. Thus, the number of clopen sets and the number of open sets equals $2^{\beth_\kappa}$.

In $\mathcal{N}$ the number of basic clopen sets is $\kappa^{<\kappa}$. The number of open sets is at most $2^{\kappa^{<\kappa}}$. If $\lambda < \kappa$ then there are at least 2^{κ^λ} clopen sets. Thus, the number of clopen sets and the number of open sets equals $2^{\kappa^{<\kappa}}$.

The product spaces $\mathcal{N}_g^k \times V_\kappa^l$ and $\mathcal{N}^k \times \kappa^l$ (where factors V_κ or κ are given the discrete topology) have various properties as in the case of ordinary Baire space. In particular, $\mathcal{N}_g^k$ is homeomorphic to $\mathcal{N}_g$ and $\mathcal{N}^k$ is homeomorphic to $\mathcal{N}$. Indeed, let $E_k : V_\kappa \times k \mapsto V_\kappa$ be a bijection; E_k induces a bijection $\hat{E}_k : \mathcal{N}_g^k \mapsto \mathcal{N}_g$, where $\hat{E}_k(\vec{F})(E_k(x,i)) = F_i(x)$. An injection (also denoted $\hat{E}_k$) $\hat{E}_k : \mathcal{A}_g^{k,\mathrm{e}} \mapsto \mathcal{A}_g$ is also induced; if $\mathrm{Dom}(\vec{f}) = x$ then $\hat{E}_k(\vec{f})$ has domain $E_k[x \times k]$ and $\hat{E}_k(\vec{f})(E_k(w,i))) = f_i(w)$. Note that there is a club subset C of $\alpha < \kappa$ such that $E[V_\alpha \times k] = V_\alpha$. For $\alpha \in C$ let $S_\alpha = \{U_{\vec{f}} : \mathrm{Dom}(\vec{f}) = V_\alpha\}$ and $T_\alpha = \{U_f : \mathrm{Dom}(f) = V_\alpha\}$. Then $\cup_\alpha S_\alpha$ is a base for the topology on $\mathcal{N}_g^k$, $\cup_\alpha T_\alpha$ is a base for the topology on $\mathcal{N}_g$, and $\hat{E}$ induces a bijection from S_α to T_α. It follows that $\hat{E}$ is a homeomorphism. The argument for $\mathcal{N}^k$ and $\mathcal{N}$ is a straightforward variation.

Various definitions and theorems concerning $\mathcal{N}_g$ can be given for $\mathcal{N}_g^k$, in a manner compatible with the homeomorphism, and similarly for $\mathcal{N}$ and $\mathcal{N}^k$. Some examples will be seen below.

In the case of $\mathcal{N}$ there is a simple bijection E_k, namely $\langle \alpha, i\rangle \mapsto k \cdot \alpha + i$. To obtain such for $\mathcal{N}_g$, let $a : \omega \mapsto V_\omega$ be a bijection; then let $E_k(x,i) = (x - V_\omega) \cup a(k \cdot a^{-1}(x \cap V_\omega) + i)$.

4. Downsets and Topology

Suppose X and Y are sets, each equipped with a partial order, which will be denoted $\leq$ in both cases. Suppose $F : X \mapsto Y$ and $G : Y \mapsto X$ are order preserving maps. Recall that the pair of maps F, G is called a Galois adjunction, with left adjoint map F and right adjoint map G, if for $x \in X$ and $y \in Y$, $F(x) \leq y$ iff $x \leq G(y)$. Given such, $x \leq G(F(x))$; an element of $\mathrm{Ran}(G)$ will be said to be A-closed. Also $F(G(y)) \leq y$; an element of $\mathrm{Ran}(F)$ will be said to be an A-kernel. The maps F restricted to the A-closed sets and G restricted to the A-kernels are the inverse maps of an order isomorphism.

Given a downset $D \subseteq \mathcal{A}_g$ let $[D] = \{F \in \mathcal{N}_g : \forall f \subseteq F(f \in D)\}$. Clearly the map $D \mapsto [D]$ is order preserving. Given $S \subseteq \mathcal{N}_g$ let $A_S = \{f \in \mathcal{A}_g : \exists F \in \mathcal{N}_g : f \subseteq F\}$. Clearly A_S is a downset and $S \mapsto A_S$ is order preserving. Say that a downset $D \subseteq \mathcal{A}_g$ is branching iff $\forall f \in D \exists F \in [D](f \subseteq F)$.

Analogous definitions for $\mathcal{N}$ are straightforward.

An example might help clarify the definition of "branching". Let $\mathcal{C}$ denote $\{0,1\}^\kappa$. In $\mathcal{C}$, let $D = \{0^i : i \in \omega\} \cup \{0^i 1 f : i \in \omega, f \in \{0,1\}^{<\kappa}\}$. D is branching, but it contains a chain of length ω with no upper bound in D. $[D]$ is in fact clopen, having complement U_{0^ω}.

Theorem 5. *For a subset $S \subseteq \mathcal{N}_g$ and a downset $D \subseteq \mathcal{A}_g$, $A_S \subseteq D$ iff $S \subseteq [D]$. In this adjunction, S is A-closed iff it is closed; and D is an A-kernel* *iff it is branching. The analogous statements hold for $\mathcal{A}$ and $\mathcal{N}$.*

Proof. For the first claim, either inclusion holds iff

$$\forall F \in \mathcal{N}_g \forall f \in \mathcal{A}_g (F \in S \wedge f \subseteq F \Rightarrow f \in D).$$

For the second claim, F is in the closure of S iff $\forall f \subseteq F \exists G \in S(f \subseteq G)$ iff F is in the A-closure of S. Hence S is closed iff it equals its topological closure iff it equals its A-closure iff it is A-closed. Clearly A_S is branching for any S, so if D is an A-kernel then D is branching. Suppose D is branching and $f \in D$, and choose $F \in [D]$ with $f \subseteq F$; then F witnesses that $f \in A_{[D]}$. The argument for $\mathcal{A}$ and $\mathcal{N}$ is a straightforward variation. □

If $D \subseteq \mathcal{A}_g$ is a downset let $U_D = \cup\{U_f : f \notin D\}$. Clearly U_D is an open subset, and $D \mapsto U_D$ is order preserving when the downsets are ordered by reverse inclusion. If $U \subseteq \mathcal{N}_g$ is an open subset let $D_U = \{f \in \mathcal{A}_g : U_f \not\subseteq U\}$. Clearly D_U is a downset, and $U \mapsto D_U$ is order preserving when the downsets are ordered by reverse inclusion. Analogous definitions for $\mathcal{N}$ are straightforward.

Theorem 6. *For a downset $D \subseteq \mathcal{A}_g$ and an open subset $U \subseteq \mathcal{N}_g$, $U_D \subseteq U$ iff $D \supseteq D_U$. In this adjunction, D is A-closed iff $\forall f(U_f \subseteq U_D \Rightarrow f \notin D)$; and U is always an A-kernel. The analogous statements hold for $\mathcal{A}$ and $\mathcal{N}$.*

Proof. For the first claim, either inclusion holds iff $\forall f \in \mathcal{A}_g(f \notin D \Rightarrow U_f \subseteq U)$. For the second claim, if $U_f \subseteq U_D \Rightarrow f \notin D$ then $f \in D \Rightarrow U_f \not\subseteq U_D \Rightarrow f \in D_{U_D}$. For the third claim, $F \in U$ iff $\exists f(f \notin D_U \wedge F \in U_f)$ iff $F \in U_{D_U}$. The argument for $\mathcal{A}$ and $\mathcal{N}$ is a straightforward variation. □

The requirement for A-closure states that if $\{U_g\}$ is a cover of U_f by subsets where $g \notin D$ then $f \notin D$. For an example, $D \subseteq \mathcal{A}_g$ is unbranched iff $\{U_f : f \notin D\}$ is an open cover of $\mathcal{N}_g$ iff $U_D = \mathcal{N}_g$ iff the A-closure of D equals $\mathcal{A}_g$ iff $\emptyset$ is an element of the A-closure.

Given a downset $D \subseteq \mathcal{A}_g^{k,\mathrm{e}}$ let $[D] = \{F \in \mathcal{N}_g^k : \forall f \subseteq_{\mathrm{c}} F(f \in D)\}$. Given $S \subseteq \mathcal{N}_g^k$ let $A_S = \{f \in \mathcal{A}_g^{k,\mathrm{e}} : \exists F \in \mathcal{N}_g^k : f \subseteq F\}$. Theorem 5 holds for these maps. The analogous facts hold for $D \subseteq \mathcal{A}^{k,\mathrm{e}}$ and $\mathcal{N}^k$.

If $D \subseteq \mathcal{A}_g^{k,\mathrm{e}}$ is a downset let $U_D = \cup\{U_{\vec{f}} : \vec{f} \notin D\}$. If $U \subseteq \mathcal{N}_g^k$ is an open subset let $D_U = \{\vec{f} \in \mathcal{A}_g^{k,\mathrm{e}} : U_{\vec{f}} \not\subseteq U\}$. Theorem 6 holds for these maps. The analogous facts hold for $D \subseteq \mathcal{A}^{k,\mathrm{e}}$ and $\mathcal{N}^k$.

5. Continuous Functions

In this section, proposition 2.6 of [15] (suitably generalized) will be shown to hold for $\mathcal{N}_g$ and $\mathcal{N}$. Suppose $A \subseteq V_\kappa$. Write $A^{\in V_\kappa}$ for $\{f : x \mapsto A : x \in V_\kappa\}$ (so that $\mathcal{A}_g = (V_\kappa)^{\in V_\kappa}$).

Say that a subset of A^{V_κ} is Γ_δ if it is of the form $\cap_{\alpha<\kappa} W_\alpha$ where W_α is open. Suppose $S \subseteq A^{\in V_\kappa}$ and $T \subseteq \mathcal{A}_g$ are downsets. A function $\phi : S \mapsto T$ is said to be monotone if $f \subseteq g \Rightarrow \phi(f) \subseteq \phi(g)$. Given such, let $D(\phi) = \{F \in [S] : \cup_{x \in V_\kappa} \mathrm{Dom}(\phi(F \restriction x)) = V_\kappa\}$. Let $\phi^* : D(\phi) \mapsto [T]$ be the function where $\phi^*(F) = \cup_{x \in V_\kappa} \phi(F \restriction x)$.

The definitions for A^κ of Γ_δ, monotone map, $D(\phi)$, and ϕ^* are analogous.

Theorem 7. *In either $\mathcal{A}_g$ or $\mathcal{A}$, $D(\phi)$ is a Γ_δ subset of $[S]$, and ϕ^* is continuous. Suppose G is a Γ_δ subset of $[S]$ and $\psi : G \mapsto [T]$ is continuous. Then there is a monotone function $\phi : S_G \mapsto T$ where $S_G = \{f \in S : U_f \cap G \neq \emptyset\}$, such that $\psi = \phi^*$. In $\mathcal{A}$, the map ϕ may be extended to S.*

Proof. The proof of proposition 2.6 of [15] can be followed quite closely; for convenience details will be given. The proof will be given for A_g; the proof for $\mathcal{A}$ is analogous..

Given ϕ, for $\alpha < \kappa$ let $W_\alpha = \{F \in [S] : \exists y(V_\alpha \subseteq \mathrm{Dom}(F \restriction y))\}$. Then W_α is open, and $D(\phi) = \cap_\alpha W_\alpha$. Thus, $D(\phi)$ is Γ_δ. Also, for $g \in \mathcal{A}_g$, $(\phi^*)^{-1}[U_g \cap [T]] = \cup_{f \in A^{\in V_\kappa}, g \subseteq \phi(f)} U_f \cap D(\phi)$. Thus, ϕ^* is continuous.

Suppose $\psi : G \in V_\kappa$. Note that S_G is a downset. Let W_α for $\alpha < \kappa$ be open subsets of $[S]$ such that $G = \cap_\alpha W_\alpha$. It may be supposed that $W_0 = [S]$, $\alpha < \beta \Rightarrow W_\alpha \supseteq W_\beta$, and for limit α $W_\alpha = \cap_{\beta<\alpha} W_\beta$. Given $f \in S$ there is a largest $\alpha < \kappa$ such that $V_\alpha \subseteq \mathrm{Dom}(f)$ and $U_f \cap [S] \subseteq W_\alpha$; let $k(f)$ denote this α. For $f \in S_G$ let $\phi(f) = \cup\{g : \exists \beta \leq k(f)(\mathrm{Dom}(g) = V_\beta \wedge \psi[U_f \cap G] \subseteq U_g)\}$.

Suppose $F \in G$. Let $g \in T$ be such that $g \subseteq \psi(F)$. There is an $f_1 \in S_G$ such that $\psi[U_{f_1} \cap G] \subseteq U_g$. It may be assumed that $\mathrm{Dom}(g) = V_\alpha$ for some α. There is an $f_2 \subseteq F$ such that $V_\alpha \subseteq \mathrm{Dom}(f_2)$ and $U_{f_2} \cap [S] \subseteq W_\alpha$. Letting $f = f_1 \cup f_2$, $f \in S_G$ and $g \subseteq \phi(f)$. This shows that $G \subseteq D(\phi)$ and $\psi(F) = \phi^*(F)$ for $F \in G$.

Suppose $F \in D(\phi)$. Then $F \in [S]$, and one readily verifies that $\forall \alpha \exists x(V_\alpha \subseteq \phi(F \restriction x))$. Then $\alpha \leq k(F \restriction x)$ and $f \in U_{F \restriction x} \cap [S] \subseteq W_\alpha$. Since α was arbitrary, $F \in G$.

In $\mathcal{N}$, if $f \in S - S_G$ then $\{g \in S_G : g \subseteq f\}$ is a chain, and $\phi(f)$ may be defined as $\sup\{\phi(g)\ g \in S_G, g \subseteq f\}$. □

This theorem for $\mathcal{N}$ is seemingly folklore; for example remark (1) following theorem 42 of [7] is readily proved using it.

6. Borel Sets and Lusin Classes

The κ-Borel, or just Borel, sets can be defined in general, as the least algebra containing the open sets and closed under unions of length at most κ and complementation. As seen in the preceding section, the "G_δ" sets may be defined in general, and have uses.

In the case of $\mathcal{N}_g$, under the assumption that κ is inaccessible, or in the case of $\mathcal{N}$, under the assumption $\kappa^{<\kappa} = \kappa$, it is easily seen that an open set is a union of at most κ basic clopen sets. As a result, a Borel hierarchy may be defined, as follows.

- A set is Σ^B_1 iff it is open.
- For $\alpha \geq 1$ a set is Π^B_α iff it is the complement of a Σ^B_α set.
- For $\alpha > 1$ a set is Σ^B_α iff it is a union of κ or fewer sets which are Π^B_β for some $\beta < \alpha$.

It is readily seen that a Σ^B_1 set Σ^B_2, whence by well-known arguments (as in [15] for example), if $\beta < \alpha$ then $\Sigma^B_\beta, \Pi^B_\beta \subseteq \Sigma^B_\alpha, \Pi^B_\alpha$. It also follows that $\cup_{\alpha<\kappa^+}\Sigma^B_\alpha$ is closed under unions and intersections of length at most κ, and complementation.

Clearly, for κ inaccessible, any homeomorphism from $\mathcal{N}$ to $\mathcal{N}_g$ preserves the levels of the Borel Hierarchy.

The definition of the Borel sets in $\mathcal{N}_g^k$ and $\mathcal{N}^k$ may be given for any regular κ, and the definition of the Borel hierarchy may be given under the same hypotheses as for $\mathcal{N}_g$ and $\mathcal{N}$.

The "Lusin classes" may be defined in general. $S \subseteq \mathcal{N}_g^k$ is said to be Σ^{1L}_1 if there is a closed subset $\hat{S} \subseteq \mathcal{N}_g^{k+1}$ such that S is the projection along the last coordinate of $\hat{S}$, i.e., $\vec{F} \in S$ iff $\exists G(\langle \vec{F}, G\rangle \in \hat{S})$. S is Π^{1L}_1 iff its complement is Σ^{1L}_1, and Δ^{1L}_1 iff it is both Σ^{1L}_1 and Π^{1L}_1. Σ^{nL}_1 sets for $n \geq 1$ may be defined, but only $n = 1$ will be considered here. These classes are defined analogously for $\mathcal{N}$. If κ is inaccessible the Lusin classes are preserved under homeomorphism between $\mathcal{N}$ and $\mathcal{N}_g$.

It is straightforward to verify that permutation of coordinates induces a homeomorphism of $\mathcal{N}_g^k$ with itself, whence a projection along any coordinate of a closed set is a Σ^{1L}_1 set and the Σ^{1L}_1 sets are closed under permutation of coordinates.

Theorem 8. *a. A projection of a Σ^{1L}_1 subset is Σ^{1L}_1.*
b. A Borel set is Δ^{1L}_1.

Proof. Some cases of this are given in [7],[12]; for convenience a proof will be given. For part a, using obvious notation (as in [20]), if $X(x) = \exists y Y(x, y)$

where $y(x, y) = \exists z Z(x, y, z)$ and Z is closed then $X(x) = \exists w W(x, w)$ where, letting p_i denote projection of the ith component and h a homeomorphism, $W(x, w) = Z(x, \pi_1 \circ h(w), \pi_2 \circ h(w))$. Thus, W is closed and part a follows. For part b it suffices to show that the Σ_1^{1L} sets are closed under unions and intersections of length κ; in fact, in the case of $\mathcal{N}_g$, they are closed under unions and intersections of length $\beth_\kappa = |V_\kappa|$. Suppose $X_x = \exists F \hat{X}_x(F, X)$ for $x \in V_\kappa$, where $\hat{X}_x$ is a closed subset of $\mathcal{N}_g^2$. The set $K = \{\langle x, F, X\rangle : \hat{X}_x(F, X)\}$ is closed. Let $f_1 : \mathcal{N}_g \mapsto V_\kappa$ be the map taking F to $F(0)$; f_1 is readily verified to be continuous. Let $f_2 : \mathcal{N}_g \mapsto \mathcal{N}_g$ be the map $F \mapsto F \circ f$, where $f(x)$ equals $x + 1$ if x is an integer, else x; this is continuous also. Then $X \in \cup_x X_x$ iff $\exists F K(f_1(F), f_2(F), X)$; thus, $\cup_x X_x$ is Σ_1^{1L}. Let N_p denote the subset of $\mathcal{N}_g$ of functions whose range is a set of ordered pairs; this is a closed subset. Let $f_3 : V_\kappa \times \mathcal{N}_g \mapsto \mathcal{N}_g$ be the map where $f_3(x, F)(y) = F(\langle x, y\rangle)$; this is continuous (proof: $f_3(x, F)(y) = z$ iff $\langle x, F\rangle \in \{x\} \times W$ where $F \in W$ iff $F(\langle x, y\rangle) = z$). Then $X \in \cap_x X_x$ iff $\forall x \exists F K(x, F, X)$ iff $\exists F \forall x (N_p(F) \wedge K(x, f_3(x, F), X))$. Since the closed sets are closed under intersection, it follows that $\cap_x X_x$ is Σ_1^{1L}. □

The Borel sets have received much attention in the literature. In particular, for $\mathcal{N}$ when $\kappa^{<\kappa} = \kappa$, not all Δ_1^{1L} sets are Borel ([7],[12]). Borel sets will not be considered further here.

7. Formulas and Topology

Let L_ϵ denote the first order language of set theory, with binary relations $=, \in$. Let $L_\in^f$ denote $L_\in$ with function variables added, considered as unary function symbols (so $F(x)$ is a term). A bounded quantifier is one of the form $\forall x \in t$ or $\exists x \in t$, where t is a term, such that x does not occur in t. A bounded, or Δ_0^0, formula is a formula where all quantifiers are bounded. For $n \geq 1$ a Σ_n^0 (resp. Π_n^0) formula is a Δ_0^0 formula, preceded by n alternating blocks of first-order quantifiers, the first of which is $\exists$ (resp. $\forall$). A Δ_0^1 formula is a formula which is Δ_0^0, Σ_n^0 for some n, or Π_n^0 for some n. For $n \geq 1$ a Σ_n^1 (resp. Π_n^1) formula is a Δ_0^1 formula, preceded by n alternating blocks of second-order quantifiers, the first of which is $\exists$ (resp. $\forall$). Note that a Δ_0^1 formula with no free function variables is a formula of $L_\in$.

Theorem 9. *Suppose ϕ is a Δ_0^0 formula in the free variables $\vec{F}, \vec{x}$, with parameters. Then the subset of $V_\kappa^l \times \mathcal{N}_g^k$ defined by ϕ is clopen.*

Proof. Suppose t is the term $F_{i_s}(\cdots(F_{i_1}(x))\cdots)$, and let a_j be the argument of F_{i_j}. Then in the open set $F_{i_j}(a_j) = a_{j+1}$ in the product space, the value of t is determined. It follows that for atomic formulas, the set of $\langle \vec{F}, \vec{x} \rangle$ where ϕ is true, and where it is false, are both open, whence both are clopen. The claim for open formulas follows by induction on formulas. If ϕ is $\exists x \in t\psi$, inductively the predicate defined by ψ is clopen. By properties of the product topology, fixing a value for x results in a clopen set. Lemma 3 and theorem 4 generalize to product spaces. It follows that the predicate defined by ϕ is clopen. □

As immediate corollaries, Σ^0_1 subsets of $\mathcal{N}_g$ are open and Π^0_1 subsets are closed, By corollary 5 of [3], Σ^1_1 subsets are Σ^{1L}_1, and Π^1_1 subsets are Π^{1L}_1.

Theorem 10. *An open subset $U \subseteq \mathcal{N}_g$ is Σ^0_1.*

Proof. Let $S \subseteq \mathcal{A}_g$ be such that $U = \cup_{f \in S} U_f$. Then $F \in U$ iff $\exists f(f \in S \wedge f \subseteq F)$. □

To define subsets of $\mathcal{N}$ an appropriate language is required. As in [2], the language L_{OS} has two sorts, one for ordinals $\alpha, \ldots$, and one for elements of $\mathcal{A}$ $s, \ldots$. The language includes the functions and relations 0, 1, $\alpha + \beta$, and $\alpha < \beta$. In addition there are the functions $\alpha = \mathrm{Dom}(s)$, $\beta = \mathrm{Eval}(s, \alpha)$, and $t = \mathrm{Rstr}(s, \alpha, \beta)$. L^f_{OS} is obtained from L_{OS} by adding function variables $F, \ldots$ for elements of $\mathcal{N}$. Bounded quantifiers are those of the form $\forall \alpha < t$ or $\exists \alpha < t$ where t is a term such that α does not occur in t. Δ^0_0 formulas, etc., are defined as for L^f_ϵ; first order quantifiers may be of either sort.

The structure OS_θ for the language L_{OS} for any cardinal θ is defined in the obvious way; see [2] for details.

$\mathcal{A}$ can be equipped with the topology whose basic open sets are $U_f = \{g \in \mathcal{A} : f \subseteq g\}$. In the following formulas are understood to be over L^f_{OS}.

Theorem 11. *Suppose ϕ is a Δ^0_0 formula in the free variables $\vec{F}, \vec{x}, \vec{f}$, with parameters. Then the subset of $\mathcal{N}^k \times \kappa^l \times \mathcal{A}^m$ defined by ϕ is clopen.*

Proof. The proof is essentially the same as that of theorem 9. □

As immediate corollaries, Σ^0_1 subsets of $\mathcal{N}$ are open and Π^0_1 subsets are closed,

A normal form theorem for L_{OS} seems problematical, since Skolem functions have arguments of both ordinal and ordinal sequence sort, and cannot readily be converted to elements of $\mathcal{N}$. As a result, there is no apparent proof that Σ^1_1 subsets are Σ^{1L}_1. It is true that if $S \subseteq \mathcal{N}$ is defined by a formula of the form

$\forall\vec{F}\exists\vec{x}\psi(\vec{F},\vec{x},G)$ (resp. $\exists\vec{F}\forall\vec{x}\psi(\vec{F},\vec{x},G)$) where ψ is Δ^0_0 then S is Π^{1L}_1 (resp. Σ^{1L}_1).

There is a well-known surjection $F_f : Ord \mapsto L$. A treatment may be found in [1], where the predicate $\alpha \in_f \beta$ is defined to hold iff $F_f(\alpha) \in F_f(\beta)$ and is shown to be Δ^0_1 (lemma 23.c). The predicate $\alpha =_f \beta$ is defined to hold iff $F_f(\alpha) = F_f(\beta)$; it is readily seen from the proof of lemma 23 of [1] that this also is Δ^0_1.

Lemma 12. *The Δ^0_1 formulas are closed under quantifiers of the form "$\exists\alpha \in_f t$" and "$\forall\alpha \in_f t$".*

Proof. Using theorem 15.23.1 of [22], $\exists\alpha \in_f t\phi$ iff $\exists\beta(\beta = t \wedge \exists\alpha < \beta(\alpha \in_f \beta \wedge \phi))$ iff $\forall\beta(\beta = t \Rightarrow \exists\alpha < \beta(\alpha \in_f \beta \wedge \phi))$. Using lemma 23.b of [1], $\exists\alpha \in_f t\phi$ is Δ^0_1. The argument for $\forall\alpha \in_f t\phi$ is similar. □

Lemma 13. *The predicate "$F_f(\alpha) = \beta$" is Δ^0_1.*

Proof. Let $\mathrm{Ord}_f(\alpha)$ hold iff $F_f(\alpha)$ is an ordinal; since $\mathrm{Ord}(x)$ is Δ_0 in $L_\in$, it is readily seen that Ord_f is Δ^0_1 (see the proof of lemma 24.b of [1]). Likewise, the predicates $\mathrm{IsSuc}_f(\alpha,\beta)$, which holds iff $F_f(\alpha)$ is an ordinal and $F_f(\beta)$ is its successor, and $\mathrm{Lim}_f(\alpha)$, which holds iff $F_f(\alpha)$ is a limit ordinal, are Δ^0_1. Let $P(\beta, s)$ be the predicate $\mathrm{Dom}(s) = \beta \wedge s(0) = 0 \wedge \forall\gamma < \beta(\exists\gamma^- < \gamma(\gamma = \gamma^- + 1) \Rightarrow \mathrm{IsSuc}_f(s(\beta^-), s(\beta)) \wedge \mathrm{Lim}_f(\gamma) \Rightarrow (\forall\delta < \gamma(s(\delta) \in_f s(\gamma)) \wedge \forall\zeta \in_f s(\gamma)\exists\eta < \gamma(s(\eta) =_f \zeta)))$; this is Δ^1_0. Then $F_f(\alpha) = \beta$ iff $\exists s(P(\beta + 1, s) \wedge s(\beta) =_f \alpha))$ iff $\forall s(P(\beta + 1, s) \Rightarrow s(\beta) =_f \alpha))$. □

Lemma 14. *The predicate "$F_f(\alpha) = f$" is Δ^0_1.*

Proof. Let $P_1(\alpha, \beta, \gamma)$ be the predicate "$F_f(\alpha)$ is the ordered pair with components $F_f(\beta)$ and $F_f(\gamma)$". Since $\alpha = \langle\beta, \gamma\rangle$ is a Δ_0 predicate in $L_\in$, P_1 is Δ^0_1. Let P_2 be the predicate $\exists\delta_1 \in_f \alpha\exists\delta_2, \delta_3 \in_f \delta_1(P_1(\alpha, \delta_1, \delta_2) \wedge F_f(\delta_2) = \beta \wedge F_f(\delta_3) = \gamma)$; using lemma 13, P_2 is Δ^0_1. Finally, $F_f(\alpha) = s$ iff $\forall\beta < \mathrm{Dom}(s)\exists\gamma \in_f \alpha(P_2(\gamma, \beta, s(\beta))) \wedge \forall\gamma \in_f \alpha\exists\beta < \mathrm{Dom}(s)(P_2(\gamma, \beta, s(\beta)))$. □

Theorem 15. *If $V = L$ then an open subset $U \subseteq \mathcal{N}$ is Σ^0_1.*

Proof. Let $S \subseteq \mathcal{A}$ be such that $U = \cup_{f\in S}U_f$. Let S' be a set of $\alpha < \kappa$ such that $S = F_f[S']$. Then $F \in U$ iff $\exists\alpha\exists f(\alpha \in S' \wedge f = F_f(\alpha) \wedge f \subseteq F)$. □

If $2^{\kappa^{<\kappa}} > 2^\kappa$ then there are more open sets than definable sets, so some additional hypothesis is necessary. It is not apparent whether $\kappa^{<\kappa} = \kappa$ suffices.

8. Formulas and Downsets

As in [3], for a Δ_0^{0f} formula $\psi(\vec{x}, \vec{F})$ let $Q_\psi(u, \vec{x}, \vec{F})$ be the Δ_0^{0f} formula defined by the following recursion. For a term t let Q_t is $\vee_s s \in u$ where s ranges over subterms other than t. $Q_{t=u} = Q_{t\in u} = Q_t \wedge Q_u$, $Q_{\neg\psi} = Q_\psi$, $Q_{\psi_1\wedge\psi_2} = Q_{\psi_1\vee\psi_2} = Q_{\psi_1} \wedge Q_{\psi_2}$, and $Q_{\exists x\in t\psi} = Q_{\forall x\in t\psi} = Q_t \wedge \forall x \in tQ_\psi$. This holds iff the arguments of function applications are all in u.

In a Δ_0^{0f} formula $\phi(\vec{x}, \vec{F}, \vec{G})$ first order variables f_i of ordinal sequence sort may be substituted for the F_i, where f_i is restricted to be a function from an ordinal to Ord. The result of this substitution may be expressed as a Δ_1^{0f} formula, which will be denoted $\phi(\vec{x}, \vec{f}, \vec{G})$. Likewise, there is a Δ_1^{0f} formula $\phi(\vec{x}, F_1 \upharpoonright u, \ldots, F_k \upharpoonright u)$.

In [3] the following are shown to hold in V_κ for a cardinal κ, where ψ is a Δ_0^{0f} formula.

- $Q_\psi \Rightarrow (\psi(\vec{x}, \vec{F}) \Leftrightarrow \psi(\vec{x}, F_1 \upharpoonright u, \ldots, F_k \upharpoonright u))$.
- If κ is regular, $\exists u Q_\psi$.
- If $\vec{f'} \subseteq_c \vec{f}$ then $Q_\psi(\mathrm{Dom}(\vec{f'}), \vec{x}, \vec{f'}) \Rightarrow Q_\psi(\mathrm{Dom}(\vec{f}), \vec{x}, \vec{f})$.

Let "$\vec{x} \in_c u$" denote that $x_1 \in u \wedge \cdots \wedge x_t \in u$. For a Δ_0^{0f} formula $\psi(\vec{x}, \vec{F}, \vec{G})$, where the G_i are parameters let D_ψ denote

$$\{\vec{f} \in \mathcal{A}_g^{k,\mathrm{e}} : \forall \vec{x} \in \mathrm{Dom}(\vec{f})(Q_\psi(\mathrm{Dom}(f), x, \vec{f}, \vec{G}) \Rightarrow \neg\psi(\vec{x}, \vec{f}, \vec{G}))\}$$

(this is a variant of $D_{@G}$ of [3]).

- If κ is regular, D_ψ is a downset, and $\forall\vec{F}\exists\vec{x}\psi$ iff D_ψ is unbranched.

By remarks in Section 4, ϕ holds in V_κ iff $\{U_f : f \notin D_\phi\}$ is a cover of $\mathcal{N}_g$.

$Q_\psi(\eta, \vec{\alpha}, \vec{s}, \vec{F})$ may be defined for a Δ_0^{0f} formula of L^f_{OS}. It holds iff for all function and ordinal sequence arguments ξ, $\xi < \eta$. In the definition of Q_t, u ranges over subterms which are arguments of a function symbol. Variables $\vec{f}$ may be taken as ordinal sequence variables. The facts given above for $L^f_\in$ may readily be seen to hold in this setting.

9. Universal Sets

Universal sets are a common tool in descriptive set theory (see [20]). Universal sets will be given for $\Pi_1^{1f}(L_\in)$ sets and $\Pi_1^{1f}(L_{\mathrm{OS}})$ sets; they have been given for

Π_1^{1L} sets in $\mathcal{N}$, for $\kappa = \aleph_1$ under the hypothesis $\kappa^{<\kappa} = \kappa$, in [19]. Let $\ulcorner\phi\urcorner$ denote an integer or hereditarily finite set coding the formula ϕ (its Godel number).

Lemma 16. *In $L_\in$, there is a Δ_1^{1f} formula $Tru_{\Delta_0^{1f}}(n, \vec{F})$ such that if κ is a regular cardinal then $\models_{V_\kappa} \phi \Leftrightarrow Tru(\ulcorner\phi\urcorner, \vec{F})$, where ϕ is a Δ_0^{1f} formula with free second order variables among $\vec{F}$.*

Proof. Let $\mathrm{Tsf}(n, \vec{F}, S)$ denote the predicate "S is a set of pairs $\langle m, \langle x_1, \ldots, x_t\rangle\rangle$ such that m is a subformula of n with t free first order variables which is true when the free variables in alphabetic order are assigned the values $x_1, \ldots, x_t$". A variation of well-known arguments shows that $\mathrm{Tsf}(n, \vec{F}, S)$ is Δ_0^{1f}. Then $\mathrm{Tru}_{\Delta_0^{1f}}(n, \vec{F})$ iff $\exists S(\mathrm{Tsf}(n, \vec{F}, S) \wedge S(\langle m, \emptyset\rangle))$ iff $\forall S(\mathrm{Tsf}(n, \vec{F}, S) \Rightarrow S(\langle m, \emptyset\rangle))$. □

Corollary 17. *There is a Σ_1^{1f} formula $Tru_{\Sigma_1^{1f}}(n, \vec{F})$ (resp. Π_1^{1f} formula $Tru_{\Pi_1^{1f}}(n, \vec{F})$) such that if κ is a regular cardinal then $\models_{V_\kappa} \phi \Leftrightarrow Tru(\ulcorner\phi\urcorner, \vec{F})$, where ϕ is a Σ_1^{1f} (resp. Π_1^{1f}) formula.*

Proof. Straightforward. □

Corollary 18. *If κ is a regular cardinal there is no Σ_1^{1f} formula $\phi(n, \vec{F})$ such that $\models_{V_\kappa} \phi \Leftrightarrow Tru_{\Pi_1^{1f}}$.*

Proof. If there were, let $n_0 = \ulcorner\phi\urcorner$. Then $\phi(n_0, \vec{F}) \Leftrightarrow \mathrm{Tru}_{\Pi_1^{1f}}(n_0, \vec{F}) \Leftrightarrow \neg\phi(n_0, \vec{F})$. □

This result can be used in a standard manner to prove facts about other predicates. For $S \subseteq V_\kappa$ let χ_S (an element of $\mathcal{N}_g$) be the characteristic function of S. In formulas of $L_\in^f$, a set S may appear, with the understanding that χ_S is meant.

Theorem 19. *In V_κ for a regular cardinal κ, $DS(D)$ is Π_1^{0f}, and $UDS(D)$ is Π_1^{1f}.*

Proof. Let $\mathrm{Func}(f)$ be the Δ_0^0 predicate "f is a function". Then $\mathrm{DS}(D)$ iff $D(f) \Rightarrow (\mathrm{Func}(f) \wedge \forall g(g \subseteq f \Rightarrow D(g)))$. For the second claim, $\mathrm{UDS}(D)$ iff $\mathrm{DS}(D)$ and $\forall F \exists x \exists f(f = F \restriction x \wedge \neg D(f))$. □

Lemma 20. *In V_κ for a regular cardinal κ, if UDS(D) is Σ_1^{1f} then $Tru_{\Pi_1^{1f}}(n)$ is Σ_1^{1f}.*

Proof. Let $\mathrm{NFm}(n, n_1)$ denote that n_1 is the Godel number of $\psi(x, F)$ where $\forall F \exists x \psi(x, F)$ is the Π_1^1 normal form for the Π_1^1 sentence ϕ with Godel number n. Let $\mathrm{DSf}(n_1, n_2)$ denote that n_2 is the Godel number of the formula in the free first order variable f, $\forall x \in \mathrm{Dom}(f)(Q_\psi(\mathrm{Dom}(f), x, f) \Rightarrow \neg\psi(x, f))$ where Q_ψ is defined in Section 8. Both these predicates are Δ_1^0. If $\mathrm{UDS}(D)$ were Σ_1^{1f} then $\exists D \exists n_1 \exists n_2 (\mathrm{NFm}(n, n1) \wedge \mathrm{DSf}(n_1, n_2) \wedge \forall f(D(f) \Leftrightarrow \mathrm{Tru}_{\Delta_0^0}(n_2, f)) \wedge \mathrm{UDS}(D))$ would be a Σ_1^{1f} formula for $\mathrm{Tru}_{\Sigma_1^{1f}}(n)$ □

Theorem 21. *In V_κ for a regular cardinal κ, UDS(D) is not Σ_1^{1f}.*

Proof. By corollary 18 and lemma 20. □

In L^f_{OS}, lemma 16 holds. Indeed, using coding methods, the predicate $\mathrm{Tsf}(n, \vec{F}, S)$ may be defined as before, where now S is an element of $\mathcal{N}$. Corollaries 17 and 18 follow.

Theorem 19 cannot even be stated in L^f_{OS}, because a tree is a set of elements of $\mathcal{A}$, and the argument of an element of $\mathcal{N}$ is an ordinal. Let $\mathrm{Tr}_f(T)$ denote that $F_f[T]$ is a tree, and similarly for UTr_f. Of course, it is independent of ZFC whether all trees have codes via Tr_f,

Theorem 22. *In OS_κ for a regular cardinal κ, $Tr_f(T)$ is Δ_1^{1f}, and $UTr_f(T)$ is Π_1^{1f}.*

Proof. $\mathrm{Tr}_f(T)$ iff $\forall \alpha, \beta, f, g(f = F_f(\alpha) \wedge \alpha \in T \wedge g = F_f(\beta) \wedge g \subseteq f \Rightarrow T(\beta)))$. UTr is left to the reader. □

A normal form theorem for Π_1^{1f} formulas of L^f_{OS} is problematical since Skolem functions can have elements of $\mathcal{A}$ as an argument or value. Thus, a version of lemma 20 for $L^1_{\mathrm{OS}}f$ is problematical also.

In [19] some facts are proved (for $\kappa = \omega_1$) under the hypothesis $\kappa^{<\kappa} = \kappa$. For convenience an outline will be given. Let $E : \kappa \mapsto \mathcal{A}^{k,\mathrm{e}}$ be a bijection (where k is determined by context). For $S \subseteq \kappa$ let $U_S = \cup\{U_f : f \in E[S]\}$. Let J_0 denote the Godel pairing function.

- Let $\mathrm{Un}(H, \vec{F})$ iff $\vec{F} \in U_H$; then Un is a universal open set.
- Let $\mathrm{Un}_{\Pi_1^{1L}}(H, \vec{F})$ iff $\forall G \mathrm{Un}(H, \vec{F}, G)$; then $\mathrm{Un}_{\Pi_1^{1L}}$ is a universal Π_1^{1L} set.
- Let $\mathrm{Tr}_\chi(T)$ hold iff $\{\langle\alpha, \beta\rangle : J_0(\alpha, \beta) \in T\}$ is a tree. Then Tr_χ is closed.

- Let $\mathrm{UTr}_\chi(T)$ iff $\forall F\exists\alpha\exists\beta<\alpha(F(\beta)\not<_T F(\alpha))$. Then UTr_χ is Π^{1L}_1.
- $\langle H,\vec{F}\rangle\in\Pi^{1L}_1$ iff $T\in\mathrm{UTr}_\chi$ where, letting $\langle\vec{f_i},g_i\rangle=E(\alpha_i)$ for $i=1,2$, $J_0(\alpha_1,\alpha_2)\in T$ iff $\vec{f_1}\subseteq\vec{F}\wedge\vec{f_2}\subseteq\vec{F}\wedge\vec{f_1}\subseteq\vec{f_2}\wedge\alpha_1\notin H\wedge\alpha_2\notin H$.
- It follows that $\mathrm{UTr}_\chi(T)$ is not Σ^{1L}_1.

10. Weak Compactness

For an inaccessible cardinal κ let $I_{\leq\kappa}$ (resp. $I_{<\kappa}$) denote the set of inaccessible cardinals λ such that $\lambda\leq\kappa$ (resp. $\lambda<\kappa$).

Suppose $\phi(\vec{X})$ is a Π^{1f}_1 formula with set second order (and no first order) free variables; by results of [3] ϕ may be assumed to be in Π^1_1 normal form $\forall\vec{F}\exists\vec{x}\psi(\vec{x},\vec{F},\vec{X})$ where ψ is Δ^{0f}_0. Replacing the free variables by subsets of V_κ, define Tsupp(ϕ) to be $\{\lambda\in I_{\leq\kappa}:\models_{V_\lambda}\phi(X_1\cap V_\lambda,\ldots,X_k\cap V_\lambda)\}$. If D is a downset over κ define Tsupp(D) to be $\{\lambda\in I_{\leq\kappa}:D\cap V_\lambda$ is unbranched$\}$.

Given a sentence ϕ let D_ϕ denote the downset D_ψ defined in Section 8. Given a downset D let ϕ_D denote the formula $\forall F\exists x(F\restriction x\notin D)$.

Theorem 23. *Suppose κ is an inaccessible cardinal.*
a. *If ϕ is a Π^1_1 normal form sentence with set second order parameters then Tsupp(D_ϕ) = Tsupp(ϕ).*
b. *If $D\subseteq\mathcal{A}_g$ is a downset then Tsupp(ϕ_D) = Tsupp(D).*

Proof. Strengthening a remark in Section 8, $\models_{V_\lambda}\phi(X_1\cap V_\lambda,\ldots,X_k\cap V_\lambda)\}$ iff $D_\phi\cap V_\lambda$ is unbranched; this proves part a. For part b, clearly $D\cap V_\lambda$ is unbranched iff $\models_{V_\lambda}\phi_D(D\cap V_\lambda)$. ☐

For an inaccessible cardinal κ, say that a bijection $E:\kappa\mapsto V_\kappa$ is ranked if, whenever $\alpha\leq\beta$, $\rho(E(\alpha))\leq\rho(E(\beta))$, where ρ is the set rank function.

Theorem 24. *If κ is an inaccessible cardinal there is a ranked bijection $E:\kappa\mapsto V_\kappa$.*

Proof. This follows by a straightforward transfinite recursion using the axiom of choice. ☐

Suppose E is a ranked bijection. It is readily seen that for $\lambda\in I_{\leq\kappa}$ $E\restriction\lambda$ is a bijection to V_λ. For a downset $D\subseteq\mathcal{A}_g$ let $T_D=\{f\in\mathcal{A}:\hat{E}(f)\in D\}$; it is readily seen that T_D is a tree. For a tree $T\subseteq\mathcal{A}$ let D_T be the downward closure of $\hat{E}[T]$; clearly D_T is a downset.

Theorem 25. *Suppose κ is an inaccessible cardinal.*
a. If $D \subseteq \mathcal{A}_g$ is a downset then Tsupp(T_D) = Tsupp(D).
b. If $T \subseteq \mathcal{A}$ is a tree then Tsupp(D_T) = Tsupp(T).

Proof. Suppose B is a branch in $D \cap V_\lambda$. If $g \subseteq \hat{E}^{-1}(B)$ then $\hat{E}(g) \subseteq B$, so $\hat{E}(g) \in D$, so $g \in T_D$; thus, $\hat{E}^{-1}(B)$ is a branch of $T_D \cap V_\lambda$. Suppose B is a branch in $T_D \cap V_\lambda$. If $g \subseteq \hat{E}(B)$ then $\exists h \subseteq B(g \subseteq \hat{E}(h))$, so $h \in T_D$, so $\hat{E}(h) \in D$, so $g \in D$; thus, $\hat{E}(B)$ is a branch of $D \cap V_\lambda$. This proves part a.

Suppose B is a branch in $T \cap V_\lambda$. If $g \subseteq \hat{E}(B)$ then $\exists h \subseteq B(g \subseteq \hat{E}(h))$, so $h \in T$, so $\hat{E}(h) \in D_T$, so $g \in D_T$; thus, $\hat{E}(B)$ is a branch of $D_T \cap V_\lambda$. Suppose B is a branch in $D_T \cap V_\lambda$. If $g \subseteq \hat{E}^{-1}(B)$ then $\hat{E}(g) \subseteq B$, so $\hat{E}(g) \in D_T$, so $g \in T$; thus, $\hat{E}^{-1}(B)$ is a branch of $T \cap V_\lambda$. This proves part b. □

By theorems 23 and 25 the notion of a T-support may be defined, as Tsupp(ϕ) for some ϕ, Tsupp(D) for some downset D, or Tsupp(T) for some tree T.

Theorem 26. *Suppose κ is an inaccessible cardinal; the following are equivalent.*
a. κ is weakly compact.
b. If $D \subseteq \mathcal{A}_g$ is an unbranched downset then for some $\lambda \in I_{<\kappa}$ $D \cap V_\lambda$ is unbranched.
c. If $T \subseteq \mathcal{A}$ is an unbranched tree then for some $\lambda \in I_{<\kappa}$ $T \cap V_\lambda$ is unbranched.

Proof. First, by well known facts and results of [3], κ is weakly compact iff for each Π^1_1 normal form sentence $\phi(\vec{X})$ with set second order parameters, if $\models_{V_\kappa} \phi(\vec{X})$ then for some inaccessible cardinal $\lambda < \kappa$ $\models_{V_\lambda} \phi(X_1 \cap V_\lambda, \ldots, X_k \cap V_\lambda)$. Then any of the three statements of the theorem is equivalent to the statement that if a T-support contains κ then it contains some $\lambda \in I_{<\kappa}$. □

11. Σ^1_1 WPS's

The notion of a WPS (well preorder on a subset) is defined in [2]. If C is a class of WPS's let $\mathrm{Ot}(C) = \sup(\mathrm{Ot}(\preceq)) : \preceq \in C\}$. The classes $\Sigma^1_1(L_\epsilon)$, $\Sigma^{1L}_1(\mathcal{N}_g)$, $\Sigma^{1L}_1(\mathcal{N})$, $\Sigma_1(H_{\kappa^+})$, $\Sigma_1(L_{\kappa^+})$,and $\mathcal{U}_{\mathrm{OS}}$ (defined in [2]), and $\mathcal{U}_{\Sigma^1_1}$ (defined in [2]), will be considered. (The definition of $\mathcal{U}_{\mathrm{OS}}$ in [2] has some typographical errors. First, it is written as $\mathcal{U}_{sOS}$. Second the formula ϕ should be required to be Σ^I_1, and ψ Π^I_1.)

Theorem 27. $Ot(\mathcal{U}_{\rm OS}) \leq Ot(\mathcal{U}_{\Sigma^1_1}) \leq Ot(\Sigma^1_1(L_\epsilon))$.

Proof. The first inequality follows by corollary 25 of [2], and the second follows immediately from the definitions. □

Lemma 28. $Ot(\Sigma_1(L_{\kappa^+})) \leq Ot(\mathcal{U}_{\rm OS})$.

Proof. Suppose $\preceq_1$ is a Σ_1 formula with parameters from L_{κ^+}, defining a WPS on L_{κ^+}, of order type α. If $\alpha < \kappa^+$ let P be a well-order on κ of order type α. Using notation of Section 12 of [2], let ϕ be the Σ^I_1 formula "$I_<(A,P) \wedge I_<(B,P) \wedge I_\leq(A,B)$", and let ψ be the Π^I_1 form; these satisfy requirements 1 and 2 for a $\mathcal{U}_{\rm OS}$ formula for α, so by theorem 31 of [2] they define a $\mathcal{U}_{\rm OS}$ order of order type α.

Suppose $\alpha \geq \kappa^+$. Let F be a Σ_1 bijection from κ^+ to L_{κ^+}. Using F, $\preceq_1$ may be transformed to a Σ_1 formula $\preceq_2$ with parameters from κ^+, which defines a WPS on κ^+ of order type α. In particular $\preceq_2$ is a well-founded relation. By results of Section 11 of [1], there is a Σ_1 formula $\preceq_3$ with domain κ^+, with parameters from κ^+, which defines a well order on κ^+ of order type β where $\beta \geq \alpha$. This has a Π_1 form. The two forms may be interpreted in L_2 to yield a $\mathcal{U}_{\rm OS}$ formula pair for β. □

Theorem 29. *Suppose κ is inaccessible. If $V = L$ then* $Ot(\Sigma^1_1(L_\epsilon)) \leq Ot(\mathcal{U}_{\rm OS})$.

Proof. By remarks follows theorem 9, and theorem 10,

$$\mathrm{Ot}(\Sigma^1_1(L_\epsilon)) = \mathrm{Ot}(\Sigma^{1L}_1(\mathcal{N}_g)).$$

Since $\mathcal{N}_g$ and $\mathcal{N}$ are homeomorphic,

$$\mathrm{Ot}(\Sigma^{1L}_1(\mathcal{N}_g)) = \mathrm{Ot}(\Sigma^{1L}_1(\mathcal{N})).$$

By lemma 2.4 of [16] and remarks following,

$$\mathrm{Ot}(\Sigma^{1L}_1(\mathcal{N})) = \mathrm{Ot}(\Sigma_1(H_{\kappa^+})).$$

By corollary 5.2.7 of [4], if $V = L$ then

$$\mathrm{Ot}(\Sigma_1(H_{\kappa^+})) = \mathrm{Ot}(\Sigma_1(L_{\kappa^+})).$$

The theorem follows by lemma 28. □

Theorem 30. *Suppose κ is inaccessible. It is consistent that*

$$Ot(\Sigma^1_1(L_\epsilon)) > Ot(\mathcal{U}_{\rm OS}).$$

Proof. As noted in [2], $\mathrm{Ot}(\mathcal{U}_{\rm OS}) \leq \kappa^{++}$, so it suffice to show that $\mathrm{Ot}(\Sigma^1_1(L_\epsilon)) > \kappa^{++}$ is consistent. This follows by theorem 1.1 of [11] (inaccessibility of κ is preserved since the forcing is $< \kappa$-closed). □

References

[1] M. Dowd, Iterating Mahlo's operation, *Int. J. Pure Appl. Math.*, bf 9, no. 4 (2003), 469-512. http://www.hyperonsoft.com/imol.pdf

[2] M. Dowd, Function Chains from Uniform Sigma-1-1 Well Orders, *Int. J. Pure Appl. Math.*, **93** no. 5 (2014), 661-684, **doi:** 10.12732/ijpam.v93i5.7.

[3] M. Dowd, Pi-1-1 Normal Form in a Regular Cardinal, *Int. J. Pure Appl. Math.*, To Appear.

[4] F. Drake, *Set Theory. An Introduction to Large Cardinals*, North Holland (1974).

[5] S. Friedman, *Descriptive Set Theory on Generalised Baire Space*, slides, 2913.

[6] S. Friedman and T. Hyttinen, On Borel equivalence relations in generalized Baire space *Archive for Mathematical Logic* **51** (2012), 299-304, **doi:** 10.1007/s00153-011-0266-3

[7] S. Friedman, T. Hyttinen, V. Kulikov, Generalized Descriptive Set Theory and Classification Theory, *Memoirs of the American Mathematical Society*, **230**, Number 1081 (2014), **doi:** 10.1090/memo/1081.

[8] S. Friedman, Y. Khomskii, V. Kulikov, Regularity Properties of the Generalized Reals, *Preprint*, 2014.

[9] A. Halko, *Negligible Subsets of the Generalized Baire Space*, Ph.D. thesis, University of Helsinki, 1996.

[10] A. Halko and S. Shelah, On strong measure zero subsets of ${}^{\kappa}2$, *Fundamenta Mathematicae* **170** (2001), 219-229, **doi:** 10.4064/fm170-3-1.

[11] P. Holy and P. Lucke, Locally Σ_1-Definable Well-Orders of $\mathrm{H}(\kappa^+)$, *Fund. Math.* **226** (2014), 221-236, **doi:** :10.4064/fm226-3-2.

[12] T. Hyttinen and V. Kulikov, Borel* Sets in the Generalized Baire Space, *ArXiv:* org/pdf/1209.3933.

[13] T. Hyttinen and V. Kulikov, On Σ_1^1-complete Equivalence Relations on the Generalized Baire Space, *ArXiv*: org/abs/1209.3932.

[14] A. Kanamori, *The Higher Infinite*, Springer-Verlag, Germany (2003).

[15] A. Kechris, *Classical Descriptive Set Theory*, Springer-Verlag, New York, 1995.

[16] P. Lucke, Σ^1_1-definability at uncountable regular cardinals, *J. Symbolic Logic* **77** no. 3 (2012), 1011-1046, **doi:** 10.2178/jsl/1344862172.

[17] P. Lucke, The influence of closed maximality principles on generalized Baire spaces, Slides, 2012.

[18] P. Lucke and P. Schlicht, Continuous Images of Closed Sets in Generalized Baire Spaces, *Israel J. Mathematics*, To Appear.

[19] A. Mekler and J. Vaananen, Trees and Π^1_1 subsets of ${}^{\omega_1}\omega_1$, *Journal of Symbolic Logic* **58** (1993), 1052-1070, **doi:** 10.2307/2154287.

[20] Y. Moschavakis, *Descriptive Set Theory*, North-Holland (1980).

[21] P. Schlicht, Perfect subsets of generalized Baire spaces and Banach-Mazur games, Slides, 2012.

[22] G. Takeuti and W. Zaring, *Introduction to Axiomatic Set Theory*, Springer-Verlag (1971).

[23] J. Vaananen, A Cantor Bendixson theorem for the space ${}^{\omega_1}\omega_1$, *Fund. Math.* **137** (1991) 41-53.