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AN IMPROVED SUB-EQUATION METHOD FOR SOLVING NONLINEAR FRACTIONAL EQUATIONS

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Abstract: In this paper we present a generalization of the fractional sub-equation method, which can be used to solve nonlinear fractional partial differential equations. The generalization is based on the use of a general fractional Riccati equation, the improved tanh-coth method and the Jumarie's modified Riemann-Lioville derivative of order α. As illustration, the nonlinear fractional Sharma-Tasso-Olver equation and the fractional Burger's equation are solved by using the method.

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1. Introduction

Several fractional differential partial equations have been used for modeling nonlinear phenomenon in physics, biology, fluid flow, control theory as well as in some social sciences such as food supplement, climate, finance and economy (see [1] and references therein). One of the methods implemented to solve this type of equations is known as the fractional sub-equation method which is based in the use of the solutions of the standard fractional Riccati equation

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$$D_\xi^\alpha \phi(\xi) = r + \phi(\xi)^2, \tag{1}$$

combined with the tanh method or the extended tanh method (see for instance [2] and [3]). The previous method have been used by several authors in a satisfactory way for obtain exact solutions of several nonlinear fractional partial differential equations (see [4], [5], [2], [6], [7]). The main objective of this paper consists in the use of solutions of the following more general fractional Riccati equation

$$D_\xi^\alpha \phi(\xi) = r + q\phi(\xi) + p\phi(\xi)^2, \tag{2}$$

where r, q, p are constants, combined with a generalization of the extended tanh method [8], for obtain exact solutions to the nonlinear fractional Sharma-Tasso-Olver equation [5], [2]

$$\frac{\partial^\alpha u}{\partial t^\alpha} + 3\rho u^2 \frac{\partial^\alpha u}{\partial x^\alpha} + 3\rho([\frac{\partial^\alpha u}{\partial x^\alpha}]^2 + u\frac{\partial^{2\alpha} u}{\partial x^{2\alpha}}) + \rho \frac{\partial^{3\alpha} u}{\partial x^{3\alpha}} = 0, \tag{3}$$

and to the fractional Burger's equation [7]

$$\frac{\partial^\alpha u}{\partial t^\alpha} + \rho u \frac{\partial^\alpha u}{\partial x^\alpha} - \delta \frac{\partial^{2\alpha} u}{\partial x^{2\alpha}} = 0, \quad t > 0, \quad 0 < \alpha \leq 1. \tag{4}$$

In the two previous models, ρ and δ are constants.

2. On the Solution of the General Fractional Riccati Equation

The Jumarie's modified Riemann-Lioville derivative is defined as

$$D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\xi)^{-\alpha} [f(\xi) - f(0)] d\xi, \quad 0 < \alpha < 1. \tag{5}$$

Some relations obtained are the following:

$$\begin{cases} D_t^\alpha t^\gamma = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)} t^{\gamma-\alpha}, \\ D_t^\alpha (u(t)v(t)) = [D_t^\alpha u(t)]v(t) + u(t)[D_t^\alpha v(t)], \\ D_t^\alpha (f(u(t))) = f'_u(u(t)) D_t^\alpha u(t) = D_t^\alpha f(u(t))(u'(t))^\alpha. \end{cases} \tag{6}$$

For more details about the previous formulas and other important relations see [9] and [10]. Now, taking into account the previous formulas, can be verified that the expression

$$\begin{cases} \phi(\xi) = -\frac{\sqrt{q^2-4rp}\tanh\left[\frac{1}{2}\sqrt{q^2-4rp}\,\frac{\xi^\alpha}{\Gamma(1+\alpha)} + \xi_0\right] - q}{2p}, \quad (q^2 - 4rp) \neq 0, \\ \frac{1}{p}(-\frac{\Gamma(1+\alpha)}{\xi^\alpha + \xi_0} - \frac{q}{2}), \quad (q^2 - 4rp) = 0, \end{cases} \tag{7}$$

being ξ_0 arbitrary constant is the solution of the general fractional Riccati equation (2). It is clear that the solutions of equation (1) presented in [2] are particular cases of this more general solution.

3. The Nonlinear Fractional Sharma-Tasso-Olver Equation

With the aim to solve the fractional Sharma-Tasso-Olver equation (3), first we use the transformation

$$u(x,t) = v(\xi), \quad \xi = k\,x + \lambda\,t. \tag{8}$$

Then the equation (3) reduces to ordinary differential equation in the variable ξ

$$\lambda^\alpha D_\xi^\alpha v(\xi) + 3k^\alpha \rho v(\xi)^2 D_\xi^\alpha v(\xi) + 3k^{2\alpha}\rho (D_\xi^\alpha v(\xi))^2 + 3k^{2\alpha}\rho v(\xi) D_\xi^{2\alpha} v(\xi) + k^{3\alpha}\rho D_\xi^{3\alpha} v(\xi) = 0. \tag{9}$$

Now, we use the generalized tanh-coth method (see [8]) which consists of assume the following expansion

$$v(\xi) = \sum_{i=0}^{M} a_i \phi(\xi)^i + \sum_{i=M+1}^{2M} a_i \phi(\xi)^{M-i}, \tag{10}$$

as solution of (Eq. 9). Here, M is a positive integer that will be determined later as well as the coefficients a_i. The function $\phi = \phi(\xi)$ satisfies the generalized fractional Riccati equation (Eq. 2).

Substituting (10) into (9) and balancing the linear terms of highest order $D^{3\alpha}v(\xi)$ with the highest order nonlinear term $v(\xi)^2 D_\xi^\alpha v(\xi)$, we have

$$M + 3 = 2M + (M+1),$$

so that

$$M = 1,$$

then, (10) reduces to

$$v(x,t) = a_0 + a_1\phi(\xi) + a_2\phi(\xi)^{-1}. \tag{11}$$

Substituting (11) into (9) and taking into account (2), (5) and (6) we obtain a polynomial in the variable $\phi(\xi)$. Equating to zero the coefficients of this polynomial we have the following algebraic system (in the unknowns a_i $i = 1, 2$, k, λ, r, q and p):

- $12a_1qp^2\rho k^{3\alpha} + 15a_1^2qp\rho k^{2\alpha} + 6a_0a_1p^2\rho k^{2\alpha} + 3a_1^3q\rho k^\alpha + 6a_0a_1^2p\rho k^\alpha = 0,$
- $6a_1p^3\rho k^{3\alpha} + 9a_1^2p^2\rho k^{2\alpha} + 3a_1^3p\rho k^\alpha = 0,$
- $8ra_1qp\rho k^{3\alpha}+9ra_1^2q\rho k^{2\alpha}+6ra_0a_1p\rho k^{2\alpha}+a_1q^3\rho k^{3\alpha}+3a_0a_1q^2\rho k^{2\alpha}+6ra_0a_1^2\rho k^\alpha +3a_0^2a_1q\rho k^\alpha + 3a_1^2a_2q\rho k^\alpha + a_1q\lambda^\alpha = 0,$
- $8ra_1p^2\rho k^{3\alpha}+12ra_1^2p\rho k^{2\alpha}+7a_1q^2p\rho k^{3\alpha}+6a_1^2q^2\rho k^{2\alpha}+9a_0a_1qp\rho k^{2\alpha}+3ra_1^3\rho k^\alpha +6a_0a_1^2q\rho k^\alpha + 3a_0^2a_1p\rho k^\alpha + 3a_1^2a_2p\rho k^\alpha + a_1p\lambda^\alpha = 0,$
- $-8ra_2qp\rho k^{3\alpha} + 6ra_0a_2p\rho k^{2\alpha} - a_2q^3\rho k^{3\alpha} + 3a_0a_2q^2\rho k^{2\alpha} + 9a_2^2qp\rho k^{2\alpha} - 3a_1a_2^2q\rho k^\alpha - 3a_0^2a_2q\rho k^\alpha - 6a_0a_2^2p\rho k^\alpha - a_2q\lambda^\alpha = 0,$
- $2r^2a_1p\rho k^{3\alpha}+3r^2a_1^2\rho k^{2\alpha}+ra_1q^2\rho k^{3\alpha}+3ra_0a_1q\rho k^{2\alpha}-2ra_2p^2\rho k^{3\alpha}-a_2q^2p\rho k^{3\alpha} +3a_0a_2qp\rho k^{2\alpha}+3a_2^2p^2\rho k^{2\alpha}+3ra_0^2a_1\rho k^\alpha+3ra_1^2a_2\rho k^\alpha-3a_1a_2^2p\rho k^\alpha-3a_0^2a_2p\rho k^\alpha +ra_1\lambda^\alpha - a_2p\lambda^\alpha = 0,$
- $-6r^3a_2\rho k^{3\alpha} + 9r^2a_2^2\rho k^{2\alpha} - 3ra_2^3\rho k^\alpha = 0,$
- $-12r^2a_2q\rho k^{3\alpha} + 6r^2a_0a_2\rho k^{2\alpha} + 15ra_2^2q\rho k^{2\alpha} - 6ra_0a_2^2\rho k^\alpha - 3a_2^3q\rho k^\alpha = 0,$
- $-8r^2a_2p\rho k^{3\alpha} - 7ra_2q^2\rho k^{3\alpha} + 9ra_0a_2q\rho k^{2\alpha} + 12ra_2^2p\rho k^{2\alpha} + 6a_2^2q^2\rho k^{2\alpha} - 3ra_1a_2^2\rho k^\alpha - 3ra_0^2a_2\rho k^\alpha - 6a_0a_2^2q\rho k^\alpha - 3a_2^3p\rho k^\alpha - ra_2\lambda^\alpha = 0.$

Solving the previous system with aid of *Mathematica* we obtain a lot of solutions. However, for sake of simplicity, we consider only the following nine from which we can obtain the most general expressions for the solution of (3):

- first set:

$$a_0 = \frac{k^{-\alpha}\left(3q\rho k^{2\alpha} - \sqrt{3}\sqrt{4rp\rho^2k^{4\alpha} - q^2\rho^2k^{4\alpha} - 4\rho k^\alpha\lambda^\alpha}\right)}{6\rho},$$

$$a_1 = 0, \;\; a_2 = rk^\alpha.$$

- second set:

$$a_0 = \frac{-3q\rho k^\alpha - \sqrt{3}k^{-\alpha}\sqrt{4rp\rho^2k^{4\alpha} - q^2\rho^2k^{4\alpha} - 4\rho k^\alpha\lambda^\alpha}}{6\rho},$$

$$a_1 = p\left(-k^\alpha\right), \;\; a_2 = 0.$$

- third set:

$$p=\frac{2\left(k^{-3\alpha}\left(-\sqrt{-155r^2q^4\rho^2k^{6\alpha}-108r^2q^2\rho k^{3\alpha}\lambda^\alpha}\right)-14rq^2\rho\right)}{27r^2\rho},$$

$$a_0=\frac{1}{18}\left(-\frac{k^{-2\alpha}\sqrt{r^2q^2\rho\left(-k^{3\alpha}\right)\left(155q^2\rho k^{3\alpha}+108\lambda^\alpha\right)}}{rq\rho}-5qk^\alpha\right),$$

$$a_1=\frac{2}{27}\left(\frac{k^{-2\alpha}\sqrt{r^2q^2\rho\left(-k^{3\alpha}\right)\left(155q^2\rho k^{3\alpha}+108\lambda^\alpha\right)}}{r^2\rho}+\frac{14q^2k^\alpha}{r}\right),\quad a_2=0.$$

- fourth set:

$$r=\frac{2\left(k^{-3\alpha}\sqrt{-155q^4p^2\rho^2k^{6\alpha}-108q^2p^2\rho k^{3\alpha}\lambda^\alpha}-14q^2p\rho\right)}{27p^2\rho},$$

$$a_0=\frac{1}{18}\left(5qk^\alpha-\frac{k^{-2\alpha}\sqrt{q^2p^2\rho\left(-k^{3\alpha}\right)\left(155q^2\rho k^{3\alpha}+108\lambda^\alpha\right)}}{qp\rho}\right),\quad a_1=0,$$

$$a_2=\frac{2}{27}\left(\frac{k^{-2\alpha}\sqrt{q^2p^2\rho\left(-k^{3\alpha}\right)\left(155q^2\rho k^{3\alpha}+108\lambda^\alpha\right)}}{p^2\rho}-\frac{14q^2k^\alpha}{p}\right).$$

- fifth set:

$$q=0,\;\; a_0=-\frac{k^{-\alpha/2}\sqrt{4rp\rho k^{3\alpha}-\lambda^\alpha}}{\sqrt{3}\sqrt{\rho}},\;\; a_1=p\left(-k^\alpha\right),\;\; a_2=rk^\alpha.$$

- sixth set:

$$r=\frac{q^2\rho+k^{-3\alpha}\lambda^\alpha}{4p\rho},\;\; a_0=0,\;\; a_1=p\left(-k^\alpha\right),\;\; a_2=\frac{k^{-2\alpha}\left(q^2\rho k^{3\alpha}+\lambda^\alpha\right)}{4p\rho}.$$

- seventh set:

$$q=\frac{ik^{-3\alpha/2}\lambda^{\alpha/2}}{\sqrt{21}\sqrt{\rho}},\;\; p=\frac{2k^{-3\alpha}\lambda^\alpha}{21r\rho},\;\; a_0=\frac{ik^{-\alpha/2}\lambda^{\alpha/2}}{\sqrt{21}\sqrt{\rho}},$$

$$a_1=-\frac{2k^{-2\alpha}\lambda^\alpha}{21r\rho},\;\; a_2=2rk^\alpha.$$

- eighth set:

$$r=\frac{2k^{-3\alpha}\lambda^{\alpha}}{21p\rho},\;\; q=\frac{ik^{-3\alpha/2}\lambda^{\alpha/2}}{\sqrt{21}\sqrt{\rho}},\;\; a_0=-\frac{ik^{-\alpha/2}\lambda^{\alpha/2}}{\sqrt{21}\sqrt{\rho}},$$

$$a_1=-2pk^{\alpha},\;\; a_2=\frac{2k^{-2\alpha}\lambda^{\alpha}}{21p\rho}.$$

- ninth set:

$$r=\frac{k^{-3\alpha}\lambda^{\alpha}}{16p\rho},\;\; q=0,\;\; a_0=0,\;\; a_1=-2pk^{\alpha},\;\; a_2=\frac{k^{-2\alpha}\lambda^{\alpha}}{8p\rho}.$$

With respect to seventh set and in accordance with (11), (7) and (8) we have the following solution to (3)

$$u(x,t)=\frac{ik^{-\alpha/2}\lambda^{\alpha/2}}{\sqrt{21}\sqrt{\rho}}$$
$$-\frac{k^{\alpha}}{2}\Big(\sqrt{-\frac{3\lambda^{\alpha}}{7\rho k^{3\alpha}}}\tanh\Big[\frac{1}{2}\sqrt{-\frac{3\lambda^{\alpha}}{7\rho k^{3\alpha}}}\frac{(kx+\lambda t)^{\alpha}}{\Gamma(1+\alpha)}+\xi_0\Big]-\frac{ik^{-3\alpha/2}\lambda^{\alpha/2}}{\sqrt{21}\sqrt{\rho}}\Big)+$$
$$\frac{8\lambda^{\alpha}}{21\rho k^{2\alpha}}\Big(\sqrt{-\frac{3\lambda^{\alpha}}{7\rho k^{3\alpha}}}\tanh\Big[\frac{1}{2}\sqrt{-\frac{3\lambda^{\alpha}}{7\rho k^{3\alpha}}}\frac{(kx+\lambda t)^{\alpha}}{\Gamma(1+\alpha)}+\xi_0\Big]-\frac{ik^{-3\alpha/2}\lambda^{\alpha/2}}{\sqrt{21}\sqrt{\rho}}\Big)^{-1}. \quad (12)$$

Here, λ, k, and ξ_0 are arbitrary parameters. The other solutions can be derived in a similar way. As before, can be verified that the solutions obtained in [2] are particular cases of those obtained here.

4. Exact Solution to the Fractional Burgers Equation

Using (8) we have that (4) converts to

$$\lambda^{\alpha}D_{\xi}^{\alpha}v(\xi)+\rho k^{\alpha}v(\xi)D_{\xi}^{\alpha}v(\xi)-\delta k^{2\alpha}D_{\xi}^{2\alpha}v(\xi)=0. \quad (13)$$

Integrating (13) once with respect to $(d\xi)^{\alpha}$ we have

$$\alpha!(\lambda^{\alpha}v(\xi)+\frac{\rho}{2}v^2(\xi)-\delta k^{2\alpha}D_{\xi}^{\alpha}v(\xi))=k_1, \quad (14)$$

being k_1 an arbitrary constant. Without lost of generality we can write (14) as

$$\lambda^{\alpha}v(\xi)+\frac{\rho}{2}v^2(\xi)-\delta k^{2\alpha}D_{\xi}^{\alpha}v(\xi)=c, \quad (15)$$

with c any constant. The Eq.(15) have the form of the generalized fractional Riccati equation (2) where $r = -\frac{c}{\delta k^{2\alpha}}$, $q = \frac{\lambda^\alpha}{\delta k^{2\alpha}}$ and $p = \frac{\rho}{2\delta k^{2\alpha}}$ and therefore, the solution to fractional Burgers equation (4) is given by

$$u(x,t) = -\frac{\delta k^{2\alpha}}{\rho}\left(\sqrt{\frac{\lambda^{2\alpha}+2c\rho}{\delta^2 k^{4\alpha}}}\tanh\left[\frac{1}{2}\sqrt{\frac{\lambda^{2\alpha}+2c\rho}{\delta^2 k^{4\alpha}}}\frac{(kx+\lambda t)^\alpha}{\Gamma(1+\alpha)}+\xi_0\right]-\frac{\lambda^\alpha}{\delta k^{2\alpha}}\right). \quad (16)$$

Here, c, ξ_0, λ and k are arbitrary constants.

5. Conclusions

We have presented a generalized fractional Riccati equation and the respective solution which can be used with the improved tanh-coth method to obtain a generalization of the fractional-sub-equation method used for solve nonlinear fractional partial differential equations. The effectiveness of the method is showed with the fractional Sahrma-Taso-Olver equation and the fractional Burgers equation.

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