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RECONSTRUCTION OF AN ELLIPSE FROM ITS RASTER IMAGE

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Abstract: Motivated by practical application, we investigate the ways how to obtain a description of an ellipse whose circumscribed rectangle is known exactly and whose coordinates of touching points are known up to a certain inaccuracy. We show the sufficient condition for determination of such an ellipse, derive the implicit and parametric equations and show a proposal of the solution by the modification of the standard regression method.

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1. Motivation – The Picture Analysis

Suppose there is a circular object taken by a digital camera from a non-frontal view. On a picture, the circle is skewed to form an ellipse-like shape. Its boundary is a curve determined by a number of parameters of the position of the camera with respect to the object, the angle from which the object is taken, the optical and digital parameters of the camera etc. One often needs to analyze the picture automatically to be able to locate the object. Therefore we need to know the exact mathematical description of the curve. If we admit no more distortion than skewing and perspective, this curve is an ellipse.

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Fitting of an ellipse through the set of points is one of the classical problems of applied mathematics. In general, there are two approaches. The algebraic one involves the system of five linear equations derived from the knowledge of the coordinates of five points - see also [8, 4, 2]. The geometric approach, uses the least-square method which can be solved numerically. Actual problems solutions mostly prefer applications of geometric approach since it is stable and provides satisfactory results. The regression method is described e.g. in [5].

Using graphical tools one may usually find quite accurately the bounding values in direction of x and y axes. Hence the problem can be restated as finding an ellipse inscribed into the rectangle, which is also studied in [6]. Moreover, we may use possible knowledge of the position of some base point on the ellipse, namely the touching point. In the paper, we derive the determination of the ellipse by these data.

In general, however, we may find only an interval in which the value of the second coordinate of the touching point occurs. Therefore it makes sense to ask for procedures how to obtain the formulas for the ellipse with such a way of description. If we take these properties in account, one can see that an advantageous choice is to take the midpoint of the interval. This procedure provides results of rather good accuracy which is demonstrated on the real-data example, where we compare the procedure with the regression method.

Finally we study other possible choices of the base points. In Appendix, we recall some relationships between the equations for ellipse and their marginal points.

2. An Ellipse Inscribed in the Rectangle

Let there be an ellipse $\mathcal{E}_0$ where the values x_{max}, x_{min}, y_{max}, y_{min} of maximum and minimum in x and y direction are determined, i.e. the ellipse is inscribed into the rectangle of known lengths. There are still more ellipses with this property. To determine the ellipse entirely, we need to know one of the following:

- the second coordinate of some of the touching points,
- the intersection of the ellipse with the rectangle's diagonal.

For other point of the ellipse there are general two possible ellipses satisfying the conditions.

Suppose y_r is the y-coordinate of the point $X_{max} = [x_{max}, y_r]$ on ellipse (the rightmost, i.e. the x-maximal point). We will show that these data determine the ellipse entirely.

Firstly, we find the center $S = [z_1, z_2]$ of the ellipse. It can be easily obtained as a center of the circumscribed rectangle because the ellipse is centrally symmetric. Hence

$$z_1 = \frac{x_{max} + x_{min}}{2}, \; z_2 = \frac{y_{max} + y_{min}}{2}. \tag{1}$$

Now, for the convenience, we move the center of ellipse to the origin so the marginal values are transformed as follows:

$$x_1 = \frac{x_{max} - x_{min}}{2}, \qquad y_2 = \frac{y_{max} - y_{min}}{2}, \tag{2}$$

$$y_1 = y_r - z_2. \tag{3}$$

Clearly $x_1, y_2 > 0$. The new origin-centered ellipse will be denoted by $\mathcal{E}_c$.

The general equation of an ellipse (or a hyperbola) is

$$Ax^2 + By^2 + 2Cxy + Dx + Ey + F = 0, \tag{4}$$

where $AB \neq 0$. The coefficients D and E determine the translation from the origin. Thus any origin-centered ellipse is fully determined by

$$Ax^2 + By^2 + 2Cxy + F = 0, \tag{5}$$

and the equation may be divided by A, so we get the form

$$x^2 + By^2 + 2Cxy + F = 0. \tag{6}$$

Lemma 1. *Let the ellipse $\mathcal{E}_c$ be described by (6) with the maximal points $R = [x_1, y_1]$ and $T = [x_2, y_2]$ in direction of x and y coordinates, respectively. Then*

$$x_1 y_1 = x_2 y_2 \tag{7}$$

and the coefficients from (6) satisfy

$$C = -\frac{x_2}{y_2}, \tag{8}$$

$$B = \frac{x_1^2}{y_2^2}, \tag{9}$$

$$F = x_2^2 - x_1^2. \tag{10}$$

Proof. From (6) we get by implicit differentiation

$$\frac{dy}{dx} = -\frac{x + Cy}{By + Cx}, \quad \frac{dx}{dy} = -\frac{By + Cx}{x + Cy}$$

and the points P and V will be obtained by the solution of equations $\frac{dx}{dy} = 0$, $\frac{dy}{dx} = 0$, respectively. Hence $x_2 + Cy_2 = 0$ and $By_1 + Cx_1 = 0$. Assume the values y_2 and x_1 are nonzero (otherwise the situation is trivial). We get

$$C = -\frac{x_2}{y_2}, \quad B = -C\frac{x_1}{y_1} = \frac{x_1x_2}{y_1y_2}.$$

Since $P \in \mathcal{E}_c$, it satisfies the equation (6), hence by substitution we get:

$$x_1^2 + \frac{x_1x_2}{y_1y_2}y_1^2 - 2\frac{x_2}{y_2}x_1y_1 + F = 0,$$

hence

$$F = \frac{x_1x_2y_1}{y_2} - x_1^2.$$

Since $V \in \mathcal{E}_c$, we may put it in the equation (6) to get

$$x_2^2 + \frac{x_1x_2y_2}{y_1} - 2\frac{x_1x_2y_2}{y_1} + \frac{x_1x_2y_1}{y_2} - x_1^2 = 0.$$

This can be rewritten as

$$(x_2y_1 - x_1y_2)(x_1y_1 - x_2y_2) = 0.$$

In non-degenerate case, i.e. when the smaller of semiaxis is non-zero, we have strict inequalities $|x_2| > |x_1|$, $|y_1| > |y_2|$. Then $|x_2y_1| > |x_1y_2|$ and $|x_2y_1| \neq |x_1y_2|$, hence $x_2y_1 - x_1y_2 \neq 0$. Therefore $x_1y_1 - x_2y_2$ must be equal to zero. In degenerate case, the ellipse collapses into a segment line, hence the maximal points in both directions coincide.

Therefore $x_1y_1 = x_2y_2$ holds in both cases. Now by substitution $x_2 = \frac{x_1y_1}{y_2}$ in the equations for B, C, F we get the required relationships. $\square$

As a consequence, we get that B is positive, hence generally $AB > 0$.

Theorem 1. Duality property
Let the ellipse $\mathcal{E}_c$ be given by (6) with $A, B > 0$. Let $\gamma = \sqrt{\frac{A}{B}}$ (this will be called a duality ratio *for $\mathcal{E}_c$). Given a point $X = [x_0, y_0]$, then the point $X' = [\frac{1}{\gamma}y_0, \gamma x_0]$ (which will be called the* dual point *for X) satisfies:*

$$X \in \mathcal{E}_c \iff X' \in \mathcal{E}_c$$

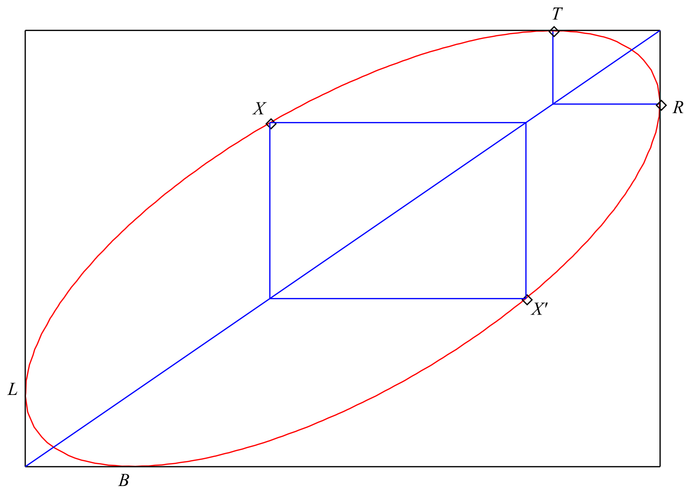

Figure 1: Duality property on the ellipse.

Proof. Since $X'' = [\frac{1}{\gamma}\gamma x_0, \gamma\frac{1}{\gamma}y_0] = [x_0, y_0] = X$, the duality is an involution and it suffices to prove just one direction. Let $X \in \mathcal{E}_c$. Then $Ax_0^2 + By_0^2 + 2Cx_0y_0 + F = 0$. Then

$$\begin{aligned} A(\frac{1}{\gamma}y_0)^2 + B(\gamma x_0)^2 + 2C\frac{1}{\gamma}y_0\gamma x_0 + F &= A(\sqrt{\frac{B}{A}}y_0)^2 + B(\sqrt{\frac{A}{B}}x_0)^2 \\ &\quad + 2Cy_0x_0 + F \\ &= By_0^2 + Ax_0^2 + 2Cy_0x_0 + F \\ &= 0 \end{aligned}$$

hence $X' \in \mathcal{E}_c$. $\square$

From now on, let ρ be an origin-centered basic (parallel with coordinate axes) rectangle with lengths 2α and 2β of horizontal and vertical, respectively, sides and $\mathcal{E}$ be its inscribed ellipse (clearly origin-centered) with the marginal points $T = [\xi, \beta]$, B, R and L (top, bottom, right and left, respectively). Hence $x_1 = \alpha, y_2 = \beta, x_2 = \xi$ and, due to (6), the ellipse can be described by the equation

$$\beta^2x^2 + \alpha^2y^2 - 2\xi\beta xy + \beta^2(\xi^2 - \alpha^2) = 0. \tag{11}$$

Then the property 1 can be applied on the marginal points of the ellipse as follows: due to 1 $\gamma = \sqrt{\frac{A}{B}} = \sqrt{\frac{\frac{1}{x_1^2}}{\frac{1}{y_2^2}}} = \sqrt{\frac{\beta^2}{\alpha^2}} = \frac{\beta}{\alpha}$. Hence γ is the tangent of the diagonal with the positive slope. Therefore two points are mutually dual if they have common projection points on the rectangle's diagonal (see Figure 1). One can easily see that the marginal points T and R are dual.

Remark 1. Using the theory of conjugate diameters (see e.g. [7]), one can see that the diagonal chords of the rectangle are exactly the conjugate diameters and then the statement follows from the well-known fact that the midpoints of the parallel chords lie on the diameter, which is conjugate to the longest of these chords being the diameter.

2.0.1. Expression of the Marginal Values

Since $\mathcal{E}$ can be described by (6), using the formulas from the Lemma 1 and introducing the variable $Q = \frac{C^2-B}{F}$ (used analogously in the Appendix) one can easily get the marginal values in the form

$$x_1 = \sqrt{\frac{B}{Q}}, \quad y_1 = \frac{|C|}{\sqrt{BQ}}, \quad x_2 = \frac{|C|}{\sqrt{Q}}, \quad y_2 = \sqrt{\frac{1}{Q}}. \tag{12}$$

3. Vagueness of the Touching Point

While the previous section was based on the knowledge of the exact values, now we will describe the more usual condition. It is usually not possible to read from the bitmap the exact value of the x-coordinate for the topmost (or y-coordinate for the rightmost) point. Let us focus on the topmost point. What one can get from the raster image is only a pixel range where it occurs (referred to as an *occurrence range*)– this is given by the resolution of the bitmap image. While the height of the range may be considered negligible, the problem of its width still needs to be solved. However, whatever the process of rasterizing of the ellipse was, the bounds of this range arose from the ellipse by replacement of a certain curve segment by a set of neighboring pixels. Hence we may expect that exact ellipse (after a slight change of the size) intersects the two-dimensional pixel range in the bounding values of x-coordinate. Though the original ellipse may not have satisfy this exactly, such a change will cause only minor differences. In fact, this change is negligible since it is certainly

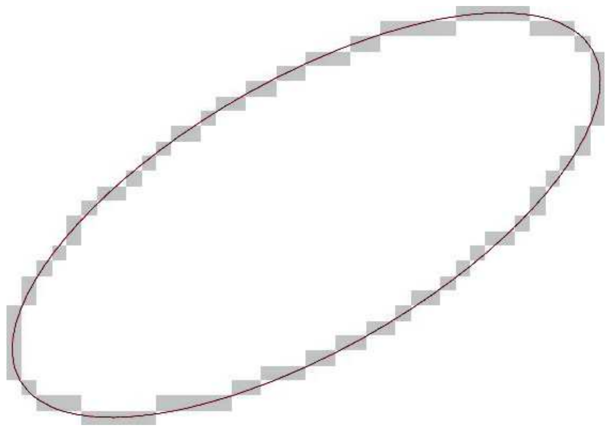

Figure 2: Fitting the ellipse into the raster image.

bellow the resolution of the image. On the Figure 2, the thin original ellipse and its raster image are depicted. In that case of scale, the rightmost and the leftmost ranges are intersected nearby the bounding values.

Now let the situation be the following: Let the topmost point $T = [\xi, \beta]$ of $\mathcal{E}$ be somewhere between the points $T_0 = [x_0, \beta]$ and $T_1 = [x_1, \beta]$ (the bounding points of the range). The segment line connecting T_0 and T_1 will be identified with the interval $\langle x_0, x_1 \rangle$ of its x-values and the same convention will be used for bottom side and for the vertical sides and y-coordinate.

Due to the previous discussion, we may consider the best solution being an ellipse which can be rescaled so it intersects the points T_0 and T_1. Hence we are looking for an ellipse $\mathcal{E}'$ satisfying:

- $T_0, T_1 \in \mathcal{E}'$,
- $\mathcal{E}'$ is origin-centered,
- the basic circumscribed rectangle has the ratio $\gamma = \frac{\beta}{\alpha}$ of the lengths of vertical and horizontal side.

Such an ellipse can be obtained as follows. Using the duality one may find the points $R_0 = T_0'$, $R_1 = T_1'$ with the duality ratio given by γ. To find the coefficients B, C, F for the equation (6) one solves the corresponding system of 3 linear equations for some triple of the points. The solution is $B = \frac{\alpha^2}{\beta^2}$, $C = -\frac{x_0+x_1}{2\beta}$, $F = x_0 x_1 - \alpha^2$. Since here $A = 1$, the ellipse $\mathcal{E}'$ has the following equation:

$$\beta^2 x^2 + \alpha^2 y^2 - (x_0 + x_1)\beta xy + \beta^2(x_0 x_1 - \alpha^2) = 0. \tag{13}$$

One may easily check that it is satisfied by the points T_0, T_1, R_0, R_1. Observe that if $x_0 = x_1$, we get exactly the equation (11), since $T_0 = T_1$ becomes the topmost point T.

Now, from $\mathcal{E}'$ we can derive the description of the ellipse $\mathcal{E}$. We need to change the scale in order to fit the ellipse into the rectangle ρ. But all we need to know is the x-value of the topmost point, i.e. the value of ξ. Due to the central symmetry, the change of the scale can be managed by translation of the points along the rays of the form $y = \delta x$. A point $[x, y]$ is then mapped onto $[x', y']$ where $x' = y' \frac{x}{y}$.

Let $[x_v, y_v]$ be the topmost point of the ellipse $\mathcal{E}'$. It will be translated to the point which will be the topmost point of $\mathcal{E}$, i.e. the point $T = [\xi, \beta]$. Therefore $\xi = \beta \frac{x_v}{y_v}$.

From the paragraph 2.0.1 we get the coordinates $x_v = \frac{|C|}{\sqrt{Q}}$, $y_v = \sqrt{\frac{1}{Q}}$. The nonzero coefficient Q is not needed because we will use only the ratio $r = \frac{x_v}{y_v} = \frac{\frac{|C|}{\sqrt{Q}}}{\sqrt{\frac{1}{Q}}} = |C|$. Hence $r = \frac{|-\frac{x_0+x_1}{2\beta}|}{1} = \frac{x_0+x_1}{2\beta}$ and we get finally $\xi = \beta r = \frac{x_0+x_1}{2}$.

The result can be stated as

Observation 1. *The best ellipse (in sense of the discussion above) inscribed in the basic origin-centered rectangle has the topmost point in the middle of the occurrence range.*

3.1. Regression Method

In general, the situation can be even worse. Since each of the four marginal points yields an occurrence range, we have four ways how to describe the ellipse which may not yield the same results. There are more ways how to handle this situation - one can interpolate either the initial values or the results or to start over using a regression method.

Due to the central symmetry and the duality, we can map all the 8 bounding points onto the same, say the upper, side. Then there are several "reasonable" ways how to assign one particular point for all these 8 points, namely we can take their (arithmetical) average, which is exactly the same as the average of the mapped midpoints for each pair of bounding points. Other options are to make the intersection or the union of all the mapped occurrence ranges and to take their center.

If we are not satisfied with the interpolation of the initial values, we may use the regression method. The problem stands as: To fit an ellipse in the set of the given 8 points. It can be solved by the standard least square method. However, it involves equations which are not directly solvable so we use Gauss-Newton algorithm for numerical approximations. The result will be obtained in the parametric form, which includes the expression of the lengths of semiaxes and the tilt angle.

After its application, the semiaxes of the resulting ellipse will be transformed in order to get the ellipse inscribed in the rectangle. It is possible to do it accurately only if the ellipse remains origin-centered, but the error becomes very small for high number of iterations. Hence it makes no problem to recenter the ellipse to the origin.

We demonstrate how the method works for rectangle ρ with the values

$$\alpha = \frac{1}{2}\sqrt{19} \qquad\qquad \beta = \frac{3}{2}$$

and the bounding points:

$$\begin{array}{lll}
T_0: & x_0 = 1.531375673 & y_0 = \beta \\
T_1: & x_1 = 1.343375673 & y_1 = \beta \\
R_0: & x_2 = \alpha & y_2 = 1.063965022 \\
R_1: & x_3 = \alpha & y_3 = 0.9245745478 \\
B_0: & x_4 = -1.343375673 & y_4 = -\beta \\
B_1: & x_5 = -1.546375673 & y_5 = -\beta \\
L_0: & x_6 = -\alpha & y_6 = -0.9245745478 \\
L_1: & x_7 = -\alpha & y_7 = -1.073965022
\end{array}$$

Let us show the samples from the first 600 steps of Gauss-Newton method. For comparison, by thick dash line we draw an ellipse of the equation $36x^2 + 76y^2 - 40\sqrt{3}xy - 96 = 0$ which is an algebraically calculated ellipse with the top-most point $T = [\frac{5}{6}\sqrt{3}, \beta]$, semiaxes $\sqrt{6} \approx 2.449489743$ and 1 and the tilt angle $\frac{\pi}{6} \approx 0.5235987758$. The right image shows the final iteration (thin solid line), recentered to the origin (dash-and-dot line) and the algebraically calculated ellipse (solid thick line).

An interpretation of the results obtained by least square method is based on the modification of the resulting ellipses. As discussed above, the obtained ellipse intersects the rectangle so that the points with the maximal value of a coordinate have the value of the other coordinate between the values of the

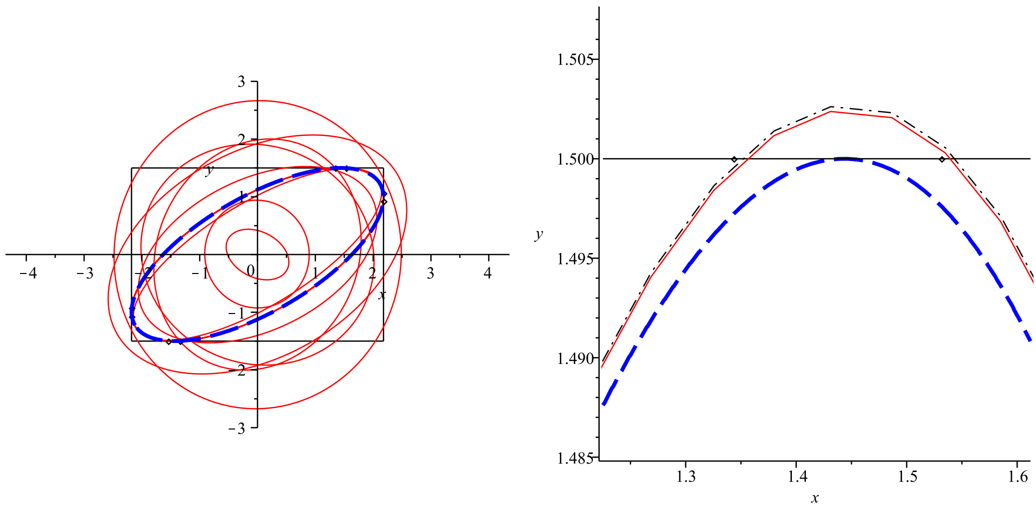

Figure 3: Demonstration of the Gauss-Newton method

defining points. Hence by slight shortening of the semiaxes we get the ellipse tangent to the rectangle. We can use the formulas derived in [6]. Namely, we shorten the obtained semiaxes the following way

$$a' = a\frac{\alpha}{\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}}, \ b' = b\frac{\beta}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}$$

– this will play the role of our new semiaxes while the angle remains the same. The resulting ellipse (obtained from the 600th iteration) will have the semiaxes $a' = 2.45018270$ $b' = 0.997965701$ and the tilt angle $\theta = 0.523920932$. As we can see, the differences between these values and the values for the algebraically calculated ellipse are smaller by degree then the lengths of the occurrence ranges. Hence, the algebraic method provides an easy way of reaching the satisfactory results.

4. Determination by Other Points

While the marginal points of $\mathcal{E}$ may not be able to obtain exactly, we still can try to obtain some other point $[x_0, y_0]$ of the ellipse more accurately. We will show that, in some cases, such a point provides a sufficient information to determine the ellipse fully.

From Lemma 1 we know that, in order to determine the ellipse, it suffices to determine the value of $\xi \in (-\alpha, \alpha)$, i.e. the x-coordinate of T. It can be obtained from the equation (11) rewritten as quadratic in variable ξ

$$\beta^2\xi^2 - 2\beta xy\xi + \beta^2x^2 + \alpha^2y^2 - \alpha^2\beta^2 = 0.$$

If $[x_0, y_0] \in \mathcal{E}_c$ is the known point, we may put it into the equation and to express ξ as

$$\xi = \frac{xy \pm \sqrt{\alpha^2 - x_0^2}\sqrt{\beta^2 - y_0^2}}{\beta}.$$

Let us look at the values of ξ for some instances of $[x_0, y_0] \in \mathcal{E}_c$.

1. Just to check the correctness, let $y_0 = \beta$. Then there is a unique solution and ξ comes up to be x_0– this is absolutely right, since the point $[\xi, \beta]$ is the only one with this y-value.

2. Similarly, let $x_0 = \alpha$. Then we get ξ comes up to be uniquely determined and equal to $\frac{\alpha y_0}{\beta}$, so we get that the point $[x_0, y_0]$ is dual to $[\xi, \beta]$, hence $[x_0, y_0]$ is the rightmost point, which is correct again.

3. Let $y_0 = \frac{\beta x_0}{\alpha}$, i.e. the point is on the diagonal with the positive slope. Then we have:

$$\begin{aligned}
\xi &= \tfrac{1}{\beta}\left(x_0\tfrac{\beta x_0}{\alpha} \pm \sqrt{\alpha^2 - x_0^2}\sqrt{\beta^2 - \tfrac{\beta^2 x_0^2}{\alpha^2}}\ \right)\\
&= \tfrac{1}{\beta}\left(x_0\tfrac{\beta x_0}{\alpha} \pm \sqrt{\alpha^2(1 - \tfrac{x_0^2}{\alpha^2})}\sqrt{\beta^2(1 - \tfrac{x_0^2}{\alpha^2})}\ \right)\\
&= \tfrac{1}{\beta}\left(x_0^2\tfrac{\beta}{\alpha} \pm \alpha\beta(1 - \tfrac{x_0^2}{\alpha^2})\right)\\
&= \tfrac{x_0^2}{\alpha} \pm \alpha\tfrac{1}{\alpha^2}(\alpha^2 - x_0^2)\\
&= \tfrac{1}{\alpha}(x_0^2 \pm \alpha^2 - x_0^2)\\
&= \begin{cases} \tfrac{1}{\alpha}(x_0^2 + \alpha^2 - x_0^2) = \alpha \\ \tfrac{1}{\alpha}(x_0^2 - \alpha^2 + x_0^2) = \tfrac{2x_0^2 - \alpha^2}{\alpha} \end{cases}
\end{aligned}$$

   Since $\xi < \alpha$, we get the unique solution $\xi = \frac{2x_0^2 - \alpha^2}{\alpha}$.

4. Analogously, we get the unique solution for the point $y_0 = -\frac{\beta x_0}{\alpha}$ on the diagonal with the negative slope.

Remark 2. The points on the diagonals are, in fact, most likely to be accurately readable from the raster image. The reason is that the tangent line of the curve is in these points generally far from being parallel to any of the coordinate axis and since these points are on the diagonal, the raster image of the curve is most even here – it admits least errors in each region bounded by the ellipse's marginal points.

Therefore, we suggest to choose the base point from the eight points of the ellipse that lie either directly on the rectangle (marginal points) or on its diagonals. The latter ones are possibly easiest readable points.

Appendix: From General Equation to the Parametric Expression

We need to find the lengths of semiaxes a, b and the angle θ of rotation from the basic position (with the semiaxes parallel to coordinate axes). Parametric expression of ellipse in the basic position is in the vector form:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a\cos\phi \\ b\sin\phi \end{pmatrix}, \ \phi \in \langle 0, 2\pi\rangle. \tag{14}$$

The rotation by an angle θ can be achieved by left multiplication by the matrix

$$\mathbf{H}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{15}$$

To find the values of a, b and θ, it is useful to know the general expression of the coefficients from the equation (4) with $D = E = 0$:

$$A = Q(a^2 - x_f^2), \tag{16}$$
$$B = Q(a^2 - y_f^2), \tag{17}$$
$$C = -Qx_f y_f. \tag{18}$$
$$F = Qa^2(x_f^2 + y_f^2 - a^2) \tag{19}$$

where $[x_f, y_f]$ are coordinates of the right-located focus and Q is a nonzero real coefficient. See e.g. the Appendix of [1] for derivation of these relations. From the properties of a general ellipse, one can easily see that

$$b^2 = a^2 - x_f^2 - y_f^2.$$

From (16) and (17) we get $x_f^2 = a^2 - \frac{A}{Q}$ and $y_f^2 = a^2 - \frac{B}{Q}$, respectively. Since $b^2 = a^2 - x_f^2 - y_f^2$, we have

$$b^2 = \frac{A+B}{Q} - a^2, \tag{20}$$

Now we take in account the equation (18) which yields $C^2 = Q^2(a^2 - \frac{A}{Q})(a^2 - \frac{B}{Q})$ hence

$$Q^2a^4 - Qa^2(A+B) + AB - C^2 \quad = \quad 0, \tag{21}$$

By substitution $S = Qa^2$ we get an equation $S^2 - (A+B)S + AB - C^2 = 0$ with the roots $\frac{A+B\pm R}{2}$ where $R = \sqrt{(A+B)^2 - 4(AB - C^2)}$. The roots are then $S_0 = \frac{A+B+R}{2}$, $S_1 = \frac{A+B-R}{2}$. Moreover, by simplification we get $R = \sqrt{(A-B)^2 + 4C^2}$. Since a has to be positive, we expect it to be either $a_0 = \sqrt{\frac{S_0}{Q}}$ or $a_1 = \sqrt{\frac{S_1}{Q}}$.

Let $\{i, i'\} = \{0, 1\}$ and $a = a_i$. Then $b^2 = \frac{A+B}{Q} - a_i^2 = \frac{A+B}{Q} - \frac{A+B+(-1)^iR}{2Q} = \frac{2A+2B-A-B-(-1)^iR}{2Q} = \frac{A+B-(-1)^iR}{2Q} = \frac{S_{i'}}{Q} = a_{i'}^2$. Since $a \geq b$, we have $a = a_0$ and $b = a_1$.

We still need to evaluate the variable Q. By the transformed equations

$$[[a^2 - [(16) + (17)]] \cdot a^2 - (19) \cdot Q] - (21)$$

we get $AB - C^2 + QF = 0$, i.e., $Q = \frac{C^2 - AB}{F}$. Collecting all together we have the formulas:

$$Q \quad = \quad \frac{C^2 - AB}{F}, \tag{22}$$

$$R \quad = \quad \sqrt{(A-B)^2 + 4C^2}, \tag{23}$$

$$S_0 = \frac{A+B+R}{2}, \qquad S_1 = \frac{A+B-R}{2}, \tag{24}$$

$$a = \sqrt{\frac{S_0}{Q}}, \qquad b = \sqrt{\frac{S_1}{Q}}. \tag{25}$$

To find the tilt angle θ, just observe that $\tan\theta = \frac{y_f}{x_f}$. Hence $\tan^2\theta = \frac{a^2 - \frac{B}{Q}}{a^2 - \frac{A}{Q}} = \frac{\frac{Qa^2 - B}{Q}}{\frac{Qa^2 - A}{Q}}$, i.e.,

$$\tan^2\theta \quad = \quad \frac{S_0 - B}{S_0 - A}. \tag{26}$$

The signum of the tangent of the main axis of the ellipse is given by the deviation of the maximal x-value from the x-axis, i.e., it is the same as the signum of y_1, hence of $\frac{y_1}{x_1} = -\frac{C}{B}$ (see (7,8,9)), since $x_1 > 0$. Hence

$$\tan\theta = -\mathrm{sgn}(\frac{C}{B})\sqrt{\frac{S_0 - B}{S_0 - A}}.$$

We may assume $\theta \in \langle -\frac{\pi}{2}, \frac{\pi}{2} \rangle$, hence by

$$\cos\theta = \frac{1}{\sqrt{1+\tan^2\theta}}, \tag{27}$$

$$\sin\theta = \frac{\tan\theta}{\sqrt{1+\tan^2\theta}} \tag{28}$$

we get the entries of the matrix $\mathbf{H}(\theta)$. The shifting to the original center is achieved by addition of the vector $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$. Now we have the ellipse described by the vector equation:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{H}(\theta) \begin{pmatrix} a\cos\phi \\ b\sin\phi \end{pmatrix} + \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \ \phi \in \langle 0, 2\pi \rangle. \tag{29}$$

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