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BÄCKLUND TRANSFORMATIONS FOR SOME NON-LINEAR EVOLUTION EQUATIONS USING PAINLEVÉ ANALYSIS

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Abstract: We prove that symmetric coupled Burger's system, claimed by Burger, to pass the Painlevé test for integrability, actually succeed the test at the highest resonance of the generic branch and therefore must be integrable.

Key Words: symmetric coupled Burger's system, Painlevé test, integrability, KdV-Burger's equations, (2+1)-dimensional breaking soliton equations, (2+1)-dimensional dispersive long wave equations, coupled Konno-Oono equations, (2+1)-dimensional breaking soliton equations

1. Introduction

In this paper, we present explicit Painlevé test for Symmetric coupled Burger's system equations, (1+1)-dimensional partial differential equations, variant Coupled KdV-Burger's equations. The associated Bäcklund transformations are obtained directly from the Painlevé test.

We use the KdV-Burger$'$s equations to illustrate the following two aspects of the Painlevé analysis of nonlinear PDEs. First, if a nonlinear equation passes

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the Painlevé test for integrability, the singular expansions of its solutions around characteristic hyper surfaces can be neither single-valued functions of independent variables nor single-valued functions of data. Second, if the truncation of singular expansions of solutions are consistent, the truncation not necessarily leads to the simplest, or elementary, auto-Bäcklund transformation related to the Lax pair.

Painlevé analysis is a powerful tool in investigating the integrability properties of differential equations [1], [2], [3]. For systems with the with the Painlevé property, BT can be defined. These appear as truncations of certain expansions of solutions about its singular manifold. Many methods have been established to study characteristic properties for integrable NLEEs and their interrelations. Some of the most important methods are the inverse scattering method (IST) [4], [5], [6], [7], the bilinear method [8], BTs [9], [10] and the Painlevé analysis method [11], [12], [13].

The Bäcklund transformation is a transformation that relates pairs of solutions of non-linear evolution equations. Bäcklund transformation is a useful tool for generating solutions to certain NLPDEs specially soliton equations. In this paper we shall use Painlevé analysis study to obtain some Bäcklund transformations for the studied equations.

1.1. Painlevé Analysis Method

Painlevé property is a method of investigation for the integrability properties of many NLEEs . If a PDE which has no points such as movable branch, algebraic and logarithmic then is called P-type. An ordinary differential equation (ODE) might still admit movable essential singularities without movable branch points. This method does not identify essential singularities and therefore it provides only necessary conditions for an ODE to be of P-type. Singularity structure analysis admitting the P-property advocated by Ablowitz et al. For ODEs and extended to PDE by Weiss, Tabor and Carnevale (WTC), plays a key role of investigating the integrability properties of many NLEEs. The well-known procedure of WTC requires:

1. The determination of leading orders Laurent series,

2. The identification of powers at which the arbitrary functions can enter into the Laurent series called resonances,

3. Verifying that , at the resonance values , sufficient number of arbitrary functions exist without introducing the movable critical manifold.

According to the WTC method, the general solution of PDEs are in the below from

$$u(x,t) = \varphi^{\alpha}(x,t)\sum_{j=0}^{\infty} u_j(x,t)\varphi^j(x,t),$$
$$v(x,t) = \varphi^{\beta}(x,t)\sum_{j=0}^{\infty} v_j(x,t)\varphi^j(x,t), \tag{1.1}$$

where α and β are negative integers, $\varphi(x,t) = 0$ is the equation of singular manifold. The functions u_j and v_j $(j = 0, 1, 2, ...)$ have to be determined by substitution of expansion (1.1) into the PDEs, so PDEs becomes

$$\sum_{j=0}^{\infty} E_j(u_0, \ldots, u_j, \varphi)\varphi^{j+q}(x,t) = 0, \tag{1.2}$$

where q is some negative constant. E_j depends on φonly by the derivatives of φ. The successive practical steps of Painlevé analysis are the following:

1. Determine the possible leading orders α by balancing two or more terms of the PDEs and expressing that they dominate.

2. Solve equation $E_0 = 0$ for non-zero values of u_0. This may lead to several solutions, called branches.

3. Find the resonances, i.e. the values of $j$for which $u_j$cannot be determined from equation $E_j = 0$. This last the other terms.

Equation has generally the form

$$E_j = (j+1)p(j)\varphi_x^i\varphi_t^{n-i}u_j + Q(u_0, \ldots, u_{j-1}, \varphi) = 0, \quad \forall j<0, \tag{1.3}$$

Here $n$is order of the PDEs, $0 \le i \le n$ and p is a polynomial of degree $n-1$.

The values of the resonances are the zeros of p. 4. Determine whether the

resonances are, compatible, or not. At resonance, after substitution in (1.3) of the previously computed u i, $i \le j-1$, the function Q is either zero or non-zero then in the case $u_j$can be arbitrarily chosen and the expansion (1.2) does not exist for arbitrary φ, so the resonance is called compatible.

5. All resonances occur at positive integer values of j and are compatible.

2. Coupled Non-Linear Evolution equations, (1 + 1)-Dimensional Partial Differential Equation

Symmetric coupled Burger's system (see [1], [2]) is

$$\begin{aligned} \frac{1}{2}u_{xx} + \frac{1}{2}v_{xx} + 5uu_x + vu_x + 2vv_x - u_t &=0, \\ \frac{1}{2}v_{xx} + \frac{1}{2}u_{xx} + 5vv_x + uv_x + 2uu_x - v_t &=0. \end{aligned} \tag{2.1}$$

We first present the Painlevé test of the Burger's system. According to the WTC method, the general solution of PDEs are in the form

$$\begin{aligned} u(x,t) &= \varphi^{\alpha}(x,t)\sum_{j=0}^{\infty} u_j(x,t)\varphi^j(x,t), \\ v(x,t) &= \varphi^{\alpha}(x,t)\sum_{j=0}^{\infty} v_j(x,t)\varphi^j(x,t), \end{aligned} \tag{2.2}$$

where α, β are negative integers, $\varphi(x,t) = 0$ is the equation of singular manifold. The function u_j $(j = 0, 1, 2, ...)$ and v_j $(\ j = 0, 1, 2, ...)$ have to be determined by substitution of expansions into the system, so it becomes

$$\sum_{j=0}^{\infty} E_j(u_0, v_0, \ldots, u_j, v_j, \varphi)\varphi^{j+q}(x,t) = 0,$$

where q is some negative constant. E_j depends on φ only (by the derivatives of φ).

The leading order of solution of equations (2.2) are assumed as

$$u \approx u_o\varphi^{\square}, \quad v \approx v_o\varphi^{\beta}. \tag{2.3}$$

Substituting equations (2.3) into (2.1) and equating the most dominant terms, the following results are obtained,

$$\alpha = \beta = -1, \quad u_0 = v_0 = \frac{\varphi_x}{5}. \tag{2.4}$$

To find the resonances, the full Laurent series

$$\begin{aligned} u &= u_0\varphi^{-1} + \sum_{j=1}^{\infty} u_j\varphi^{j-1}, \\ v &= v_0\varphi^{-1} + \sum_{j=1}^{\infty} v_j\varphi^{j-1}, \end{aligned} \tag{2.5}$$

are substituted into equations (2.1) and by equating the coefficients of φ^{j-5}, the polynomial equation in j is derivated as

$$(j+1)(j-2)=0 \tag{2.6}$$

using the previous equation (2.6), the resonances are found to be $j=-1,2$.

As usual, the resonance at $j=-1$ corresponds to the arbitrariness of singular manifold $\varphi(x,t)=0$. In order to check the existence of sufficient number of arbitrary functions at the other resonance values, the full Laurent expansions (2.5) are substituted in equations (2.1). From the coefficient of φ^{-3}, the explicit value of u_0, v_0 are obtained as given in equation (2.4). Collecting the coefficient of φ^{-2}, the following equations are obtained to give u_1 and v_1 as solving these algebraic equations by Maple program , we obtain the results:

$$\left\{u_1=-\frac{1}{10}\frac{u_0\phi_{x,x}-10u_{0,x}u_0+2u_{0,x}\phi_x}{u_0\phi_x}\right\}=\nu_1.$$

Collecting the coefficient of φ^{-1}, the result is obtained as zero. The absence of u_2 and v_2 proves that u_2 and v_2 may be arbitrary. This corresponds to the resonance value at $j=2$. From the coefficient of φ^0, the value of u_3 and v_3 are obtained as

$$u_3=v_3=-\frac{1}{2\varphi_x(\varphi_x+5u_0)}(2u_{2,x}\varphi_x+10u_{2,x}u_0+u_2\varphi_{xx}+10u_{0,x}u_2 + u_{1,xx}+10u_1u_{1,x}+10u_1u_2\varphi_x), \tag{2.8}$$

Proceeding further to the coefficient of φ^{-2}, the value of u_4 and v_4 are obtained as

$$u_4=v_4=-\frac{1}{2}\frac{1}{\varphi_x\,(\,10\,u_0+3\,\varphi_x\,)}(2u_3\varphi_{xx}+4u_{3,x}\varphi_x+10u_{3,x}u_0 + 10u_{0,x}u_3+10u_2u_{1,x}+10u_2^2\varphi_x+10u_1u_{2,x}+20u_1u_3\varphi_x+u_{2,xx}),$$

and so on. we conclude that a nonlinear equations passes the Painlevé test for integrability.

To construct a Bäcklund transformation of equations (2.1), let us truncate the Laurent series at the constant level term to give

$$u=u_0\varphi^{-2}+u_1\varphi^{-1},\quad v=v_0\varphi^{-2}+v_1\varphi^{-1}. \tag{2.9}$$

Hence

$$\begin{aligned} u&=-2\alpha\varphi_x\varphi^{-2}+u_1\varphi^{-1}+u_2+u_3\varphi+u_4\varphi^2,\\ v&=-12\beta\varphi_x^2\varphi^{-2}+v_1\varphi^{-1}+v_2+v_3\varphi+v_4\varphi^2, \end{aligned} \tag{2.9}$$

where the pair of function (u, v) and (u_1, v_1) satisfy equations (2.1) and hence equations (2.8) are the associated Bäcklund transformation of equations (2.1) relating a solution u with a known solution u_1 of the equations (2.1) which can be taken to be a known solution.

We can also construct another BT of equations (2.1) to be

$$u = u_0\varphi^{-2} + u_1\varphi^{-1} + u_2, \quad v = v_0\varphi^{-2} + v_1\varphi^{-1} + v_2, \tag{2.10}$$

where (u, v) and (u_1, v_1) satisfy equations (2.1) while (u_2, v_2) satisfies equations (2.7) and hence equations (2.9) are a BT too. Also let us truncate the Laurent series again we get

$$u = u_0\varphi^{-2} + u_1\varphi^{-1} + u_2 + u_3\varphi, \quad v = v_0\varphi^{-2} + v_1\varphi^{-1} + v_2 + v_3\varphi, \tag{2.11}$$

where u, u_1 satisfy equations (2.1) while u_2 and u_3 are given by Eqns (2.7) and (2.8). We can make more truncations to the Laurent series at the constant level term u_4 and it will produce another BT for equations (2.1).

3. Applications: Finding of New Exact Solutions

Consider KdV-Burger's equations (see [19, 20]):

$$\begin{aligned} u_t + uu_x - \lambda_1 u_{xx} =& 0,\\ v_t + vv_x + \lambda_2 v_{xxx} =& 0,\\ u_t + uu_x - \lambda_1 u_{xx} + \lambda_2 u_{xxx} =& 0, \end{aligned} \tag{3.1}$$

where λ_1 and λ_2 are arbitrary constant, with $\lambda_1, \lambda_2 \neq 0$.

We first present the Painlevé test of the KdV-Burger's equations. According to the WTC method, the general solution of PDEs are in the below from

$$\begin{aligned} u(x,t) =& \varphi^{\alpha}(x,t)\sum_{j=0}^{\infty} u_j(x,t)\varphi^j(x,t),\\ v(x,t) =& \varphi^{\beta}(x,t)\sum_{j=0}^{\infty} v_j(x,t)\varphi^j(x,t), \end{aligned} \tag{3.2}$$

where α, β are negative integers, $\varphi(x,t) = 0$ is the equation of singular manifold. The functions u_j and v_j $(j = 0, 1, 2, ...)$ have to be determined by substitution of expansions (3.2) into the system. Therefore, PDEs become to

$$\sum_{j=0}^{\infty} E_j(u_0, v_0, \dots, u_j, v_j, \varphi)\varphi^{j+q}(x,t) = 0, \tag{3.3}$$

where q is some negative constant. E_j depends on φ only by the derivatives of φ. The leading order of solution of equations (3.2) are assumed as

$$u \approx u_o\varphi^{\square}, \quad v \approx v_o\varphi^{\beta}. \tag{3.4}$$

Substituting equation (3.3) into (3.1) and equating the most dominant terms, the following results are obtained

$$\begin{aligned} &\alpha = -2, \quad \beta = -2, \\ &u_0 = -2\alpha\varphi_x, \quad v_0 = -12\beta\varphi_x^2, \end{aligned} \tag{3.5}$$

The full Laurent series:

$$\begin{aligned} u &= u_0\varphi^a - 2 + \sum_{j=1}^{\infty} u_j\varphi^{j-2}, \\ v &= v_0\varphi^{-2} + \sum_{j=1}^{\infty} v_j\varphi^{j-2}, \end{aligned} \tag{3.6}$$

We substitute into equations (3.1) and by equating the coefficients of φ^{j-5}, the polynomial equation in j is derivated as

$$(j+1)(j-4) = 0, \tag{3.7}$$

using the previous equation (3.7), the resonances are found to be $j = -1, 4$.

As usual, the resonance at $j = -1$ corresponds to the arbitrariness of singular manifold $\varphi(x,t) = 0$. In order to check the existence of sufficient number of arbitrary functions at the other resonance values, the full Laurent expansions (3.5) are substituted in equations (3.1). From the coefficient of φ^{-5}, the explicit values of u_0 and v_0 are obtained as given in equations (3.4). Collecting the coefficient of φ^{-4}, the following equations are obtained to give u_1 and v_1 as solving these algebraic equations by Maple program, we obtain the results:

$$\left\{\nu_1 = \frac{6\lambda 2 v_0\phi_{x,x}}{2\lambda 2\phi_x^2 + v_0}\right\}, \left\{u_1 = \frac{\lambda_1 u_0\phi_{x,x} + 2\lambda_1 u_{0,x}\phi_x + u_{0,x}u_0}{u_0\phi_x}\right\}.$$

Collecting the coefficient of φ^{-3}, the following equation is obtained to give v_2

$$v_2 = \frac{1}{2\phi_x u_0}(-2\lambda_2 u_0\phi_{xxx} - \phi_x u_1^2 + 6\lambda_2 u_1\phi_x\phi_{xx}).$$

From the coefficient of φ^{-2}, the value of u_3 and v_3 are obtained as

$$u_3 = \frac{1}{\phi_x(2\lambda_1\phi_x - u_0)}(-\lambda_1 u_2\phi_{xx} - 2\lambda_1 u_{2,x}\phi_x - \lambda_1 u_{1,xx} + u_{0,x}u_2 + u_{2,x}u_0 + u_1 u_{1,x} + u_1 u_2\phi_x)$$

$$\left\{u_3 = -\frac{v_1(\phi_x v_2 + \lambda 2\phi_{x,x,x})}{\phi_x v_0}\right\}.$$

Proceeding further to the coefficient of φ^{-1}, the value of u_4 and $v_4 = 0$. The absence of u_4 and v_4 proves that u_4 and v_4 are arbitrary. This corresponds to the resonance value at $j = 4$. So equations (3.1) admits sufficient number of arbitrary functions.

$$v_5 = \frac{-1}{\phi_x(u_0 + 6\lambda_2\phi_x^2)}(\phi_x u_3 u_2 + 6\lambda_2 u_4\phi_x\phi_{xx} + \lambda_2 u_3\phi_{xxx} + \phi_x u_1 u_4),$$

$$u_5 = \frac{1}{3\phi_x(4\lambda_1\phi_x - u_0)}(-\lambda_1 u_{3,xx} - 6\lambda_1 u_{4,x}\phi_x - 3\lambda_1 u_x\phi_{xx} + u_2 u_{2,x} + 3u_2 u_3\phi_x + u_1 u_{3,x} + 3u_1 u_4\phi_x + u_{0,x}u_4 + u_3 u_{1,x} + u_{4,x}u_0),$$

and so on. We conclude that the equations be satisfy to integration possible.

To construct the Bäcklund transformation of equations (3.1), let us truncate the Laurent series

$$u = u_0\varphi^{-2} + u_1\varphi^{-1} + u_2 + u_3\varphi + u_4\varphi^2,$$

$$v = v_0\varphi^{-2} + v_1\varphi^{-1} + v_2 + v_3\varphi + v_4\varphi^2.$$

Hence

$$u = -2\,\alpha\,\varphi_x\varphi^{-2} + u_1\varphi^{-1} + u_2 + u_3\varphi + u_4\varphi^2,$$

$$v = -12\beta\varphi_x^2\varphi^{-2} + v_1\varphi^{-1} + v_2 + v_3\varphi + v_4\varphi^2, \tag{3.8}$$

where the pair of function (u,v) and (u_4, v_4)satisfy equations (3.1) and hence equation (3.1) may be the associated Bäcklund transformation of equations (3.1).

4. The (2 + 1)-Dimensional Dispersive Long Wave Equations (see [11, 12])

We consider

$$
\begin{aligned}
u_{yt} + v_{xx} + u_x u_y + u u_{yx} =& 0, \\
v_t + u_x + (uv)_x + u_{xxy} =& 0.
\end{aligned}
\tag{4.1}
$$

We first present the Painlevé test of the Burger's equations. According to the WTC method, the general solution of PDEs are in the form

$$
\begin{aligned}
u(x,t) =& \varphi^{\alpha}(x,t) \sum_{j=0}^{\infty} u_j(x,t)\varphi^{j}(x,t), \\
v(x,t) =& \varphi^{\beta}(x,t) \sum_{j=0}^{\infty} v_j(x,t)\varphi^{j}(x,t),
\end{aligned}
\tag{4.2}
$$

where α, β are negative integers, $\varphi(x,y,t) = 0$ is the equation of singular manifold. The function u_j and v_j $(j = 0, 1, 2, ...)$ have to be determined by substitution of expansions (4.2) into the system(4.1), so it becomes

$$
\sum_{j=0}^{\infty} E_j(u_0, v_0, \ldots, u_j, v_j, \varphi)\varphi^{j+q}(x,t) = 0,
$$

where q is some negative constant. E_j depends on φ only by the derivatives of φ.

The leading order of solution of equations (4.2) are assumed as

$$
u \approx u_o \varphi^{\square}, \quad v \approx v_o \varphi^{\beta}. \tag{4.3}
$$

Substituting equation (4.3) into (4.1) and equating the most dominant terms, the following results are impossible, because it is impossible to balance two or more terms of the PDEs

$$
\alpha = \beta = 0,
$$

5. The Coupled Konno-Oono Equations (see [16], [17])

Let us consider

$$\begin{aligned} u_{xt} - 2uv &= 0, \\ v_t + 2uu_x &= 0. \end{aligned} \tag{5.1}$$

We first present the Painlevé test of the Burger's equations. According to the WTC method, the general solution of PDEs are in the form

$$\begin{aligned} u(x,t) &= \varphi^{\alpha}(x,t) \sum_{j=0}^{\infty} u_j(x,t)\varphi^j(x,t), \\ v(x,t) &= \varphi^{\beta}(x,t) \sum_{j=0}^{\infty} v_j(x,t)\varphi^j(x,t), \end{aligned} \tag{5.2}$$

where α and β are negative, $\varphi(x,t) = 0$ is the equation of singular manifold. The function u_j and v_j $(j = 0, 1, 2, ...)$ have to be determined by substitution of expansions (5.2) into the PDEs. So it becomes in the form

$$\sum_{j=0}^{\infty} E_j(u_0, v_0, \ldots, u_j, v_j, \varphi)\varphi^{j+q}(x,t) = 0,$$

where q is some negative constant. E_j depends on φ only by the derivatives of φ.

The leading order of solution of equations (5.2) are assumed as

$$u \approx u_o\varphi^{\square}, \quad v \approx v_o\varphi^{\beta}. \tag{5.3}$$

Substituting equations (5.3) into (5.1) and equating the dominant terms, the following results are impossible, because it is impossible to balancing two or more terms of the PDEs:

$$\alpha = \beta = 0.$$

Then

$$u = u_0, \quad v = v_0.$$

6. The (2+1)-Dimensional Breaking Soliton Equations (see [11], 13])

Consider

$$\begin{aligned} u_t + au_{xxy} + 4auv_x + 4au_x v = 0, \\ u_y = v_x. \end{aligned} \tag{6.1}$$

We first present the Painlevé test for known as breaking soliton system.

According to the WTC method, the general solution of PDEs is in the below from

$$\begin{aligned} u(x,t) &= \varphi^{\alpha}(x,t)\sum_{j=0}^{\infty} u_j(x,t)\varphi^j(x,t), \\ v(x,t) &= \varphi^{\beta}(x,t)\sum_{j=0}^{\infty} v_j(x,t)\varphi^j(x,t), \end{aligned} \tag{6.2}$$

where α is negative integer, $\varphi(x,y,t) = 0$ is the equation of singular manifold. The functions u_j and v_j $(j = 0,1,2,...)$ have to be determined by substitution of expansions (6.2) into the PDEs. So PDEs becomes

$$\sum_{j=0}^{\infty} E_j(u_0, v_0, \ldots, u_j, v_j, \varphi)\varphi^{j+q}(x,t) = 0,$$

where q is some negative constant. E_j depends on φ only by the derivatives of φ. The leading order of solution of equations (6.2) is assumed as

$$u \approx u_0\varphi^{\square}, \quad v \approx v_0\varphi^{\beta}. \tag{6.3}$$

Substituting equations (6.3) into (6.1) and equating the dominant terms, the following results are obvious $\alpha = -2$, $\beta = -2$,

$$\left\{u_0 = \frac{u_1\phi_x^2\phi_y}{2\phi_x\phi_{x,y} + \phi_{x,x}\phi_y}\right\},$$

$$v_0 = \frac{u_1(2\phi_{xy}\phi_x + \phi_y\phi_{xx})}{\phi_{xxy}}. \tag{6.4}$$

For finding the resonances, the full Laurent series:

$$\begin{aligned} u &= u_0\varphi^{-2} + \sum_{j=1}^{\infty} u_j\varphi^{j-2}, \\ v &= v_0\varphi^{-2} + \sum_{\mathbf{j=1}}^{\infty} \mathbf{v}_j\varphi^{j-2}, \end{aligned} \tag{6.5}$$

are substituted into equations (6.1) and by equating the coefficients of φ^{j-5}, the polynomial equation in j is derivated as

$$(j+1)(j-4)=0, \tag{6.6}$$

using the previous equation (6.6), the resonances are found to be $j=-1,4$.

As usual, the resonance at $j=-1$ corresponds to the arbitrariness of singular manifold $\varphi(x,y,t)=0$. In order to check the existence of sufficient number of arbitrary functions at the other resonance values, the full Laurent expansion (6.5) is substituted in equations (6.1). From the coefficient of φ^{-5}, the explicit values of u_0 and v_0 are obtained as given in equation (6.4). Collecting the coefficient of φ^{-4}, the following equations are obtained to give u_1 and v_1 as solving these algebraic equations by Maple program , we obtain the results:

$$\left\{u_1=\frac{u_0\phi_{x,x,y}}{2\phi_x\phi_{x,y}+\phi_{x,x}\phi_y}\right\},v_1=\frac{u_0\left(2\phi_x\phi_{x,y}+\phi_{x,x}\phi_y\right)}{\phi_x^2\phi_y}.$$

Proceeding further to the coefficient of φ^{-3}, the values of v_2 and u_2 are obtained as:

$$\begin{gathered}v_2=\frac{-1}{2}u_4\phi_{xy}\phi_x-\frac{1}{4}u_4\phi_y\phi_{xx}-\frac{3}{4}u_5\phi_x^2\phi_y-\frac{1}{8}u_3\phi_{xxy},\\ \left\{u_3=-\frac{3}{2}u_5\phi_x\phi_{x,y}-\frac{3}{4}\phi_{x,x}\phi_y-\frac{1}{4}u_4\phi_{x,x,y}-3u_6\phi_x^2\phi_y\right\}.\end{gathered} \tag{6.7}$$

From the coefficient of φ^{-2}, the values of u_3 and v_3 are obtained as

$$\begin{gathered}v_3=\frac{-1}{\phi_{xxy}}(2(2u_4\phi_{xy}\phi_x+u_4\phi_y\phi_{xx}+3u_5\phi_x^2\phi_y+4u_2)),\\ \left\{u_3=-\frac{3}{2}u_5\phi_x\phi_{x,y}-\frac{3}{4}\phi_{x,x}\phi_y-\frac{1}{4}u_4\phi_{x,x,y}-3u_6\phi_x^2\phi_y\right\}.\end{gathered}$$

Proceeding further to the coefficient of φ^{-1}, the value of u_4 and $v_4=0$. The absence of u_4 and v_4 yield that u_4 and v_4 are arbitrary. This corresponds to the resonance value at $j=4$. So equations (6.1) admits sufficient number of arbitrary functions:

$$\begin{gathered}\left\{u_5=-\frac{1}{3}\frac{12u_6\phi_x^2\phi_y+u_4\phi_{x,x,y}+4u_3}{2\phi_x\phi_{x,y}+\phi_{x,x}\phi_y}\right\},\\ v_5=\frac{-1}{6\phi_x^2\phi_y}(4u_4\phi_{xy}\phi_x+2u_4\phi_{xx}\phi_y+u_3\phi_{xxy}+8u_2),\end{gathered} \tag{6.8}$$

and so on. we conclude that the equations satisfies the Painlevé test and is integrable.

To construct a Bäcklund transformation of equations (6.1), let us truncate the Laurent series at the constant level term to give

$$u = u_0\varphi^{-2} + u_1\varphi^{-1}, \quad v = v_0\varphi^{-2} + v_1\varphi^{-1}.$$

Hence

$$\begin{aligned} u &= -2\alpha\varphi_x\varphi^{-2} + u_1\varphi^{-1} + u_2 + u_3\varphi + u_4\varphi^2, \\ v &= -12\beta\varphi_x^2\varphi^{-2} + v_1\varphi^{-1} + v_2 + v_3\varphi + v_4\varphi^2, \end{aligned} \tag{6.9}$$

where the pair of function (u, v) and (u_1, v_1) satisfy equations (6.1) and hence equations (6.8) are the associated Bäcklund transformation of equations (6.1) relating a solution u with a known solution u_1 of the equations (6.1) which can be taken to be a known solution.

We can also construct another BT of equations (6.1) to be

$$u = u_0\varphi^{-2} + u_1\varphi^{-1} + u_2, \quad v = v_0\varphi^{-2} + v_1\varphi^{-1} + v_2, \tag{6.10}$$

where (u,v) and (u_1,v_1) satisfy equations (6.1) while (u_2,v_2) and (u_3,v_3) satisfies equations (6.7) and hence equations (6.10) are a BT too.

Also let us truncate the Laurent series again we get

$$u = u_0\varphi^{-2} + u_1\varphi^{-1} + u_2 + u_3\varphi, \quad v = v_0\varphi^{-2} + v_1\varphi^{-1} + v_2 + v_3\varphi, \tag{6.11}$$

where u, u_1 satisfy equations (6.1) while u_2 and u_3 are given by equations (6.7) and (6.8). We can make more truncations to the Laurent series at the constant level term u_4 and it will produce another BT for equations (6.1).

7. Conclusion

We conclude that some of the partial differential evolution equations be amenable to integration possible by using Painlevé test, singular manifold, and consequently, these equations be satisfy to integration belongs to the class of C-integrable equations.

The Painlevé analysis easily detects that this is a typical C-integrable system in the Calogero sense and rediscovers its linearizing transformation. The Painlevé test easily detects that a system is a typical C-integrable system, in the terminology of Calogero.

There is a strong empirical evidence that any nonlinear differential equation which passed the Painlevé test must be integrable. The test itself, however, does not tell whether the equation is C-integrable (solvable by quadratures or exactly linearizable) or S-integrable (solvable by an inverse scattering transform technique). Often some additional information on integrability of the studied equation, such as its linearizing transformation, Lax pair, Bäcklund transformation, etc., can be obtained by truncation of the Laurent-type expansion representing the equation's general solution, see [14], [15], [16], [17], [18], [19], [20], [21].

The Painlevé analysis is used for some NLEEs arising in mathematical physics namely, the coupled Burger's system, the $(1+1)$-dimensional partial differential equations, KdV-Burger's equations, The $(2+1)$-dimensional dispersive long wave equation, to obtain associated Bäcklund transformations directly from it. Also, the Painlevé analysis is used for some NLEEs namely, The Coupled Konno-Oono equations, The $(2+1)$-dimensional breaking soliton equations. These equations have the same polynomial equation. The Bäcklund transformations for each equation are directly obtained from the Painlevé test. Finally we make more than one truncation to the Laurent series and get other Bäcklund transformations for the equations under consideration.

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