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A VARIATION OF (G'/G)-EXPANSION METHOD AND NEW EXACT TRAVELLING WAVE SOLUTIONS OF NONLINEAR REACTION-DIFFUSION MODEL

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Abstract: In this paper, we investigate the nonlinear reaction-diffusion (NRD) model. The Fisher equation, Burgers-Fisher equation and FitzHugh-Nagumo equation are simplest examples of NRD model. A variation of (G'/G)-expansion method is proposed to seek exact travelling wave solutions of nonlinear partial differential equations, and it is applied to construct a new exact travelling wave solutions for simplest examples of (NRD) model.

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1. Introduction

Many transport phenomena arising in natural science have been successfully described by the nonlinear reaction-diffusion (NRD) model. The applications NRD model have had wide variety including population dynamics, transport in porous medium, combustion theory and plasma physics[1-4]. The NRD equation is formulated by:

$$u_t = [D(u)u_x]_x + R(u), \tag{1}$$

where $u := u(x,t) \geq 0$, depending on the system, is density or temperature at the position $x \in (-\infty, \infty)$ and the time $t \in [0, \infty)$. $D(u)$ and $R(u)$ are the diffusion coefficient and the reaction term respectively. The simplest form of NRD equation (1) is the well known Fisher equation [5], as well as the Kolmogoroff-Petrovsky-Piscounoff (KPP) equation, which was originally derived for the propagation of a gene in a population. In this classic model, the reaction term is represented by the self-limiting Pearl-Verhulst logistic law $R(u) = \alpha u(1 - u^2/\sigma)$ and the diffusion is represented by the Brownian process, reflected through the term $D(u) = k$, where $k > 0$ is the diffusion constant, $\alpha > 0$ is rate constant and $\sigma = \lim_{t\to\infty} u(x,t)$ is equilibrium density.

Recently, the NRD model has been employed to describe the spatiotemporal pattern formation in bacterial colonies which is the elegant self-organization observed in the biological system [6-8]. For each species, the bacterial colonies generate various patterns depending on surrounding environments, mainly, including the nutrient concentration and the softness of the medium. This cooperative behavior in adaptation to survival in the environment reflects bacterial communication capabilities and social intelligence [7, 9]. The understanding of the underly mechanism is not only important to the biotechnology but also the basic science of the living organisms.

Recently, the (G'/G)-expansion method, firstly introduced by Wang et al. [10] has become widely used to search for various exact solutions of nonlinear evolution equations (NLEEs). This method is straightforward, concise and capable of producing new applications. Moreover, the solutions obtained by this method are of general nature and a number of specific solutions can be deduced by putting conditions on arbitrary constants present in the general solutions. Thereafter, a number of applications of This method have also been reported [11-20]. A generalized and simplified version of (G'/G)-expansion method is also reported [21-23].

In this paper, we have developed a variation of (G'/G)-expansion method [24], in which we have used the full advantage of the well known solution of the coupled Riccati equation. The presented method is used to find new exact travelling wave solutions of the Fisher equation, Burgers-Fisher equation and FitzHugh-Nagumo equation as examples of NRD model. The paper is organized as follows :This introduction in Section 1. In Section 2, we describe the mathematical analysis of the variation of (G'/G)-expansion method. In Section 3, we applied the proposed method on the above mentioned three equations and obtained new exact travelling wave solutions. Finally, the Conclusions are given in Section 4.

2. Mathematical Analysis of the Variation of (G'/G)-Expansion Method

Consider the nonlinear partial differential equation (NPDE), say, in (n+1) independent variables x_1, x_2, x_3,....,x_n and t, as

$$P(u, u_t, u_{x_1}, u_{x_2}, u_{tt}, u_{x_1 t}, u_{x_2 t}, u_{x_1 x_1}, u_{x_2 x_2}, ...) = 0, \tag{2}$$

where $u = u(x_1, x_2, ..., x_n, t)$ is an unknown function, P is a polynomial in $u = u(x_1, x_2, ..., x_n, t)$ and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved. In the following steps, we present the main steps of the variation of (G'/G)-expansion method.

Step 1. Consider the travelling wave transformation:

$$u = u(x_1, x_2, ..., x_n, t) = u(\xi), \quad \xi = \sum_{i=1}^{n} r_i x_i + Vt, \tag{3}$$

where $r_i (i = 1, 2, ..., n)$ and V (speed of wave) are constants to be determined later. The travelling wave variable u in (3) allows us to reduce the equation (2) to an ordinary differential equation for $u = u(\xi)$:

$$P(u, Vu', r_1 u', r_2 u', V^2 u', r_1 V u'', r_2 V u'', r_1^2 u'', r_2^2 u'', ...) = 0. \tag{4}$$

Step 2. If possible integrate equation (4) term by term one or more times yields constant(s) of integration.

Step 3. Assume that the solution $u(\xi)$, of the equation (4) can be expressed as a polynomial in (G'/G) and (F'/F) as follows:

$$u(\xi) = \sum_{i=0}^{m} a_i (G'/G)^i + \sum_{i=1}^{m} b_i (G'/G)^{i-1} (F'/F), \tag{5}$$

where $G = G(\xi)$ and $F = F(\xi)$ expresses the solution of the coupled Riccati equation,

$$G'(\xi) = -G(\xi)F(\xi), \tag{6}$$

$$F'(\xi) = 1 - F^2(\xi), \tag{7}$$

where prime denotes derivative with respect to ξ, $a_i(i = 0, 1, ..., m)$, $b_i(i = 1, 2, ..., m)$ are constants to be determined later.

These governing equations lead us two types of general solutions:

$$G(\xi) = \pm sech(\xi), \quad F(\xi) = \tanh(\xi), \tag{8}$$

$$G(\xi) = \pm csch(\xi), \quad F(\xi) = \coth(\xi). \tag{9}$$

Step 4. The positive integer m can be determined by considering the homogeneous balance between the highest order derivatives and the nonlinear terms appearing in equation (4) as follows: If $D[u(\xi)] = m$, then $D[u^r(\frac{d^q u}{d\xi^q})^s] = mr + s(q + m)$, where D denotes the degree of the expression.

Step 5. Substituting equation (5) into equation (4) and using equation (6) and equation (7), collecting all terms with the same order of (G'/G) or (F) together, left-hand side of equation (4) is converted into another polynomial in (G'/G) or (F). Equating each coefficient of this polynomial to zero, yields a set of algebraic equations for $a_i(i = 0, 1, ..., m)$, $b_i(i = 1, 2, ..., m)$, $r_i(i = 1, 2, ..., n)$ and V.

Step 6. Determining the constants $a_i(i = 0, 1, ..., m)$, $b_i(i = 1, 2, ..., m)$, $r_i(i = 1, 2, ..., n)$ and V by solving the algebraic equations in step 5. As the general solutions of equation (6) and equation (7) are already known to us, then substituting $a_i(i = 0, 1, ..., m)$, $b_i(i = 1, 2, ..., m)$, $r_i(i = 1, 2, ..., n)$, V and the general solutions of equation (6) and equation (7), we obtain the travelling wave solutions of equation (2).

Finally we well consider a generalized version of the governing equations for G and F, which are as follows:

$$G'^2(\xi) = (1 - G^2(\xi)) \cdot (k'^2 + k^2 G^2(\xi)), \tag{10}$$

$$F'^2(\xi) = (1 - F^2(\xi)) \cdot (1 - k^2 F^2(\xi)), \tag{11}$$

where k is known as the elliptic modulus and $k' = \sqrt{1 - k^2}$ is known as complementary elliptic modulus of the jacobi elliptic functions [25].

The governing equations lead us to the following jacobi elliptic expressions for G and F:

$$G(\xi) = cn(\xi, k), \tag{12}$$

$$F(\xi) = sn(\xi, k). \tag{13}$$

3. Applications of the Method

3.1. Example 1: The Fisher Equation

From the NRD equation (1), take $D(u) = 1$ and $R(u) = u - u^3$, we find the Fisher equation in the form

$$u_t - u_{xx} - u + u^3 = 0. \tag{14}$$

To obtain a new travelling wave solution of Fisher equation (14), take the transformation $u(x,t) = u(\xi)$, $\xi = rx + Vt$, then Fisher equation (14) reduces to

$$Vu' - r^2u'' - u + u^3 = 0. \tag{15}$$

Balancing the order of u'' and u^3, we get $m = 1$.

Hence the solution of equation (15) as described in step 3, can be expressed as,

$$u(\xi) = a_0 - a_1F + b_1(F^{-1} - F). \tag{16}$$

Substituting from (16) into (15), collecting the coefficients of $(F)^i$ ($i = 0, \pm 1$, $\pm 2, \pm 3$), and letting it be zero, yields a set of simultaneous algebraic equations for a_0, a_1, b_1, r and V as follows:

$$F^{-3}: \quad b_1^3 - 2b_1r^2 = 0, \tag{17}$$

$$F^{-2}: \quad -Vb_1 + 3a_0b_1^2 = 0, \tag{18}$$

$$F^{-1}: \quad -b_1 + 3a_0^2b_1 - 3a_1b_1^2 - 3b_1^3 + 2b_1r^2 = 0, \tag{19}$$

$$F^{0}: \quad -a_0 + a_0^3 - Va_1 - 6a_0a_1b_1 - 6a_0b_1^2 = 0, \tag{20}$$

$$F^{1}: \quad a_1 - 3a_0^2a_1 + b_1 - 3a_0^2b_1 + 3a_1^2b_1 + 6a_1b_1^2 + 3b_1^3 - 2a_1r^2 - 2b_1r^2 = 0, \tag{21}$$

$$F^{2}: \quad Va_1 + 3a_0a_1^2 + Vb_1 + 6a_0a_1b_1 + 3a_0b_1^2 = 0, \tag{22}$$

$$F^{3}: \quad -a_1^3 - 3a_1^2b_1 - 3a_1b_1^2 - b_1^3 + 2a_1r^2 + 2b_1r^2 = 0. \tag{23}$$

To solve this set of algebraic equations for a_0, a_1, b_1, r and V by use of Mathematica, we get:

Case 1.

$$a_0 = -\frac{1}{2}, \quad a_1 = -\frac{1}{2}, \quad b_1 = 0, \quad V = -\frac{3}{4}, \quad r = \mp\frac{1}{2\sqrt{2}}. \tag{24}$$

Case 2.

$$a_0 = -\frac{1}{2}, \quad a_1 = -\frac{1}{2}, \quad b_1 = \frac{1}{4}, \quad V = -\frac{3}{8}, \quad r = \mp\frac{1}{4\sqrt{2}}. \tag{25}$$

Case 3.

$$a_0 = -\frac{1}{2}, \quad a_1 = -\frac{1}{2}, \quad b_1 = \frac{1}{2}, \quad V = -\frac{3}{4}, \quad r = \mp\frac{1}{2\sqrt{2}}. \tag{26}$$

Case 4.

$$a_0 = -\frac{1}{2}, \quad a_1 = \frac{1}{2}, \quad b_1 = -\frac{1}{2}, \quad V = \frac{3}{4}, \quad r = \mp\frac{1}{2\sqrt{2}}. \tag{27}$$

Case 5.

$$a_0 = 0, \quad a_1 = -1, \quad b_1 = 0, \quad V = 0, \quad r = \mp\frac{1}{\sqrt{2}}. \tag{28}$$

Case 6.

$$a_0 = 0, \quad a_1 = -1, \quad b_1 = \frac{1}{2}, \quad V = 0, \quad r = \mp\frac{1}{2\sqrt{2}}. \tag{29}$$

Case 7.

$$a_0 = 0, \quad a_1 = 0, \quad b_1 = \frac{i}{\sqrt{2}}, \quad V = 0, \quad r = \mp\frac{i}{2}. \tag{30}$$

Case 8.

$$a_0 = \frac{1}{2}, \quad a_1 = \frac{1}{2}, \quad b_1 = -\frac{1}{4}, \quad V = -\frac{3}{8}, \quad r = \mp\frac{1}{4\sqrt{2}}. \tag{31}$$

Case 9.

$$a_0 = \frac{1}{2}, \quad a_1 = \frac{1}{2}, \quad b_1 = 0, \quad V = -\frac{3}{4}, \quad r = \mp\frac{1}{2\sqrt{2}}. \tag{32}$$

Substituting (24-32) into (16), we get two types of travelling wave solutions for all cases of the Fisher equation (14):

According to Case 1.

Type 1:

$$u_1(x,t) = -\frac{1}{2}(1 - \tanh[\frac{1}{2\sqrt{2}}x - \frac{3}{4}t]), \tag{33}$$

Type 2:

$$u_2(x,t) = -\frac{1}{2}(1 - \coth[\frac{1}{2\sqrt{2}}x - \frac{3}{4}t]). \tag{34}$$

According to Case 2.

Type 1:

$$u_3(x,t) = \frac{1}{4}(-2 + \coth[\frac{1}{4\sqrt{2}}x - \frac{3}{8}t] + \tanh[\frac{1}{4\sqrt{2}}x - \frac{3}{8}t]), \tag{35}$$

Type 2:

$$u_4(x,t) = \frac{1}{4}(-2 + \coth[\frac{1}{4\sqrt{2}}x - \frac{3}{8}t] + \tanh[\frac{1}{4\sqrt{2}}x - \frac{3}{8}t]). \tag{36}$$

According to Case 3.

Type 1:

$$u_5(x,t) = \frac{1}{2}(-1 + \coth[\frac{1}{2\sqrt{2}}x - \frac{3}{4}t]), \tag{37}$$

Type 2:

$$u_6(x,t) = \frac{1}{2}(-1 + \tanh[\frac{1}{2\sqrt{2}}x - \frac{3}{4}t]). \tag{38}$$

According to Case 4.

Type 1:

$$u_7(x,t) = \frac{1}{2}(-1 - \coth[\frac{1}{2\sqrt{2}}x + \frac{3}{4}t]), \tag{39}$$

Type 2:

$$u_8(x,t) = \frac{1}{2}(-1 - \tanh[\frac{1}{2\sqrt{2}}x + \frac{3}{4}t]). \tag{40}$$

According to Case 5.

Type 1:

$$u_9(x,t) = -\tanh[\frac{1}{\sqrt{2}}x], \tag{41}$$

Type 2:

$$u_{10}(x,t) = -\coth[\frac{1}{\sqrt{2}}x]. \tag{42}$$

According to Case 6.

Type 1:

$$u_{11}(x,t) = \frac{1}{2}(\tanh[\frac{1}{2\sqrt{2}}x] + \coth[\frac{1}{2\sqrt{2}}x]), \tag{43}$$

Type 2:

$$u_{12}(x,t) = \frac{1}{2}(\tanh[\frac{1}{2\sqrt{2}}x] + \coth[\frac{1}{2\sqrt{2}}x]). \tag{44}$$

According to Case 7.

Type 1:

$$u_{13}(x,t) = i\sqrt{2}csch[ix], \tag{45}$$

Type 2:

$$u_{14}(x,t) = -i\sqrt{2}csch[ix]. \tag{46}$$

According to Case 8.

Type 1:

$$u_{15}(x,t) = \frac{1}{4}(2 - \coth[\frac{1}{4\sqrt{2}}x - \frac{3}{8}t] - \tanh[\frac{1}{4\sqrt{2}}x - \frac{3}{8}t]), \tag{47}$$

Type 2:

$$u_{16}(x,t) = \frac{1}{4}(2 - \coth[\frac{1}{4\sqrt{2}}x - \frac{3}{8}t] - \tanh[\frac{1}{4\sqrt{2}}x - \frac{3}{8}t]). \tag{48}$$

According to Case 9.

Type 1:

$$u_{17}(x,t) = \frac{1}{2}(1 - \tanh[\frac{1}{2\sqrt{2}}x - \frac{3}{4}t]), \tag{49}$$

Type 2:

$$u_{18}(x,t) = \frac{1}{2}(1 - \coth[\frac{1}{2\sqrt{2}}x - \frac{3}{4}t]). \tag{50}$$

We have represented this solution (49) for a set of parameter value in Figure 1.

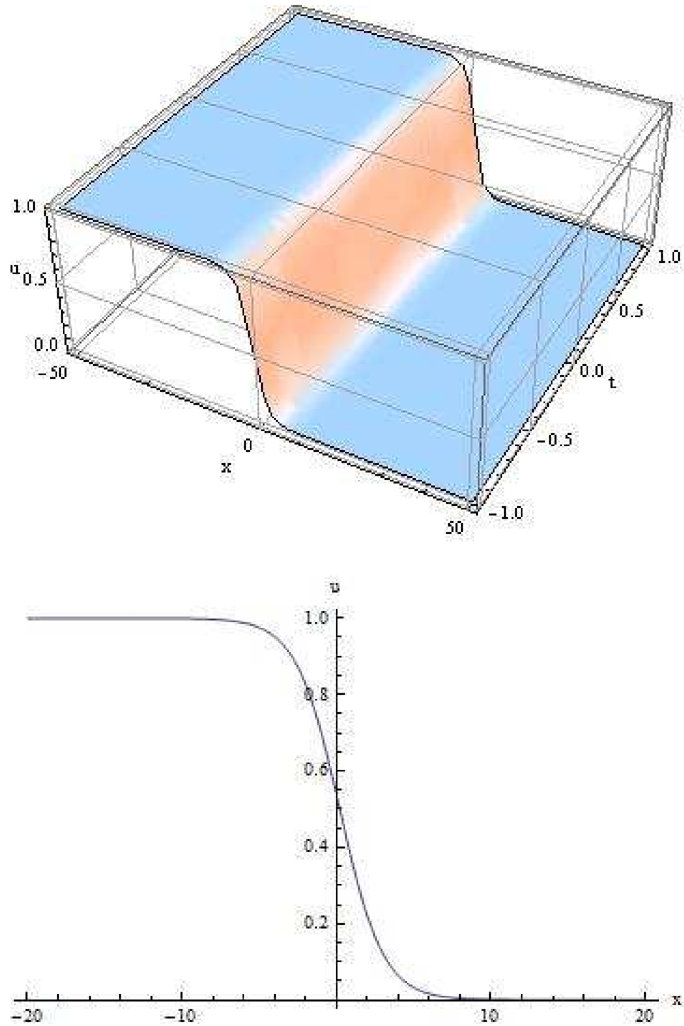

Figure 1: The travelling wave solution $u(x,t)$ of Fisher equation defined in equation (49).

3.2. Example 2: The Burgers-Fisher Equation

From the NRD equation (1), take $D(u) = 1$ and $R(u) = uu_x + u - u^3$, we find the Burgers-Fisher equation in the form

$$u_t - u_{xx} - uu_x - u + u^3 = 0. \tag{51}$$

To obtain a new travelling wave solution of Burgers-Fisher equation (51), take the transformation $u(x,t) = u(\xi)$, $\xi = rx + Vt$, then Burgers-Fisher equation (51) reduces to

$$Vu' - r^2u'' - ruu' - u + u^3 = 0. \tag{52}$$

Balancing the order of u'' and uu', we get $m = 1$.

Hence the solution of equation (52) as described in step 3, can be expressed as

$$u(\xi) = a_0 - a_1F + b_1(F^{-1} - F). \tag{53}$$

Substituting from (53) into (52), collecting the coefficients of $(F)^i$ $(i = 0, \pm1, \pm2, \pm3)$, and letting it be zero, yields a set of simultaneous algebraic equations for a_0, a_1, b_1, r and V as follows:

$$F^{-3}: \quad b_1^3 + b_1^2r - 2b_1r^2 = 0, \tag{54}$$

$$F^{-2}: \quad -Vb_1 + 3a_0b_1^2 + a_0b_1r = 0, \tag{55}$$

$$F^{-1}: \quad -b_1 + 3a_0^2b_1 - 3a_1b_1^2 - 3b_1^3 - b_1^2r + 2b_1r^2 = 0, \tag{56}$$

$$F^{0}: \quad -a_0 + a_0^3 - Va_1 - 6a_0a_1b_1 - 6a_0b_1^2 + a_0a_1r = 0, \tag{57}$$

$$F^{1}: \quad a_1 - 3a_0^2a_1 + b_1 - 3a_0^2b_1 + 3a_1^2b_1 + 6a_1b_1^2 + 3b_1^3 - a_1^2r - 2a_1b_1r - b_1^2r - 2a_1r^2 - 2b_1r^2 = 0, \tag{58}$$

$$F^{2}: \quad Va_1 + 3a_0a_1^2 + Vb_1 + 6a_0a_1b_1 + 3a_0b_1^2 - a_0a_1r - a_0b_1r = 0, \tag{59}$$

$$F^{3}: \quad -a_1^3 - 3a_1^2b_1 - 3a_1b_1^2 - b_1^3 + a_1^2r + 2a_1b_1r + b_1^2r + 2a_1r^2 + 2b_1r^2 = 0. \tag{60}$$

To solve this set of algebraic equations for a_0, a_1, b_1, r and V by use of Mathematica, we get:

Case 1.

$$a_0 = -\frac{1}{2}, \quad a_1 = -\frac{1}{2}, \quad b_1 = 0, \quad V = -1, \quad r = \frac{1}{2}. \tag{61}$$

Case 2.

$$a_0 = -\frac{1}{2}, \quad a_1 = -\frac{1}{2}, \quad b_1 = \frac{1}{4}, \quad V = -\frac{1}{2}, \quad r = \frac{1}{4}. \tag{62}$$

Case 3.

$$a_0 = -\frac{1}{2}, \quad a_1 = \frac{1}{2}, \quad b_1 = -\frac{1}{2}, \quad V = \frac{5}{8}, \quad r = \frac{1}{4}. \tag{63}$$

Case 4.

$$a_0 = -\frac{1}{2}, \quad a_1 = \frac{1}{2}, \quad b_1 = -\frac{1}{4}, \quad V = \frac{5}{16}, \quad r = \frac{1}{8}. \tag{64}$$

Case 5.

$$a_0 = 0, \quad a_1 = -1, \quad b_1 = 0, \quad V = 0, \quad r = 1. \tag{65}$$

Case 6.

$$a_0 = 0, \quad a_1 = -1, \quad b_1 = \frac{1}{2}, \quad V = 0, \quad r = \frac{1}{2}. \tag{66}$$

Case 7.

$$a_0 = \frac{1}{2}, \quad a_1 = -\frac{1}{2}, \quad b_1 = 0, \quad V = 1, \quad r = \frac{1}{2}. \tag{67}$$

Case 8.

$$a_0 = \frac{1}{2}, \quad a_1 = -\frac{1}{2}, \quad b_1 = \frac{1}{4}, \quad V = \frac{1}{2}, \quad r = \frac{1}{4}. \tag{68}$$

Case 9.

$$a_0 = \frac{1}{2}, \quad a_1 = \frac{1}{2}, \quad b_1 = -\frac{1}{2}, \quad V = -\frac{5}{8}, \quad r = \frac{1}{4}. \tag{69}$$

Case 10.

$$a_0 = \frac{1}{2}, \quad a_1 = \frac{1}{2}, \quad b_1 = -\frac{1}{4}, \quad V = -\frac{-5}{16}, \quad r = \frac{1}{8}. \tag{70}$$

Case 11.

$$a_0 = \frac{1}{2}, \quad a_1 = \frac{1}{2}, \quad b_1 = 0, \quad V = -\frac{5}{8}, \quad r = \frac{1}{4}. \tag{71}$$

Substituting (61-71) into (53), we get two types of travelling wave solutions for all cases of the Burgers-Fisher equation (51):

According to

Case 1.

Type 1:

$$u_1(x,t) = \frac{1}{2}(-1 + \tanh[\frac{1}{2}x - t]), \tag{72}$$

Type 2:

$$u_2(x,t) = \frac{1}{2}(-1 + \coth[\frac{1}{2}x - t]). \tag{73}$$

According to

Case 2.

Type 1:

$$u_3(x,t) = \frac{1}{4}(-2 + \coth[\frac{1}{4}x - \frac{1}{2}t] + \tanh[\frac{1}{4}x - \frac{1}{2}t]), \tag{74}$$

Type 2:

$$u_4(x,t) = \frac{1}{4}(-2 + \coth[\frac{1}{4}x - \frac{1}{2}t] + \tanh[\frac{1}{4}x - \frac{1}{2}t]). \tag{75}$$

According to

Case 3.

Type 1:

$$u_5(x,t) = \frac{1}{2}(-1 - \coth[\frac{1}{4}x + \frac{5}{8}t]), \tag{76}$$

Type 2:

$$u_6(x,t) = \frac{1}{2}(-1 - \tanh[\frac{1}{4}x + \frac{5}{8}t]). \tag{77}$$

According to

Case 4.

Type 1:

$$u_7(x,t) = \frac{1}{4}(-2 - \coth[\frac{1}{8}x + \frac{5}{16}t] - \tanh[\frac{1}{8}x + \frac{5}{16}t]), \tag{78}$$

Type 2:

$$u_8(x,t) = \frac{1}{4}(-2 - \coth[\frac{1}{8}x + \frac{5}{16}t] - \tanh[\frac{1}{8}x + \frac{5}{16}t]). \tag{79}$$

According to

Case 5.

Type 1:

$$u_9(x,t) = \tanh[x], \tag{80}$$

Type 2:

$$u_{10}(x,t) = \coth[x]. \tag{81}$$

According to

Case 6.

Type 1:

$$u_{11}(x,t) = \frac{1}{2}(\tanh[\frac{1}{2}x] + \coth[\frac{1}{2}x]), \tag{82}$$

Type 2:

$$u_{12}(x,t) = \frac{1}{2}(\tanh[\frac{1}{2}x] + \coth[\frac{1}{2}x]). \tag{83}$$

According to

Case 7.

Type 1:

$$u_{13}(x,t) = \frac{1}{2}(1 + \tanh[\frac{1}{2}x + t]), \tag{84}$$

Type 2:

$$u_{14}(x,t) = \frac{1}{2}(1 + \coth[\frac{1}{2}x + t]). \tag{85}$$

According to

Case 8.

Type 1:

$$u_{15}(x,t) = \frac{1}{4}(2 + \coth[\frac{1}{4}x + \frac{1}{2}t] + \tanh[\frac{1}{4}x + \frac{1}{2}t]), \tag{86}$$

Type 2:

$$u_{16}(x,t) = \frac{1}{4}(2 + \coth[\frac{1}{4}x + \frac{1}{2}t] + \tanh[\frac{1}{4}x + \frac{1}{2}t]). \tag{87}$$

According to

Case 9.

Type 1:

$$u_{17}(x,t) = \frac{1}{2}(1 - \coth[\frac{1}{4}x - \frac{5}{8}t]), \tag{88}$$

Type 2:

$$u_{18}(x,t) = \frac{1}{2}(1 - \tanh[\frac{1}{4}x - \frac{5}{8}t]). \tag{89}$$

According to

Case 10.

Type 1:

$$u_{19}(x,t) = \frac{1}{4}(2 - \coth[\frac{1}{8}x - \frac{5}{16}t] - \tanh[\frac{1}{8}x - \frac{5}{16}t]), \tag{90}$$

Type 2:

$$u_{20}(x,t) = \frac{1}{4}(2 - \coth[\frac{1}{8}x - \frac{5}{16}t] - \tanh[\frac{1}{8}x - \frac{5}{16}t]). \tag{91}$$

According to

Case 11.

Type 1:

$$u_{21}(x,t)=\frac{1}{2}(1-\tanh[\frac{1}{4}x-\frac{5}{8}t]), \tag{92}$$

Type 2:

$$u_{22}(x,t)=\frac{1}{2}(1-\coth[\frac{1}{4}x-\frac{5}{8}t]). \tag{93}$$

We have represented this solution (74) for a set of parameter value in Figure 2.

3.3. Example 3: The FitzHugh-Nagumo Equation

From the NRD equation (1), take $D(u)=1$ and $R(u)=-au+u^2+au^2-u^3$, we find the FitzHugh-Nagumo equation in the form

$$u_t-u_{xx}+au-u^2-au^2+u^3=0, \quad a\neq 0. \tag{94}$$

To obtain a new travelling wave solution of FitzHugh-Nagumo equation (94), take the transformation $u(x,t)=u(\xi)$, $\xi=rx+Vt$, then FitzHugh-Nagumo equation (94) reduces to

$$Vu'-r^2u''+au-u^2-au^2+u^3=0. \tag{95}$$

Balancing the order of u'' and u^3, we get $m=1$.

Hence the solution of equation (95) as described in step 3, can be expressed as,

$$u(\xi)=a_0-a_1F+b_1(F^{-1}-F). \tag{96}$$

Substituting from (96) into (95), collecting the coefficients of $(F)^i$ ($i=0,\pm1,\pm2,\pm3$), and letting it be zero, yields a set of simultaneous algebraic equations for a_0, a_1, b_1, r and V as follows:

$$F^{-3}:\quad b_1^3-2b_1r^2=0, \tag{97}$$

$$F^{-2}:\quad -Vb_1-b_1^2-ab_1^2+3a_0b_1^2=0, \tag{98}$$

$$F^{-1}:\quad ab_1-2a_0b_1-2aa_0b_1+3a_0^2b_1-3a_1b_1^2-3b_1^3+2b_1r^2=0, \tag{99}$$

$$F^{0}:\quad aa_0-a_0^2-aa_0^2+a_0^3-Va_1+2a_1b_1+2aa_1b_1-6a_0a_1b_1 \tag{100}$$
$$+2b_1^2+2ab_1^2-6a_0b_1^2=0,$$

$$F^{1}:\quad -aa_1+2a_0a_1+2aa_0a_1-3a_0^2a_1-ab_1+2a_0b_1+2aa_0b_1 \tag{101}$$

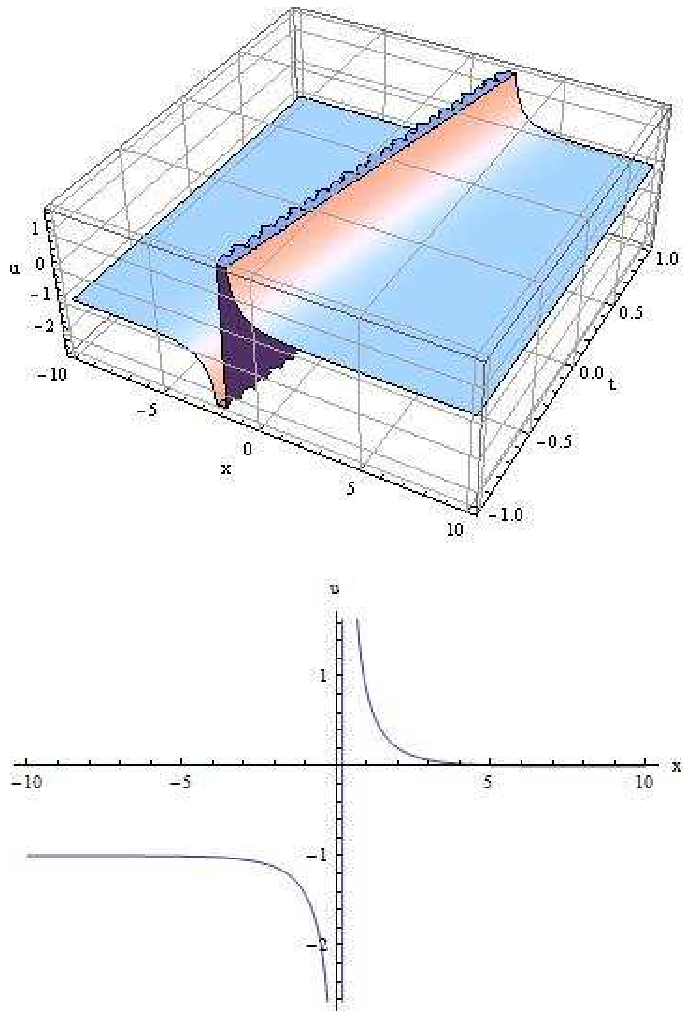

Figure 2: The travelling wave solution $u(x,t)$ of Burgers-Fisher equation defined in equation (74).

$$
\begin{aligned}
& -3a_0^2b_1 + 3a_1^2b_1 + 6a_1b_1^2 + 3b_1^3 - 2a_1r^2 - 2b_1r^2 = 0, \\
F^2: \quad & Va_1 - a_1^2 - aa_1^2 + 3a_0a_1^2 + Vb_1 - 2a_1b_1 - 2aa_1b_1 \qquad (102) \\
& + 6a_0a_1b_1 - b_1^2 - ab_1^2 + 3a_0b_1^2 = 0, \\
F^3: \quad & -a_1^3 - 3a_1^2b_1 - 3a_1b_1^2 - b_1^3 + 2a_1r^2 + 2b_1r^2 = 0. \qquad (103)
\end{aligned}
$$

To solve this set of algebraic equations for a_0, a_1, b_1, r and V by use of Mathematica, we get:

Case 1.

$$a_0 = \frac{1}{2}, \quad a_1 = -\frac{1}{2}, \quad b_1 = 0, \quad V = \frac{1-2a}{4}, \quad r = \mp\frac{1}{2\sqrt{2}}. \tag{104}$$

Case 2.

$$a_0 = \frac{1}{2}, \quad a_1 = -\frac{1}{2}, \quad b_1 = \frac{1}{4}, \quad V = \frac{1-2a}{8}, \quad r = \mp\frac{1}{4\sqrt{2}}. \tag{105}$$

Case 3.

$$a_0 = \frac{1}{2}, \quad a_1 = \frac{1}{2}, \quad b_1 = -\frac{1}{2}, \quad V = \frac{-1+2a}{4}, \quad r = \mp\frac{1}{2\sqrt{2}}. \tag{106}$$

Case 4.

$$a_0 = \frac{a}{2}, \quad a_1 = -\frac{a}{2}, \quad b_1 = 0, \quad V = \frac{-2a+a^2}{4}, \quad r = \mp\frac{a}{2\sqrt{2}}. \tag{107}$$

Case 5.

$$a_0 = \frac{a}{2}, \quad a_1 = \frac{a}{2}, \quad b_1 = -\frac{a}{2}, \quad V = \frac{-2a+a^2}{4}, \quad r = \mp\frac{a}{2\sqrt{2}}. \tag{108}$$

Case 6.

$$a_0 = \frac{1+a}{2}, \quad a_1 = \frac{1-a}{2}, \quad b_1 = 0, \quad V = \frac{-1+a^2}{4}, \quad r = \mp\frac{-1+a}{2\sqrt{2}}. \tag{109}$$

Case 7.

$$a_0 = \frac{1+a}{2},\ a_1 = \frac{1-a}{2},\ b_1 = \frac{-1+a}{4},\ V = \frac{-1+a^2}{8},\ r = \mp\frac{-1+a}{4\sqrt{2}}. \tag{110}$$

Case 8.

$$a_0 = \frac{1+a}{2},\ a_1 = \frac{-1+a}{2},\ b_1 = 0,\ V = \frac{1-a^2}{4},\ r = \mp\frac{-1+a}{2\sqrt{2}}. \tag{111}$$

Case 9.

$$a_0 = \frac{1+a}{2},\ a_1 = \frac{-1+a}{2},\ b_1 = \frac{1-a}{4},\ V = -\frac{-1+a^2}{8},\ r = \mp\frac{-1+a}{4\sqrt{2}}. \tag{112}$$

Case 10.

$$a_0 = \frac{1+a}{2},\ a_1 = \frac{-1+a}{2},\ b_1 = \frac{1-a}{2},\ V = -\frac{-1+a^2}{4},\ r = \mp\frac{-1+a}{2\sqrt{2}}. \tag{113}$$

Substituting (104-113) into (96), we get two types of travelling wave solutions for all cases of the FitzHugh-Nagumo equation (94):

According to

Case 1.

Type 1:

$$u_1(x,t) = \frac{1}{2}(1 + \tanh[\frac{1}{2\sqrt{2}}x + \frac{1-2a}{4}t]), \tag{114}$$

Type 2:

$$u_2(x,t) = \frac{1}{2}(1 + \coth[\frac{1}{2\sqrt{2}}x + \frac{1-2a}{4}t]). \tag{115}$$

where a is constant.

According to

Case 2.

Type 1:

$$u_3(x,t) = \frac{1}{4}(2 + \coth[\frac{1}{4\sqrt{2}}x + \frac{1-2a}{8}t] + \tanh[\frac{1}{4\sqrt{2}}x + \frac{1-2a}{8}t]), \tag{116}$$

Type 2:

$$u_4(x,t) = \frac{1}{4}(2 + \coth[\frac{1}{4\sqrt{2}}x + \frac{1-2a}{8}t] + \tanh[\frac{1}{4\sqrt{2}}x + \frac{1-2a}{8}t]). \tag{117}$$

where a is constant.

According to

Case 3.

Type 1:

$$u_5(x,t) = \frac{1}{2}(1 - \coth[\frac{1}{2\sqrt{2}}x + \frac{-1+2a}{4}t]), \tag{118}$$

Type 2:

$$u_6(x,t) = \frac{1}{2}(1 - \tanh[\frac{1}{2\sqrt{2}}x + \frac{-1+2a}{4}t]). \tag{119}$$

where a is constant.

According to

Case 4.

Type 1:

$$u_7(x,t) = \frac{a}{2}(1 + \tanh[\frac{a}{2\sqrt{2}}x + \frac{-2a+a^2}{4}t]), \tag{120}$$

Type 2:

$$u_8(x,t) = \frac{a}{2}(1 + \coth[\frac{a}{2\sqrt{2}}x + \frac{-2a+a^2}{4}t]). \tag{121}$$

where a is constant.

According to

Case 5.

Type 1:

$$u_9(x,t) = \frac{a}{2}(1 - \coth[\frac{a}{2\sqrt{2}}x - \frac{-2a+a^2}{4}t]), \tag{122}$$

Type 2:

$$u_{10}(x,t) = \frac{a}{2}(1 - \tanh[\frac{a}{2\sqrt{2}}x - \frac{-2a+a^2}{4}t]). \tag{123}$$

where a is constant.

According to

Case 6.

Type 1:

$$u_{11}(x,t) = \frac{1}{2}(1 + a + (-1 + a)\tanh[\frac{-1 + a}{2\sqrt{2}}x + \frac{-1 + a^2}{4}t]), \tag{124}$$

Type 2:

$$u_{12}(x,t) = \frac{1}{2}(1 + a + (-1 + a)\coth[\frac{-1 + a}{2\sqrt{2}}x + \frac{-1 + a^2}{4}t]). \tag{125}$$

where a is constant.

According to

Case 7.

Type 1:

$$\begin{aligned} u_{13}(x,t) = \frac{1}{4}(2 + 2a + (-1 + a)\coth[\frac{-1 + a}{4\sqrt{2}}x + \frac{-1 + a^2}{8}t] \\ + (-1 + a)\tanh[\frac{-1 + a}{4\sqrt{2}}x + \frac{-1 + a^2}{8}t]), \end{aligned} \tag{126}$$

Type 2:

$$\begin{aligned} u_{14}(x,t) = \frac{1}{4}(2 + 2a + (-1 + a)\coth[\frac{-1 + a}{4\sqrt{2}}x + \frac{-1 + a^2}{8}t] \\ + (-1 + a)\tanh[\frac{-1 + a}{4\sqrt{2}}x + \frac{-1 + a^2}{8}t]). \end{aligned} \tag{127}$$

where a is constant.

According to

Case 8.

Type 1:

$$u_{15}(x,t) = \frac{1}{2}(1 + a + (1 - a)\tanh[\frac{-1 + a}{2\sqrt{2}}x + \frac{1 - a^2}{4}t]), \tag{128}$$

Type 2:

$$u_{16}(x,t) = \frac{1}{2}(1 + a + (1-a)\coth[\frac{-1+a}{2\sqrt{2}}x + \frac{1-a^2}{4}t]). \tag{129}$$

where a is constant.

According to

Case 9.

Type 1:

$$u_{17}(x,t) = \frac{1}{4}(2 + 2a + (1-a)\coth[\frac{-1+a}{4\sqrt{2}}x - \frac{-1+a^2}{8}t] + (1-a)\tanh[\frac{-1+a}{4\sqrt{2}}x - \frac{-1+a^2}{8}t]), \tag{130}$$

Type 2:

$$u_{18}(x,t) = \frac{1}{4}(2 + 2a + (1-a)\coth[\frac{-1+a}{4\sqrt{2}}x - \frac{-1+a^2}{8}t] + (1-a)\tanh[\frac{-1+a}{4\sqrt{2}}x - \frac{-1+a^2}{8}t]). \tag{131}$$

where a is constant.

According to

Case 10.

Type 1:

$$u_{19}(x,t) = \frac{1}{2}(1 + a + (1-a)\coth[\frac{-1+a}{2\sqrt{2}}x - \frac{-1+a^2}{4}t]), \tag{132}$$

Type 2:

$$u_{20}(x,t) = \frac{1}{2}(1 + a + (1-a)\tanh[\frac{-1+a}{2\sqrt{2}}x - \frac{-1+a^2}{4}t]). \tag{133}$$

where a is arbitrary constant.

We have represented this solution (124) for a set of parameter value in Figure 3.

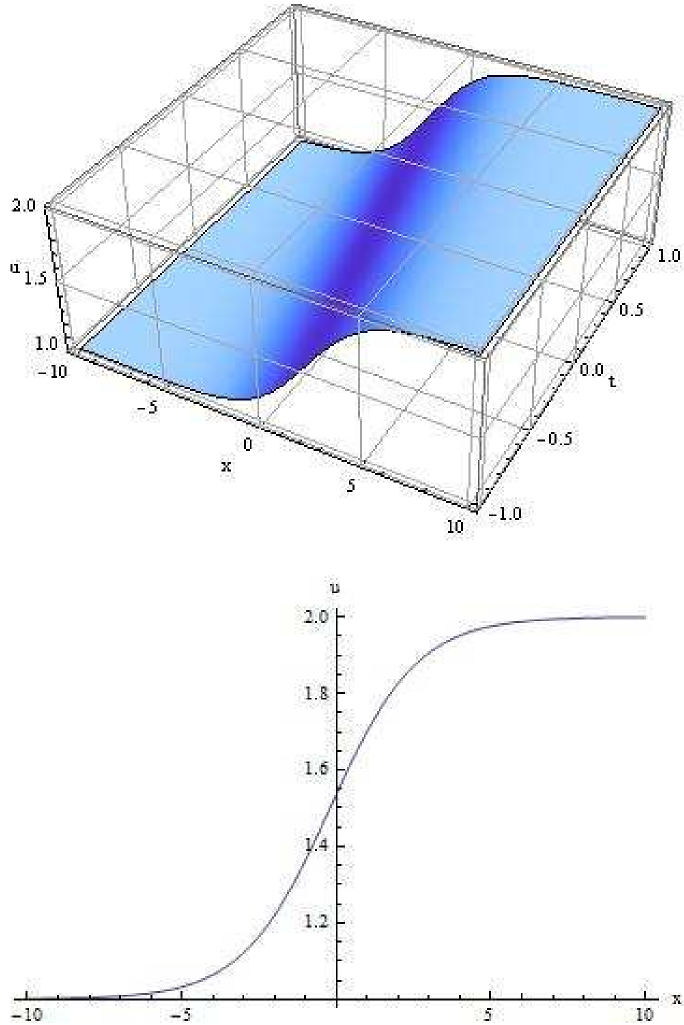

Figure 3: The travelling wave solution $u(x,t)$ of FitzHugh-Nagumo equation defined in equation (124) with $a = 2$.

4. Conclusion

In this article, the variation of (G'/G)-expansion method is developed, in which we have used the full advantage of the well known solution of the coupled Riccati equation. The presented method is used to find new exact travelling wave solutions of the Fisher equation, Burgers-Fisher equation and FitzHugh-

Nagumo equation as examples of NRD model. The obtained results shows that the method is a powerful mathematical tool to solve a huge variety of nonlinear partial differential equations in mathematical physics.

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