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ASYMPTOTICS OF TAIL PROBABILITY FOR MAXIMUM WITH DEPENDENT SUBEXPONENTIAL RANDOM VARIABLES

Xijun Liu[1 §], Chunyan Du[2]
[1,2]Foundation Department
The First Aeronautical College of Air Force
Hangkong Road, Xinyang, 464000, P.R. CHINA

Abstract: Let random variables $X_1, X_2, \cdots, X_n$ be dependent r.v.s on $(-\infty, +\infty)$. This paper is concerned with the asymptotic behavior of tail probabilities of maximum for subexponential random variables, which develops the conclusion of [1].

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1. Introduction

It is well-known that the heavy tailed distribution classes have been extensively investigated in many literatures. See, for example, [1], [2], [3] and references therein. In this paper, we will consider the asymptotics for tail probability of maximum with dependent subexponential random variables. Firstly, we introduce some notions and notations. For two positive functions $a(\cdot)$ and $b(\cdot)$, we write $a(x) \lesssim b(x)$ if $\limsup_{x\to\infty} a(x)/b(x) \le 1$, write $a(x) \gtrsim b(x)$ if $\liminf_{x\to\infty} a(x)/b(x) \ge 1$, write $a(x) \sim b(x)$ if $a(x) \lesssim b(x)$ and $a(x) \gtrsim$

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[§]Correspondence author

$b(x)$, write $a(x) = o(1)b(x)$ if $\lim_{x\to\infty} a(x)/b(x) = 0$, write $a(x) = O(1)b(x)$ if $\limsup_{x\to\infty} a(x)/b(x) < \infty$.

Let F and G be the distributions of random variables X and Y, respectively. $F * G$ denotes the convolution of F and G, in particular, F^{*n} denotes the n-fold convolution of F, $n \geq 1$, $F^{*1} = F$, and F^{*0} is the degenerate distribution at 0.

For a proper distribution F on $(-\infty, \infty)$, denote its tail by $\overline{F}(x) = 1 - F(x)$ and its upper Matuszewska index by

$$J_F^+ = -\lim_{y\to\infty} \frac{\log \overline{F}_*(y)}{\log y} \quad \text{with} \quad \overline{F}_*(y) := \liminf_{x\to\infty} \frac{\overline{F}(yx)}{\overline{F}(x)}, \quad y > 1.$$

Now we present some common heavy-tailed distribution classes. Say that a distribution F on $(-\infty, \infty)$ belongs to the dominated variation class, denoted by $F \in \mathscr{D}$, if $\overline{F}_*(y) > 0$ for any $y > 1$; belongs to the long-tailed class, denoted by $F \in \mathscr{L}$, if $\overline{F}(x+y) \sim \overline{F}(x)$ for any $y > 0$; belongs to the consistent variation class, denoted by $F \in \mathscr{S}$, if $\lim_{x\to\infty} F^{*2}(x)/\overline{F}(x) = 2$; belongs to the $\mathscr{R}$ class of regularly-varying-tailed distributions, if there exist some $0 < \alpha < \infty$ such that $\lim \overline{F}(xy)/\overline{F}(x) = y^\alpha$ for any $y \geq 1$.

It is well known that

$$\mathscr{R} \subset \mathscr{L} \cap \mathscr{D} \subset \mathscr{S} \subset \mathscr{L},$$

For more details on heavy-tailed distributions and their applications, we refer to [4], [5] and [6].

Write $X^+ = \max\{0, X\}, X \wedge Y = \min\{X, Y\}, X \vee Y = \max\{X, Y\}$, $X_{(n)} = \max_{1\leq i\leq n} X_i, S_n = \sum_{i=1}^n X_i, S_{(n)} = \max_{1\leq i\leq n} S_i$.

Now we consider a dependence structure of r.v.s, namely upper tail asymptotic independence (UTAI) structure.

Say r.v.s $\{X_n, n \geq 1\}$ are UTAI, if $P(X_n > x) > 0$ for all $x \in (-\infty, \infty)$, $n \geq 1$, and

$$\lim_{\min\{x_i, x_j\}\to\infty} P(X_i > x_i | X_j > x_j) = 0 \quad \text{for all} \quad 1 \leq i \neq j < \infty. \tag{1.1}$$

If we change (1.1) into the following

$$\lim_{\min\{x_i, x_j\}\to\infty} P(|X_i| > x_i | X_j > x_j) = 0 \quad \text{for all} \quad 1 \leq i \neq j < \infty,$$

then say $\{X_n, n \geq 1\}$ are tail asymptotically independent (TAI, see e.g. [7]).

Remark 1.1. The term UTAI and TAI is named by [8]. Clearly, the UTAI r.v.s can properly cover both negatively dependent and positively dependent r.v.s, see, e.g. Example 3.1 of [8].

In [1] one may obtain the following result.

Theorem A. *Let $X_1, X_2, \cdots, X_n$ be n TAI r.v.s with distributions $F_1, F_2, \cdots, F_n$ on $(-\infty,+\infty)$. If $F_k \in \mathscr{L} \cap \mathscr{D}, 1 \leq k \leq n$, then*

$$P(S_n) \sim \sum_{k=1}^{n} \overline{F}_k(x). \tag{1.2}$$

It is well-known that $\mathscr{L} \cap \mathscr{D} \subset \mathscr{S}$, that is, some important subexponential distributions such as lognormal and Weibull distributions are unfortunately excluded. [1] also gave a result for the case of subexponential marginal distributions. In doing so, they had to strengthen the dependence from TAI to the negative (or positive) regression dependence (NPRD) (see [9]).

Say r.v.s $\{X_n, n \geq 1\}$ are NPRD, if there exist positive constants x_0 and c such that the inequality

$$P(|X_i| > x_i | X_j = x_j, j \in J) \leq c\overline{F}_i(x)$$

holds for all $1 \leq i \leq n, \phi \neq J \subset \{1, 2, \cdots, n\} \backslash \{i\}, x_i > x_0$ and $x_j > x_0, j \in J$.

Theorem B. *Let $X_1, X_2, \cdots, X_n$ be NPRD r.v.s with distributions $F_1, F_2, \cdots, F_n$ on $(-\infty,+\infty)$, respectively. If $F_k \in \mathscr{S}$, $1 \leq k \leq n$ and $F_i * F_j \in \mathscr{S}$, $1 \leq i \neq j \leq n$, then (1.2) still holds.*

In this paper, we consider asymptotics of r.v.s $S_{(n)}$ and $X_{(n)}$ under the dependence structure TAI or NPRD, that is

$$P(S_n) \sim P(S_{(n)}) \sim P(X_{(n)}) \sim \sum_{k=1}^{n} \overline{F}_k(x). \tag{1.3}$$

Clearly, (1.3) holds under the dependence structure TAI, so we omit the proof. Next we give the main result of this paper.

Theorem 1.1. *Let $X_1, X_2, \cdots, X_n$ be NPRD r.v.s with distributions $F_1, F_2, \cdots, F_n$ on $(-\infty,+\infty)$, respectively. Then (1.3) holds if one of the following conditions is true:*

(i)If $F_k \in \mathscr{S}$, $1 \leq k \leq n$ and $\overline{F}_i(x) = O(\overline{F}_j(x))$ or $\overline{F}_j(x) = O(\overline{F}_i(x))$, $1 \leq i, j \leq n$ holds for all $i, j = 1, 2, \cdots, n$;

(ii)If $F_k \in \mathscr{S}$, $1 \leq k \leq n$ and $F_i * F_j \in \mathscr{S}$, $1 \leq i \neq j \leq n$.

The following part of this paper is organized as follows: We will give some lemmas in Section 2, and prove the main result in Section 3.

2. Some Lemmas

In order to prove the main result, we first give some lemmas. For simplicity, let r.v.s X_1^*, $X_2^*, \cdots, X_n^*$ be independent copies of $X_1, X_2, \cdots, X_n$, and write $S_{n,k} = S_n - X_k$, $S_{n,k}^* = S_n^* - X_k^*, 1 \leq k \leq n$.

Lemma 2.1. *Let $X_1, X_2, \cdots, X_n$ be TAI r.v.s with distributions $F_1, F_2, \cdots,$ F_n on $(-\infty, +\infty)$, respectively. Then*

$$P(X_{(n)}) \sim \sum_{k=1}^{n} \overline{F}_k(x). \tag{2.1}$$

Proof. Clearly, by TAI property, we have

$$\begin{aligned} P(X_{(2)}) &= P(X_1 > x) + P(X_2 > x) - P(X_1 > x, X_2 > x) \\ &= P(X_1 > x) + P(X_2 > x) - P(X_1 > x | X_2 > x) P(X_2 > x) \\ &\sim P(X_1 > x) + P(X_2 > x). \end{aligned} \tag{2.2}$$

Therefor, (2.1) holds for $n = 2$.

We assume that (2.1) holds for $n-1$, that is,

$$P(X_{(n-1)}) \sim \sum_{k=1}^{n-1} \overline{F}_k(x). \tag{2.3}$$

Then, by TAI property, (2.2) and (2.3), we have

$$P(X_{(n)}) \sim P(X_{(n-1)} > x) + P(X_n > x) \sim \sum_{k=1}^{n} \overline{F}_k(x).$$

This ends the proof of Lemma 2.1. □

The following lemma is from Lemma 4.3 of [1].

Lemma 2.2. Let $X_1, X_2, \cdots, X_n$ be n TAI r.v.s with distributions $F_1 \in \mathscr{L}, F_2 \in \mathscr{L}, \cdots, F_n \in \mathscr{L}$, respectively. Then

$$P(S_n > x) \gtrsim \sum_{k=1}^{n} \overline{F}_k(x).$$

The following lemma is from Lemma 5.1 of [1].

Lemma 2.3. *Let $X_1, X_2, \cdots, X_n$ be n NPRD r.v.s with distributions $F_1 \in \mathscr{L}, F_2 \in \mathscr{L}, \cdots, F_n \in \mathscr{L}$, respectively. Then there exist positive constants x_0 and d_n such that*

$$P(S_{n,k} > x | X_k = x_k) \leq d_n P(S^*_{n,k} > x).$$

Lemma 2.4. *Let X_1, X_2 be nonnegative and independent r.v.s. If $F_1 \in \mathscr{S}$, $F_2 \in \mathscr{L}$ and $\overline{F}_2(x) = O(\overline{F}_1(x))$, then*

$$P(X_1 + X_2 > x) \sim \overline{F}_1(x) + \overline{F}_2(x).$$

Proof. We first prove that

$$P(X_1 + X_2 > x) \lesssim \overline{F}_1(x) + \overline{F}_2(x). \tag{2.4}$$

By $F_2 \in \mathscr{L}$, we have

$$\begin{aligned} P(X_1 + X_2 > x) &\leq P(X_1 > x) + P(X_2 > x - h(x)) \\ &\quad + P(X_1 + X_2 > x, h(x) < X_1 \leq x) \\ &\sim P(X_1 > x) + P(X_2 > x)) \\ &\quad + \int_{h(x)}^{x} P(X_2 > x - y) dF_1(y). \end{aligned} \tag{2.5}$$

By $F_1 \in \mathscr{S}$ and $\overline{F}_2(x) = O(\overline{F}_1(x))$, we have

$$\begin{aligned} \int_{h(x)}^{x} P(X_2 > x - y) dF_1(y) &= O(1) \int_{h(x)}^{x} P(X_1 > x - y) dF_1(y) \\ &= o(\overline{F}_1(x)). \end{aligned} \tag{2.6}$$

Therefor, by (2.5) and (2.6), the relation (2.4) holds.

Next we prove that

$$P(X_1 + X_2 > x) \gtrsim \overline{F}_1(x) + \overline{F}_2(x). \tag{2.7}$$

Clearly,

$$\begin{aligned} P(X_1 + X_2 > x) &\geq P(X_{(2)} > x) \\ &= \overline{F}_1(x) + \overline{F}_2(x) - \overline{F}_1(x)\overline{F}_2(x) \\ &\sim \overline{F}_1(x) + \overline{F}_2(x). \end{aligned}$$

This ends the proof of Lemma 2.4. □

Lemma 2.5. *Let $X_1, X_2, \cdots, X_n$ be n nonnegative and independent r.v.s with distributions $F_k \in \mathscr{S}, 1 \le k \le n$ and $\overline{F}_i(x) = O(\overline{F}_j(x))$ or $\overline{F}_j(x) = O(\overline{F}_i(x)), 1 \le i, j \le n$ holds, then*

$$P(S_n > x) \sim \sum_{k=1}^{n} \overline{F}_k(x). \tag{2.8}$$

Proof. For $n = 2$, by Lemma 2.4, the relation (2.8) holds. We assume that the relation (2.8) holds for $n-1$, that is,

$$P(S_{n-1} > x) \sim \sum_{k=1}^{n-1} \overline{F}_k(x). \tag{2.9}$$

By $F \in \mathscr{S} \subset \mathscr{L}$, we have $S_{n-1} \in \mathscr{L}$. By $\overline{F}_i(x) = O(\overline{F}_j(x))$ or $\overline{F}_j(x) = O(\overline{F}_i(x))$, we have $\overline{F}_n(x) = O(P(S_{n-1} > x))$ and $P(S_{n-1} > x) = O(\overline{F}_n(x))$. Therefor, by Lemma 2.4 and (2.9), we have

$$P(S_n > x) = P(S_{n-1} + X_n > x) \sim P(S_{n-1} > x) + P(X_n > x) \sim \sum_{k=1}^{n} \overline{F}_k(x).$$

This ends the proof of Lemma 2.5. □

Lemma 2.6. *let $X_1, X_2, \cdots, X_n$ are n nonnegative and independent r.v.s with distributions $F_k \in \mathscr{S}, 1 \le k \le n$ and $\overline{F}_i(x) = O(\overline{F}_j(x))$ or $\overline{F}_j(x) = O(\overline{F}_i(x)), 1 \le i, j \le n$ holds, then*

$$P(S_n^* > x, h(x) < X_j^* \le x) = o(1) \sum_{k=1}^{n} \overline{F}_k(x). \tag{2.10}$$

Proof. Clearly,

$$\begin{aligned}
& P(S_n^* > x, h(x) < X_j^* \le x) \\
\le\ & \int_0^x P(x - y < S_{n,j}^* \le x) dF_j(y) + P(S_{n,j}^* > x)\overline{F}_j(h(x)) \\
=\ & P(S_n^* > x) - P(S_{n,j}^* \vee X_j^* > x) + P(S_{n,j}^* > x)\overline{F}_j(h(x)) \qquad (2.11)
\end{aligned}$$

By Lemma 2.1, Lemma 2.5 and (2.11), we have that the relation (2.10) holds. This ends the proof of Lemma 2.6. □

3. Proof of the Main Result

Proof of Theorem 1.1. (i)By lemma 2.1, we only need to prove that

$$P(S_n > x) \sim P(S_{(n)} > x) \sim \sum_{k=1}^{n} \overline{F}_k(x). \tag{3.1}$$

Firstly, we prove that

$$P(S_n > x) \sim \sum_{k=1}^{n} \overline{F}_k(x).$$

By Lemma 2.2, we only need to prove that

$$P(S_n > x) \lesssim \sum_{k=1}^{n} \overline{F}_k(x).$$

Clearly, $P(S_n > x) \leq P(S_n^+ > x)$. Therefor, Without loss of generality, we assume that $X_k, 1 \leq k \leq n$ are nonnegative r.v.s. By Lemmas 2.5 and 2.6, we have

$$\begin{aligned}
P(S_n > x) &\leq P(\bigcup_{k=1}^{n}(X_k > x - h(x))) + P(S_n > x, \bigcap_{k=1}^{n}(X_k \leq x - h(x))) \\
&\leq \sum_{k=1}^{n} P(X_k > x - h(x)) + P(S_n > x, h(x) < X_{(n)} \leq x - h(x)) \\
&\lesssim \sum_{k=1}^{n} \overline{F}_k(x) + \sum_{k=1}^{n} P(S_n > x, h(x) < X_k \leq x - h(x)) \\
&= \sum_{k=1}^{n} \overline{F}_k(x) + \sum_{k=1}^{n} \int_{h(x)}^{x-h(x)} P(S_{n,k} > x - y | X_k = y) dF_k(y) \\
&\leq \sum_{k=1}^{n} \overline{F}_k(x) + d_n \sum_{k=1}^{n} \int_{h(x)}^{x-h(x)} P(S_{n,k}^* > x - y) dF_k(y) \\
&= \sum_{k=1}^{n} \overline{F}_k(x) + d_n \sum_{k=1}^{n} P(S_n^* > x, h(x) < X_k^* \leq x - h(x)) \\
&\sim \sum_{k=1}^{n} \overline{F}_k(x).
\end{aligned}$$

Next we prove that

$$P(S_{(n)} > x) \sim \sum_{k=1}^{n} \overline{F}_k(x). \tag{3.2}$$

By $P(S_{(n)} > x) \geq P(S_n > x) \sim \sum_{k=1}^{n} \overline{F}_k(x)$ and $P(S_{(n)} > x) \leq P(S_n^+ > x) \sim \sum_{k=1}^{n} \overline{F}_k(x)$, the relation (3.2) holds.

(ii)By Lemma 2.1 and Theorem B, we only need to prove that

$$P(S_{(n)} > x) \sim \sum_{k=1}^{n} \overline{F}_k(x). \tag{3.3}$$

By Theorem B, we have that $P(S_{(n)} > x) \geq P(S_n > x) \sim \sum_{k=1}^{n} \overline{F}_k(x)$ and $P(S_{(n)} > x) \leq P(S_n^+ > x) \sim \sum_{k=1}^{n} \overline{F}_k(x)$. Thus, the relation (3.3) holds. □

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