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TAIL BEHAVIOR OF THE SUPREMUM OF A RANDOM WALK WITH HEAVY-TAILED INCREMENTS AND PERTURBATIONS

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Abstract: Let $\{\xi_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables with a common distribution F and $\{\eta_n, n \geq 1\}$ a sequence of independent normal random variables with zero means and different variances. Set $T_n = \sum_{i=1}^{n} X_i, n \geq 1$ and $T_0 = 0$, where $X_i = \xi_i + \eta_i, 1 \leq i \leq n$. Under conditions that the two sequences of random variables are independent to each other and the integrated tail distribution of F belongs to the subexponential distribution class, we derive the asymptotic tail behavior of the supremum of the partial sums $T_n, n \geq 0$.

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1. Motivation and Main Result of the Paper

Let $\{\xi_n, n \geq 1\}$ be a sequence of independent and identically distributed random

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variables (r.v.s) with a common distribution F and a finite mean $-\mu$, $\mu > 0$. For all $n \geq 0$, let $S_n = \sum_{i=1}^{n} \xi_i$, where a summation over an empty index set is understood as 0. Then $\{S_n, n \geq 0\}$ is usually called a random walk generated by $\{\xi_n, n \geq 1\}$.

An elementary goal in random walk theory is to derive the distribution of the supremum $\sup_{n\geq 0} S_n$. However, it is extremely difficult to get the exact distribution except for some special cases. And in many cases, what we are interested in is, the probability of the supremum exceeding a high level, which, can be interpreted as the ruin probability of the renewal risk model or the maximum waiting time of the GI/G/1 queuing system. Thus many researchers turn to study the asymptotic tail behavior of the supremum.

In order to state our motivation and the main result of this paper, we first introduce some notation, notions and related results. Unless otherwise stated, in this paper a limit is taken as $x \to \infty$. For two positive functions $a(\cdot)$ and $b(\cdot)$, we write $a(x) = O(b(x))$ if $\limsup a(x)/b(x) < \infty$ and $a(x) = o(b(x))$ if $\lim a(x)/b(x) = 0$. Furthermore, we write $a(x) \lesssim b(x)$ if $\limsup a(x)/b(x) \leq 1$, $a(x) \gtrsim b(x)$ if $\liminf a(x)/b(x) \geq 1$ and $a(x) \sim b(x)$ if $\lim a(x)/b(x) = 1$. For two distributions V_1 and V_2, we denote by $V_1 * V_2$ the convolution of V_1 and V_2. In particular, for a distribution V, we denote the n-fold convolution of V by V^{*n}, $n \geq 0$, where $V^{*1} = V$ and V^{*0} is the distribution degenerated at zero. We denote the tail of the distribution V by $\overline{V}(x) = 1 - V(x)$ for any $x \in \mathbb{R}$. If $m_V := \int_0^\infty \overline{V}(y)\mathrm{d}y < \infty$, then we define the integrated tail distribution of V by

$$V^I(x) := m_V^{-1} \int_0^x \overline{V}(y)\mathrm{d}y$$

when $x \geq 0$ and $V^I(x) = 0$ when $x < 0$.

Now we introduce some important distribution classes. For a distribution V, denote its Laplace transform at point $\alpha \in \mathbb{R}$ by

$$\widehat{V}(\alpha) = \int_{-\infty}^{\infty} e^{\alpha x} V(\mathrm{d}x).$$

Write

$$\gamma_V = \sup\{\alpha \geq 0 : \widehat{V}(\alpha) < \infty\}.$$

V is said to be heavy-tailed, denoted by $V \in \mathcal{K}$, if $\gamma_V = 0$. It is obvious that if $m_V = \infty$, then V is heavy-tailed. Further, if $m_V < \infty$, then $V \in \mathcal{K}$ if and only if $V^I \in \mathcal{K}$, see Lemma 2.7 of Foss et al. (2013).

Chistyakov (1964) introduced the following two important distribution classes.

A distribution V supported on $(-\infty,\infty)$ is said to be long-tailed, denoted by $V\in\mathcal{L}$, if $\overline{V}(x)>0$ for all $x\in(-\infty,\infty)$ and $\overline{V}(x-1)\sim\overline{V}(x)$. A distribution V supported on $[0,\infty)$ is said to be subexponential, denoted by $V\in\mathcal{S}$, if $\overline{V}(x)>0$ for all $x\in(-\infty,\infty)$ and $\overline{V^{*2}}(x)\sim 2\overline{V}(x)$.

The class $\mathcal{S}$ contains a lot of common distributions, such as Pareto distributions, log-normal distributions, Benktander Types I and II distributions and Weibull distributions with parameters in $(0,1)$. For excellent discussion on the two distribution classes, we refer the reader to Embrechts et al. (1997), Asmussen (2000) and Foss et al. (2013), among many others.

We now introduce some related results on the supremum of the random walk. It is well-known that if $F^I\in\mathcal{S}$, then

$$P\left(\sup_{n\geq 0} S_n > x\right)\sim \frac{1}{\mu}\int_x^{\infty}\overline{F}(t)\mathrm{d}t. \tag{1.1}$$

For more details and some other equivalent conditions, we refer the reader to Pakes (1975) and Veraverbeke (1977). Further extensions may be found in Bertoin and Doney (1996), Korshunov (1997), and Denisov et al. (2008).

However, in many cases, the assumptions that the increments are independent and identically distributed are too unrealistic. So some researchers tried to remove or weaken these two conditions. See, for instance, Tang and Su (2002), Foss et al. (2007), Wang et al. (2007), Gao and Wang (2009) and Yang et al. (2011), among many others.

Among the above references, Tang and Su (2002) studied the asymptotics of the ruin probability of the ordinary delayed renewal risk model, Gao and Wang (2009) studied the asymptotics of the ruin probability of a random multi-delayed renewal risk model. In the proofs of the two papers, the asymptotic behavior of the supremum of a random walk with non-identically distributed increments was essentially studied. Foss et al. (2007) derived the asymptotics of $\sup_{n\geq 0} S_n$ under conditions that the increments were non-identically distributed and modulated by a regenerative process. Wang et al. (2007) derived the asymptotics of $\sup_{n\geq 0} S_n$ for negatively associated increments. Yang et al. (2011) derived the ruin probability of some generalized renewal risk models. Motivated by the above results, this paper aims to relax the restrictive condition that the summands are identically distributed.

Our model is as follows.

(i) Let $\{\xi_n, n\geq 1\}$ be a sequence of independent and identically distributed r.v.s with a common distribution F and a finite mean $-\mu$, $\mu>0$;

(ii) Let $\{\eta_n, n \geq 1\}$ be a sequence of independent normal r.v.s with zero-means and variances $\sigma_n^2, n \geq 1$;

(iii) The two sequences of r.v.s are independent to each other;

(iv) The n-th increment is $X_n = \xi_n + \eta_n, n \geq 1$;

(v) $T_n = \sum\limits_{i=1}^{n} X_i, n \geq 0$, where we always assume that $T_0 = 0$ in this paper.

The rationality of such a model can be expressed in insurance. For each $n \geq 1$, the increment X_n can be regarded as the net loss of the insurance company in the n-th period, which consists of two parts, namely, the routine part ξ_n, and the unexpected part η_n. The routine part contains the regular claim amount, the average premium income of each period, and some daily expenses of the company. While the unexpected part, may include the fluctuation of the claims (such as the periodic fluctuation caused by seasons or some other reasons) and the expenses or premium caused by accidents. The means of the unexpected parts are zero (since they include both of the unexpected expenses and the unexpected premium), but in different periods, the variances may differ from each other. It might be more and more difficult to control this part as time goes by, so in our main result the variances of individual variables may be allowed to tend to infinite, see Remark 1.1 below. But in general, the variances of the fluctuations should not be too large to ensure that the routine parts play a major role in the risk model. So condition (1.2) below is reasonable in certain sense.

Denote $W(x) = P\left(\sup\limits_{n \geq 0} T_n \leq x\right), x \geq 0$. Our main result is as follows.

Theorem 1.1. *If $F^I \in \mathcal{S}$ and*

$$\limsup_{n \to \infty} \frac{\sum\limits_{i=1}^{n} \sigma_i^2}{n} < \infty, \tag{1.2}$$

then $W \in \mathcal{S}$ and

$$\overline{W}(x) \sim \frac{1}{\mu} \int_x^\infty \overline{F}(t)dt.$$

Remark 1.1. Suppose that $\sigma_n^2 = n$ if $n = 2^k$ for some positive integer k, and $\sigma_n^2 = 1$ otherwise, then (1.2) holds. However, for any positive integer k,

$$P(\eta_{2^k} > \sqrt{k}) = \overline{\Phi}(1),$$

where $\Phi(\cdot)$ denotes the standard normal distribution. So for any k,

$$P(X_{2^k} > \sqrt{k}) > \overline{F}(0)\overline{\Phi}(1),$$

which contradicts condition (D1) of Foss et al. (2007), so Theorem 1.1 cannot be covered by Theorem 2.1 of Foss et al. (2007).

We will present some preliminaries in Section 2 and give the proof of Theorem 1.1 in Section 3.

2. Some Preliminaries

In order to prove Theorem 1.1, we need some lemmas. The first two lemmas are due to Lemma 4 and Proposition 1(i) of Embrechts and Goldie (1979) respectively, which characterize some properties of the subexponential distributions. For further extensions on the two lemmas, we refer the reader to Lemma 2.4 of Pakes (2004).

Lemma 2.1. *Let G_1 and G_2 be two distributions supported on $[0,\infty)$. If $G_1 \in \mathcal{S}$ and $\overline{G_2}(x) \sim c\overline{G_1}(x)$ for some $c \in (0,\infty)$, then $G_2 \in \mathcal{S}$.*

Lemma 2.2. *Let G_1 and G_2 be two distributions supported on $[0,\infty)$. If $G_1 \in \mathcal{S}$ and $\overline{G_2}(x) = o(\overline{G_1}(x))$, then $G_1 * G_2 \in \mathcal{S}$ and $\overline{G_1 * G_2}(x) \sim \overline{G_1}(x)$.*

The following lemma plays an important role in the proof of the main result, which has its own independent interest.

Lemma 2.3. *Let $\{\eta_n, n \geq 1\}$ be a sequence of independent normal r.v.s with zero-means and standard deviations $\sigma_n \geq 0, n \geq 1$. If (1.2) holds, then for any $\varepsilon > 0$, there exists some $\delta > 0$ and $C = C(\delta,\varepsilon)$ such that, for all $x \geq 0$,*

$$P\left(\sup_{n\geq 0}\sum_{i=1}^{n}(\eta_i - \varepsilon) > x\right) < Ce^{-\delta x}. \tag{2.1}$$

Proof. For all $\delta > 0$ and each $n \geq 1$, we have

$$\begin{aligned} E\exp\{\delta\eta_n\} &= \frac{1}{\sqrt{2\pi}\sigma_n}\int_{-\infty}^{\infty}\exp\{\delta y\}\exp\left\{-\frac{y^2}{2{\sigma_n}^2}\right\}\mathrm{d}y \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\{\delta\sigma_n u\}\exp\left\{-\frac{u^2}{2}\right\}\mathrm{d}u \\ &= \exp\left\{\frac{\delta^2\sigma_n^2}{2}\right\}. \end{aligned} \tag{2.2}$$

Choose $C_1 > 0$ such that for all $n \geq 1$, $\sum\limits_{i=1}^{n}\sigma_i^2 \leq C_1 n$. Then for any $\delta > 0$, $\varepsilon > 0$

and $x \geq 0$, by Markov inequality and (2.2), we have

$$\begin{aligned} P\left(\sum_{i=1}^{n} \eta_i > x + n\varepsilon\right) &\leq \exp\{-\delta(x+n\varepsilon)\} \prod_{i=1}^{n} E\exp\{\delta\eta_i\} \\ &\leq \exp\{-\delta(x+n\varepsilon)\} \exp\left\{\frac{\delta^2}{2}\sum_{i=1}^{n}\sigma_i^2\right\} \\ &\leq \exp\left\{\frac{C_1 n\delta^2 - 2n\delta\varepsilon}{2}\right\} \exp\{-\delta x\}. \qquad (2.3) \end{aligned}$$

Letting $\delta < \frac{\varepsilon}{C_1}$ in (2.3), we get

$$\begin{aligned} P\left(\sup_{n\geq 0}\sum_{i=1}^{n}(\eta_i - \varepsilon) > x\right) &\leq \sum_{n=1}^{\infty} \exp\left\{-\frac{n\delta\varepsilon}{2}\right\}\exp\{-\delta x\} \\ &< C\exp\{-\delta x\}, \end{aligned}$$

where $C = \frac{1}{1-\exp\{-\frac{\delta\varepsilon}{2}\}}$. This completes the proof. □

By Lemma 2.3, We immediately get the following result.

Lemma 2.4. *Let $\{\eta_n, n \geq 1\}$ be a sequence of independent normal r.v.s with zero-means and standard deviations $\sigma_n \geq 0, n \geq 1$. If (1.2) holds, then for any $\varepsilon > 0$, there exists some $\delta > 0$ and $C = C(\delta, \varepsilon)$ such that, for all $x \geq 0$,*

$$P\left(\inf_{n\geq 0}\sum_{i=1}^{n}(\eta_i + \varepsilon) < -x\right) < C\exp\{-\delta x\}.$$

3. Proof of the Main Result

We now give the proof of Theorem 1.1.

Proof of Theorem 1.1. We first study the upper limit. For any $\varepsilon \in (0, \mu)$,

$$\begin{aligned} & P\left(\sup_{n\geq 0} T_n > x\right) \\ \leq\ & P\left(\sup_{n\geq 0}\sum_{i=1}^{n}(\xi_i + \varepsilon) + \sup_{n\geq 0}\sum_{i=1}^{n}(\eta_i - \varepsilon) > x\right). \qquad (3.1) \end{aligned}$$

Since $F^I \in \mathcal{S}$, by Lemma 2.1 and (1.1), the distribution of $\sup\limits_{n\geq 0}\sum\limits_{i=1}^{n}(\xi_i+\varepsilon)$ is subexponential and

$$P\left(\sup_{n\geq 0}\sum_{i=1}^{n}(\xi_i+\varepsilon)>x\right) \sim \frac{1}{\mu-\varepsilon}\int_x^\infty \overline{F}(y)\mathrm{d}y. \tag{3.2}$$

By Lemma 2.3, for any given $\varepsilon>0$, there exists some $\delta>0$ and $C=C(\varepsilon,\delta)$ such that (2.1) holds for all $x\geq 0$. Thus by Lemma 1 of Embrechts et al. (1979),

$$P\left(\sup_{n\geq 0}\sum_{i=1}^{n}(\eta_i-\varepsilon)>x\right) = o\left(\overline{F^I}(x)\right). \tag{3.3}$$

Therefore, by Lemma 2.2 and (3.1)-(3.3), we have,

$$\begin{aligned} P\left(\sup_{n\geq 0}T_n>x\right) &\lesssim P\left(\sup_{n\geq 0}\sum_{i=1}^{n}(\xi_i+\varepsilon)>x\right) \\ &\sim \frac{1}{\mu-\varepsilon}\int_x^\infty \overline{F}(y)\mathrm{d}y. \end{aligned} \tag{3.4}$$

By (3.4) and the arbitrariness of ε, we have

$$P\left(\sup_{n\geq 0}T_n>x\right) \lesssim \frac{1}{\mu}\int_x^\infty \overline{F}(y)\mathrm{d}y. \tag{3.5}$$

Now we study the lower limit. By Lemma 2.4, for any $\varepsilon>0$, there exists some $\delta>0$ and $C=C(\varepsilon,\delta)>0$ such that, for all $A>0$,

$$P\left(\inf_{n\geq 0}\sum_{i=1}^{n}(\eta_i+\varepsilon)<-A\right)<C\exp\{-\delta A\},$$

thus we have

$$\begin{aligned} &P\left(\sup_{n\geq 0}T_n>x\right) \\ \geq\ & P\left(\sup_{n\geq 0}\sum_{i=1}^{n}(\xi_i-\varepsilon)+\inf_{n\geq 0}\sum_{i=1}^{n}(\eta_i+\varepsilon)>x\right) \\ \geq\ & P\left(\sup_{n\geq 0}\sum_{i=1}^{n}(\xi_i-\varepsilon)>x+A\right)P\left(\inf_{n\geq 0}\sum_{i=1}^{n}(\eta_i+\varepsilon)\geq -A\right) \end{aligned}$$

$$\geq \ (1 - C\exp\{-\delta A\})P\left(\sup_{n\geq 0}\sum_{i=1}^{n}(\xi_i - \varepsilon) > x + A\right)$$
$$\sim \ (1 - C\exp\{-\delta A\})\frac{1}{\mu+\varepsilon}\int_{x+A}^{\infty}\overline{F}(y)\mathrm{d}y,$$

where we used (1.1) in the last step. Recall that $F^I \in \mathcal{L}$, we have

$$P\left(\sup_{n\geq 0} T_n > x\right) \gtrsim (1 - C\exp\{-\delta A\})\,\frac{1}{\mu+\varepsilon}\int_{x}^{\infty}\overline{F}(y)\mathrm{d}y. \tag{3.6}$$

Letting $A \to \infty$ in (3.6) , we have

$$P\left(\sup_{n\geq 0} T_n > x\right) \ \gtrsim \ \frac{1}{\mu+\varepsilon}\int_{x}^{\infty}\overline{F}(y)\mathrm{d}y. \tag{3.7}$$

By (3.7) and the arbitrariness of ε, we immediately have

$$P\left(\sup_{n\geq 0} T_n > x\right) \ \gtrsim \ \frac{1}{\mu}\int_{x}^{\infty}\overline{F}(y)\mathrm{d}y. \tag{3.8}$$

Combining (3.5) and (3.8), we finish the proof. □

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References

[1] S. Asmussen, *Ruin Probabilities*, World Scientific, Singapore (2000).

[2] J. Bertoin and R. A. Doney, Some asymptotic results for transient random walks, *Adv. Appl. Probab.*, **28**, No. 1 (1996), 207-226, **doi:** 10.2307/1427918.

[3] V. P. Chistyakov, A theorem on sums of independent positive random variables and its applications to branching process, *Theory Probab. Appl.*, **9**, No. 4 (1964), 640-648, **doi:** 10.1137/1109088.

[4] D. Denisov, S. Foss and D. Korshunov, On lower limits and equivalences for distribution tails of randomly stopped sums, *Bernoulli*, **14**, No. 2 (2008), 391-404, **doi:** 10.3150/07-BEJ111.

[5] P. Embrechts, C. M. Goldie and N. Veraverbeke, Subexponentiality and infinite divisibility, *Z. Wahr. Verw. Gebiete*, **49**, No. 3 (1979), 335-347, **doi:** 10.1007/BF00535504.

[6] P. Embrechts, C. Klüppelberg and T. Mikosch, *Modelling Extremal Events for Insurance and Finance*, Springer, New York (1997).

[7] S. Foss, T. Konstantopoulos and S. Zachary, Discrete and continuous time modulated random walks with heavy-tailed increments, *J. Theor. Probab.*, **20**, No. 3 (2007), 581-612, **doi:** 10.1007/s10959-007-0081-2.

[8] S. Foss, D. Korshunov and S. Zachary, *An introduction to heavy-tailed and subexponential distributions*, New York, Springer (2013).

[9] Q. Gao and Y. Wang, Ruin probability and local ruin probability in the random multi-delayed renewal risk model, *Stat. Prob. Letters*, **79**, No. 5 (2009), 588-596, **doi:** 10.1016/j.spl.2008.10.001.

[10] D. Korshunov, On distribution tail of the maximum of a random walk, *Stoch. Proc. Appl.*, **72** (1997), 97-103, **doi:** 10.1016/S0304-4149(97)00060-4.

[11] A. Pakes, On the tails of waiting time distributions, *J. Appl. Prob.*, **7**, No. 3 (1975), 745-789, **doi:** 10.2307/3212870.

[12] A. Pakes, Convolution equivalence and infinite divisibility, *J. Appl. Prob.*, **41**, No. 2 (2004), 407-424, **doi:** 10.1239/jap/1082999075.

[13] Q. Tang and C. Su, Ruin probabilities for large claims in delayed renewal risk model, *Southeast Asian Bull. Math.*, **25**, No. 4 (2002), 735-743, **doi:** 10.1007/s100120200014.

[14] N. Veraverbeke, Asymptotic behavior of Wiener-Hopf factors of a random walk, *Stoch. Proc. Appl.*, **5** (1977), 27-37, **doi:** 10.1016/0304-4149(77)90047-3.

[15] D. Wang, P. Chen, and C. Su, The supremum of random walk with negatively associated and heavy-tailed steps, *Stat. Prob. Letters*, **77**, No. 13 (2007), 1403-1412, **doi:** 10.1016/j.spl.2007.02.018.

[16] Y. Yang, Y. Wang and X. Liu, Asymptotics for ruin probabilities of two kinds of dependent risk models with NLOD inter-arrival times, *J. Syst. Sci. Complex*, **24**, No. 2 (2011), 328-334, **doi:** 10.1007/s11424-011-8107-3.