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A FAMILY OF MITTAG-LEFFLER TYPE FUNCTIONS AND ITS RELATION WITH BASIC SPECIAL FUNCTIONS

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Abstract: In this paper, we propose a new Mittag-Leffler type function named E-function that unifies many forms of Mittag-Leffler type functions and many other special functions, specially a newly defined **generalized sine function**, then derive its relation with some well-known special functions.

AMS Subject Classification: 33E12, 33C20, 33E20
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1. Introduction

Most of the practical problems of science and engineering give solutions in the form of differential and integral equations (Newton's equations of motion, Schrödinger wave equation, etc.). By solving these equations we obtain a new relation of the known and unknown variables in the form of a function that is a more convenient form of the solution of practical problems. This function is called special function [8].

The Fox's H-function [5] is the generalized special function of integer order differential equations since it contains a large number of special functions as special cases, some of these named functions are generalized hypergeometric function, Meijer's G-function, Bessel functions, etc. On the other hand, Mittag-

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Leffler function [7] is recognized to provide solutions of fractional differential and integral equations [6].

The Mittag-Leffler (M-L) function introduced in 1903 due to Gösta Mittag-Leffler is a generalization of the exponential function e^z and later its many generalizations were studied by several authors, it motivates the researcher to unify many of these generalizations to make study convenient.

1.1. Definitions

- In 1903, Gösta Mittag-Leffler [7], the Swedish mathematician introduced the function E_α(z), as follows

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(\alpha n + 1)} z^n \tag{1}$$

where $z, \alpha \in \mathbb{C}; \Re(\alpha) \geq 0 \, and \, |z| < \infty$.

- In 1905, Wiman [10], extended (1) in the form

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(\alpha n + \beta)} z^n \tag{2}$$

where $z, \alpha, \beta \in \mathbb{C}; \Re(\alpha) > 0 \, and \, \Re(\beta) > 0$.

- In 1953, Humbert and Agarwal [2], have studied the properties of a slightly more general function defined by

$$E^{\square}_{\alpha,\beta}(z) = z^{\frac{\beta-1}{\alpha}} \sum_{n=0}^{\infty} \frac{1}{\Gamma(\alpha n + \beta)} z^n \tag{3}$$

where $z, \alpha, \beta \in \mathbb{C}; \Re(\alpha) > 0 \, and \, \Re(\beta) > 0$.

- In 2000, Kiryakova [4], has studied “multiindex M-L functions” defined by

$$E_{(1/\rho_i),(\mu_i)}(z) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(\mu_1 + n/\rho_1) \ldots \Gamma(\mu_m + n/\rho_m)} z^n \tag{4}$$

where $m > 1$, is an integer, $\rho_1, \ldots, \rho_m > 0$ and $\mu_1, \ldots, \mu_m$ are arbitrary real numbers.

- In 2009, Srivastava and Tomovski [9], introduced and studied another generalization of M-L function in the form

$$\breve{E}_{\alpha,\beta}^{\gamma,\delta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n\delta}}{\Gamma(\alpha n + \beta)} \frac{z^n}{n!} \tag{5}$$

where $z, \alpha, \beta, \gamma, \delta \in \mathbb{C}; \Re(\alpha) > max\{0, \Re(\delta) - 1\}, \Re(\beta) > 0, \Re(\gamma) > 0 \, and$ $\Re(\delta) > 0$.

2. Introduction of the *E*-Function

2.1. Definition

We introduce the so-called E-function as follows

$$ {}_{\tau}E_k^h \left[z \middle| \begin{array}{c} (\rho, a); (\gamma_i, q_i, s_i)_{1,h} \\ (\alpha, \beta); (\delta_j, p_j, r_j)_{1,k} \end{array} \right] = {}_{\tau}E_k^h \left[z \middle| \begin{array}{c} (\rho, a); (\gamma_1, q_1, s_1), ..., (\gamma_h, q_h, s_h) \\ (\alpha, \beta); (\delta_1, p_1, r_1), ..., (\delta_k, p_k, r_k) \end{array} \right]$$

$$= \sum_{n=0}^{\infty} \frac{\left[(\gamma_1)_{q_1 n}\right]^{s_1} \left[(\gamma_2)_{q_2 n}\right]^{s_2} ... \left[(\gamma_h)_{q_h n}\right]^{s_h} (-1)^{\rho n} z^{an+\tau}}{\left[(\delta_1)_{p_1 n}\right]^{r_1} \left[(\delta_2)_{p_2 n}\right]^{r_2} ... \left[(\delta_k)_{p_k n}\right]^{r_k} \Gamma(\alpha n + \beta)} \tag{6}$$

where

$$z, \alpha, \beta, \gamma_i, \delta_j \in \mathbb{C}; \Re(\alpha) \geq 0, \Re(\beta) > 0, \Re(\gamma_i) > 0, \Re(\delta_j) > 0, \Re(q_i) \geq 0,$$

$$\Re(p_j) \geq 0; s_i, r_j, a, \tau \in \mathbb{R}; \rho \in \{0, 1\}, \left(\sum_{i=1}^{h} q_i s_i < \sum_{j=1}^{k} p_j r_j + \Re(\alpha) \right) or$$

$$\left(\sum_{i=1}^{h} q_i s_i = \sum_{j=1}^{k} p_j r_j + \Re(\alpha) \, when \prod_{i=1}^{h} (q_i)^{q_i s_i} \left[\alpha^{\alpha} \prod_{j=1}^{k} (p_j)^{p_j r_j} \right]^{-1} |z^a| < 1 \right)$$

$$for \;\; i = 1, 2, ..., h; j = 1, 2, ..., k. \tag{7}$$

Theorem 1. *Let convergence conditions (7) are satisfied then the E-function ${}_{\tau}E_k^h[z]$ can be represented as the Mellin-Barnes type contour integral as follows*

$$ {}_{\tau}E_k^h\left[z\left|\begin{array}{c}(\rho,a);(\gamma_1,q_1,s_1),...,(\gamma_h,q_h,s_h)\\(\alpha,\beta);(\delta_1,p_1,r_1),...,(\delta_k,p_k,r_k)\end{array}\right.\right]=\frac{\prod\limits_{v=1}^{k}[\Gamma(\delta_v)]^{r_v}}{\prod\limits_{u=1}^{h}[\Gamma(\gamma_u)]^{s_u}}\times $$

$$ \times\frac{z^{\tau}}{2\pi i}\int_{\mathsf{L}}\frac{\Gamma(\zeta)\Gamma(1-\zeta)\prod\limits_{i=1}^{h}[\Gamma(\gamma_i-q_i\zeta)]^{s_i}}{\Gamma(\beta-\alpha\zeta)\prod\limits_{j=1}^{k}\left[\Gamma\left(\delta_j-p_j\zeta\right)\right]^{r_j}}\left[(-1)^{\rho}(-z^a)\right]^{-\zeta}d\zeta \qquad (8) $$

where $\mathcal{L}$ is a suitable contour of integration that runs from $c-i\infty$ to $c+i\infty, c\in\mathbb{R}$ and intended to separate the poles of the integrand at $\zeta=-n$ for all $n\in\mathbb{N}_0\,(to\,the\,left)$ from those at $\zeta=n+1$ and at $\zeta=\frac{\gamma_i+n}{q_i}, i=1,2,...,h;$ for all $n\in\mathbb{N}_0\,(to\,the\,right)$.

Corollary 2. *Let convergence conditions (7) are satisfied then the E-function ${}_{\tau}E_k^h[z]$ can be represented as the Mellin-Barnes type contour integral as follows*

$$ {}_{\tau}E_k^h\left[z\left|\begin{array}{c}(\rho,a),(\gamma_1,q_1,s_1),...,(\gamma_h,q_h,s_h)\\(\alpha,\beta),(\delta_1,p_1,r_1),...,(\delta_k,p_k,r_k)\end{array}\right.\right]=\frac{\prod\limits_{v=1}^{k}[\Gamma(\delta_v)]^{r_v}}{\prod\limits_{u=1}^{h}[\Gamma(\gamma_u)]^{s_u}}\times $$

$$ \times\frac{(\rho+1)z^{\tau}}{2\pi i}\int_{\mathsf{L}}\frac{\Gamma[(\rho+1)\zeta]\Gamma[1-(\rho+1)\zeta]\prod\limits_{i=1}^{h}[\Gamma(\gamma_i-q_i\zeta)]^{s_i}}{\Gamma(\beta-\alpha\zeta)\prod\limits_{j=1}^{k}\left[\Gamma\left(\delta_j-p_j\zeta\right)\right]^{r_j}}(-z^a)^{-\zeta}d\zeta \quad (9) $$

where $\mathcal{L}$ is a suitable contour of integration that runs from $c-i\infty$ to $c+i\infty, c\in\mathbb{R}$ and intended to separate the poles of the integrand at $\zeta=-\frac{n}{\rho+1}$ for all $n\in\mathbb{N}_0\,(to\,the\,left)$ from those at $\zeta=\frac{n+1}{\rho+1}$ and at $\zeta=\frac{\gamma_i+n}{q_i}, i=1,2,...,h;$ for all $n\in\mathbb{N}_0\,(to\,the\,right)$.

2.2. Special Cases

1. For $h = 1, s_1 = 0$; $k = 1, r_1 = 0$, we get **generalized sine function** as

$$ {}_{\tau}E_1^1 \left[z \Big| \begin{array}{c} (\rho, a); (\gamma_1, q_1, 0) \\ (\alpha, \beta); (\delta_1, p_1, 0) \end{array} \right] = \sum_{n=0}^{\infty} (-1)^{\rho n} \frac{z^{an+\tau}}{\Gamma(\alpha n + \beta)} = sin^{\square}(z) \quad (10) $$

where $sin^{\square}(z)$ is the newly defined generalization of sine function.

3. Relation with Basic Special Functions

Theorem 3. (Generalized Hypergeometric Function) *Let condition* (7) *is satisfied with restriction* $s_i, r_j \in \mathbb{N}_0$, $i = 1, 2, ..., h; j = 1, 2, ..., k$ *then the E-function can be written as follows*

$$ {}_{\tau}E_k^h \left[z \Big| \begin{array}{c} (\rho, a); (\gamma_1, q_1, s_1), ..., (\gamma_h, q_h, s_h) \\ (\alpha, \beta); (\delta_1, p_1, r_1), ..., (\delta_k, p_k, r_k) \end{array} \right] $$

$$ = \frac{z^{\tau}}{\Gamma(\beta)} {}_{q^{\square}}F_{p^{\square}} \left[\begin{array}{c} [\Delta(q_i, \gamma_i)^{s_i}]_{1,h}, 1; \\ \Delta(\alpha, \beta), \left[\Delta(p_j, \delta_j)^{r_j}\right]_{1,k}; \end{array} \frac{z^a (-1)^{\rho} \prod_{i=1}^{h} (q_i)^{q_i s_i}}{(\alpha)^{\alpha} \prod_{j=1}^{k} (p_j)^{p_j r_j}} \right] \quad (11) $$

where

$$ q^{\square} = \sum_{i=1}^{h} q_i s_i + 1, p^{\square} = \sum_{j=1}^{k} r_j p_j + \alpha; \Delta(\alpha, \beta) = \frac{\beta}{\alpha}, \frac{\beta+1}{\alpha}, \frac{\beta+2}{\alpha}, ..., \frac{\beta+\alpha-1}{\alpha}; $$

$$ [\Delta(q_i, \gamma_i)^{s_i}]_{1,h} = \overbrace{\Delta(q_1, \gamma_1), \ldots, \Delta(q_1, \gamma_1)}^{s_1\, times}, \ldots\ldots, \overbrace{\Delta(q_h, \gamma_h), \ldots, \Delta(q_h, \gamma_h)}^{s_h\, times} $$

and

$$ \left[\Delta(p_j, \delta_j)^{r_j}\right]_{1,k} = \underbrace{\Delta(p_1, \delta_1), \ldots, \Delta(p_1, \delta_1)}_{r_1\, times}, \ldots\ldots, \underbrace{\Delta(p_k, \delta_k), \ldots, \Delta(p_k, \delta_k)}_{r_k\, times}. \quad (12) $$

Proof. The E-function is defined by (6) as follows

$$
{}_{\tau}E_k^h\left[z\left|\begin{array}{l}(\rho,a);(\gamma_1,q_1,s_1),...,(\gamma_h,q_h,s_h)\\(\alpha,\beta);(\delta_1,p_1,r_1),...,(\delta_k,p_k,r_k)\end{array}\right.\right]
$$

$$
=\sum_{n=0}^{\infty}\frac{\left[(\gamma_1)_{q_1n}\right]^{s_1}\left[(\gamma_2)_{q_2n}\right]^{s_2}...\left[(\gamma_h)_{q_hn}\right]^{s_h}(-1)^{\rho n}z^{an+\tau}}{\left[(\delta_1)_{p_1n}\right]^{r_1}\left[(\delta_2)_{p_2n}\right]^{r_2}...\left[(\delta_k)_{p_kn}\right]^{r_k}\Gamma(\alpha n+\beta)} \tag{13}
$$

Now by comparing (13) with definition of generalized hypergeometric function [8], we get

$$
=\frac{z^{\tau}}{\Gamma(\beta)}{}_{q^{\prime}}F_{p^{\prime}}\left[\begin{array}{c}[\Delta(q_i,\gamma_i)^{s_i}]_{1,h},1;\\\Delta(\alpha,\beta),\left[\Delta(p_j,\delta_j)^{r_j}\right]_{1,k};\end{array}\frac{z^a(-1)^{\rho}\prod_{i=1}^{h}(q_i)^{q_is_i}}{(\alpha)^{\alpha}\prod_{j=1}^{k}(p_j)^{p_jr_j}}\right] \tag{14}
$$

where

$$
q^{\prime}=\sum_{i=1}^{h}q_is_i+1,p^{\prime}=\sum_{j=1}^{k}r_jp_j+\alpha;\Delta(\alpha,\beta)=\frac{\beta}{\alpha},\frac{\beta+1}{\alpha},\frac{\beta+2}{\alpha},...,\frac{\beta+\alpha-1}{\alpha};
$$

$$
[\Delta(q_i,\gamma_i)^{s_i}]_{1,h}=\overbrace{\Delta(q_1,\gamma_1),\ldots,\Delta(q_1,\gamma_1)}^{s_1\,times},\ldots\ldots,\overbrace{\Delta(q_h,\gamma_h),\ldots,\Delta(q_h,\gamma_h)}^{s_h\,times}
$$

and

$$
\left[\Delta(p_j,\delta_j)^{r_j}\right]_{1,k}=\underbrace{\Delta(p_1,\delta_1),\ldots,\Delta(p_1,\delta_1)}_{r_1\,times},\ldots\ldots,\underbrace{\Delta(p_k,\delta_k),\ldots,\Delta(p_k,\delta_k)}_{r_k\,times}. \tag{15}
$$

□

Theorem 4. (Fox's H-Function and $\overline{H}$-Function) *Let condition* (7) *is satisfied with restriction* $s_i,r_j\in\mathbb{N}_0$, $i=1,2,...,h;j=1,2,...,k$ *then the E-function can be written as follows*

$$
{}_{\tau}E_k^h\left[z\left|\begin{array}{l}(\rho,a);(\gamma_1,q_1,s_1),...,(\gamma_h,q_h,s_h)\\(\alpha,\beta);(\delta_1,p_1,r_1),...,(\delta_k,p_k,r_k)\end{array}\right.\right]=z^{\tau}\frac{\prod_{m=1}^{k}[\Gamma(\delta_m)]^{r_m}}{\prod_{l=1}^{h}[\Gamma(\gamma_l)]^{s_l}}\times
$$

$$\times H^{1,n^*}_{n^*,q^*}\left[(-1)^{\rho}\left(-z^{a}\right)\Big|\begin{array}{c}(0,1),(A,B)\\(0,1),(1-\beta,\alpha),(C,D)\end{array}\right] \tag{16}$$

where

$$(A,B)=\overbrace{(1-\gamma_1,q_1),\ldots,(1-\gamma_1,q_1)}^{s_1\,times},\ldots\ldots,\overbrace{(1-\gamma_h,q_h),\ldots,(1-\gamma_h,q_h)}^{s_h\,times},$$

$$(C,D)=\underbrace{(1-\delta_1,p_1),\ldots,(1-\delta_1,p_1)}_{r_1\,times},\ldots\ldots,\underbrace{(1-\delta_k,p_k),\ldots,(1-\delta_k,p_k)}_{r_k\,times},$$

$$n^{*}=\sum_{i=1}^{h}s_i+1 \quad and \quad q^{*}=\sum_{j=1}^{k}r_j+2. \tag{17}$$

Also, let condition (7) is satisfied then the E-function can be written as follows

$${}_{\tau}E^{h}_{k}\left[z\Big|\begin{array}{c}(\rho,a);(\gamma_1,q_1,s_1),...,(\gamma_h,q_h,s_h)\\(\alpha,\beta);(\delta_1,p_1,r_1),...,(\delta_k,p_k,r_k)\end{array}\right]=z^{\tau}\frac{\prod\limits_{m=1}^{k}\left[\Gamma\left(\delta_m\right)\right]^{r_m}}{\prod\limits_{l=1}^{h}\left[\Gamma\left(\gamma_l\right)\right]^{s_l}}\times$$

$$\times\overline{H}^{1,h+1}_{h+1,k+2}\left[(-1)^{\rho}\left(-z^{a}\right)\Big|\begin{array}{c}(0,1;1),(1-\gamma_i,q_i;s_i)_1^h;\text{—}\\(0,1);(1-\beta,\alpha;1),\left(1-\delta_j,p_j;r_j\right)_1^k\end{array}\right]. \tag{18}$$

Proof. Using (8) the E-function ${}_{\tau}E^{h}_{k}[z]$ can be written as follows

$${}_{\tau}E^{h}_{k}\left[z\Big|\begin{array}{c}(\rho,a);(\gamma_1,q_1,s_1),...,(\gamma_h,q_h,s_h)\\(\alpha,\beta);(\delta_1,p_1,r_1),...,(\delta_k,p_k,r_k)\end{array}\right]$$

$$=\frac{\prod\limits_{m=1}^{k}\left[\Gamma\left(\delta_m\right)\right]^{r_m}}{\prod\limits_{l=1}^{h}\left[\Gamma\left(\gamma_l\right)\right]^{s_l}}\frac{z^{\tau}}{2\pi i}\int_{\mathsf{L}}g(\zeta)\left\{(-1)^{\rho}\left(-z^{a}\right)\right\}^{-\zeta}d\zeta \tag{19}$$

where

$$g(\zeta)=\frac{\Gamma(0+\zeta)\,\Gamma\{1-0-\zeta\}\prod\limits_{i=1}^{h}\left[\Gamma\{1-(1-\gamma_i)-q_i\zeta\}\right]^{s_i}}{\Gamma\{1-(1-\beta)-\alpha\zeta\}\prod\limits_{j=1}^{k}\left[\Gamma\left\{1-\left(1-\delta_j\right)-p_j\zeta\right\}\right]^{r_j}} \tag{20}$$

Now by comparing (19) with definition of H-function [8], we get

$$L.H.S. = z^{\tau} \frac{\prod_{m=1}^{k} [\Gamma(\delta_m)]^{r_m}}{\prod_{l=1}^{h} [\Gamma(\gamma_l)]^{s_l}} H^{1,n^*}_{n^*,q^*}\left[(-1)^{\rho}(-z^a) \Big| \begin{matrix} (0,1),(A,B) \\ (0,1),(1-\beta,\alpha),(C,D) \end{matrix}\right] \tag{21}$$

where

$$(A,B) = \overbrace{(1-\gamma_1,q_1),\ldots,(1-\gamma_1,q_1)}^{s_1\, times},\ldots\ldots,\overbrace{(1-\gamma_h,q_h),\ldots,(1-\gamma_h,q_h)}^{s_h\, times},$$

$$(C,D) = \underbrace{(1-\delta_1,p_1),\ldots,(1-\delta_1,p_1)}_{r_1\, times},\ldots\ldots,\underbrace{(1-\delta_k,p_k),\ldots,(1-\delta_k,p_k)}_{r_k\, times},$$

$$n^{\square} = \sum_{i=1}^{h} s_i + 1 \quad \text{and} \quad q^{\square} = \sum_{j=1}^{k} r_j + 2. \tag{22}$$

Again by comparing (19) with definition of $\overline{H}$-function [3], we get

$$L.H.S. = z^{\tau} \frac{\prod_{m=1}^{k} [\Gamma(\delta_m)]^{r_m}}{\prod_{l=1}^{h} [\Gamma(\gamma_l)]^{s_l}} \overline{H}^{1,h+1}_{h+1,k+2}\left[(-1)^{\rho}(-z^a) \Big| \begin{matrix} (0,1;1),(1-\gamma_i,q_i;s_i)_1^h;— \\ (0,1);(1-\beta,\alpha;1),\left(1-\delta_j,p_j;r_j\right)_1^k \end{matrix}\right]. \tag{23}$$

$\square$

Theorem 5. (*Wright Function*) *Let condition* (7) *is satisfied with restriction* $s_i, r_j \in \mathbb{N}_0$, $i = 1,2,...,h; j = 1,2,...,k$ *then the* E*-function can be written as follows*

$${}_{\tau}E^h_k\left[z \Big| \begin{matrix} (\rho,a);(\gamma_1,q_1,s_1),...,(\gamma_h,q_h,s_h) \\ (\alpha,\beta);(\delta_1,p_1,r_1),...,(\delta_k,p_k,r_k) \end{matrix}\right] = z^{\tau} \frac{\prod_{m=1}^{k} [\Gamma(\delta_m)]^{r_m}}{\prod_{l=1}^{h} [\Gamma(\gamma_l)]^{s_l}} \times$$

$$\times {}_{p^*}\Psi_{q^*}\left[\begin{array}{l} (1,1),\overbrace{(\gamma_1,q_1),\ldots,(\gamma_1,q_1)}^{s_1\,times},\ldots\ldots,\overbrace{(\gamma_h,q_h),\ldots,(\gamma_h,q_h)}^{s_h\,times}; \\ (\beta,\alpha),\underbrace{(\delta_1,p_1),\ldots,(\delta_1,p_1)}_{r_1\,times},\ldots\ldots,\underbrace{(\delta_k,p_k),\ldots,(\delta_k,p_k)}_{r_k\,times}; \end{array} (-1)^{\rho} z^{a}\right] \tag{24}$$

$$\text{where} \quad p^{\square} = \sum_{i=1}^{h} s_i + 1 \quad \text{and} \quad q^{\square} = \sum_{j=1}^{k} r_j + 1.$$

Proof. The E-function is defined by (6) as follows

$${}_{\tau}E_k^h\left[z\left|\begin{array}{l}(\rho,a);(\gamma_1,q_1,s_1),...,(\gamma_h,q_h,s_h)\\(\alpha,\beta);(\delta_1,p_1,r_1),...,(\delta_k,p_k,r_k)\end{array}\right.\right]$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{\rho n}\prod_{i=1}^{h}\left[(\gamma_i)_{q_i n}\right]^{s_i}}{\Gamma(\alpha n+\beta)\prod_{j=1}^{k}\left[(\delta_j)_{p_j n}\right]^{r_j}} z^{an+\tau} \tag{25}$$

$$= z^{\tau}\frac{\prod_{m=1}^{k}\left[\Gamma(\delta_m)\right]^{r_m}}{\prod_{l=1}^{h}\left[\Gamma(\gamma_l)\right]^{s_l}}\sum_{n=0}^{\infty}\frac{(-1)^{\rho n}\Gamma(1+n)\prod_{i=1}^{h}\left[\Gamma(\gamma_i+q_i n)\right]^{s_i}}{\Gamma(\alpha n+\beta)\prod_{j=1}^{k}\left[\Gamma(\delta_j+p_j n)\right]^{r_j}}\frac{z^{an}}{n!} \tag{26}$$

Now by comparing (26) with definition of Fox-Wright function [1], we get

$$L.H.S. = z^{\tau}\frac{\prod_{m=1}^{k}\left[\Gamma(\delta_m)\right]^{r_m}}{\prod_{l=1}^{h}\left[\Gamma(\gamma_l)\right]^{s_l}}\times$$

$$\times {}_{p^*}\Psi_{q^*}\left[\begin{array}{l} (1,1),\overbrace{(\gamma_1,q_1),\ldots,(\gamma_1,q_1)}^{s_1\,times},\ldots\ldots,\overbrace{(\gamma_h,q_h),\ldots,(\gamma_h,q_h)}^{s_h\,times}; \\ (\beta,\alpha),\underbrace{(\delta_1,p_1),\ldots,(\delta_1,p_1)}_{r_1\,times},\ldots\ldots,\underbrace{(\delta_k,p_k),\ldots,(\delta_k,p_k)}_{r_k\,times}; \end{array} (-1)^{\rho} z^{a}\right] \tag{27}$$

$$\text{where} \quad p^{\square} = \sum_{i=1}^{h} s_i + 1 \quad \text{and} \quad q^{\square} = \sum_{j=1}^{k} r_j + 1.$$

□

4. Concluding Remarks

The present paper provides a scope of defining M-L function of many parameters as a Mathematica function that will enable the researchers to solve more complex problems using Mathematica. Currently it is implemented in Mathematica as MittagLefflerE[α, z] and MittagLefflerE[α, β, z] for the M-L functions $\mathrm{E}_\alpha(z)$ and $\mathrm{E}_{\alpha,\beta}(z)$ respectively.

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