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MAXIMIZATION PROPERTIES OF AVERAGE PRODUCTION FUNCTION

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Abstract: It is well known that under classical economics assumptions production functions are concave [1]. Average production maximization as a classical economics problem has been studied in fundamental textbooks [1, **?**, 7] and in the literature [2, 3, 5, 6, 9]. However, it seems that less attention so far has been paid to properties of the average production function and its maximization methods. Aim of this paper is to fulfill this gap. First, we show that the average production functions are pseudoconcave. Second, we develop an algorithm for solving the average production maximization problem. We implement the algorithm for a mongolian company.

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1. Introduction

Production functions play important role in the firm theory. Cost function is defined with help of a production function as a solution of the following cost

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minimization problem subject to a given level of production.

$$\min(r_1y_1 + r_2y_2 + ... + r_ny_n), \tag{1}$$

subject to:

$$f(y_1, y_2, ..., y_n) = x, \tag{2}$$

where $f : \mathbb{R}^+ \to \mathbb{R}$ is a continuously differentiable strictly concave production function, $r_i(i = 1, 2, ..., n)$ are factor prices, x is output, $y_i(i = 1, 2, ..., n)$ are factors.

The solutions $y_i^*(r_1, r_2, ..., r_n, x), i = 1, 2, ..., n,$ to problem (1.1)-(1.2) constitute the cost function $\overline{C}$ as follows:

$$\overline{C}(x, r_1, r_2, ..., r_n) = \sum_{i=1}^{n} r_i y_i^*(r_1, r_2, ..., r_n, x)$$

If factor prices are fixed, i.e, $r_i = r_i^0$, $i = 1, 2, ..., n$ then one variable function $\overline{C}(x, r_1, r_2, ..., r_n)$ is a called as a short run cost function defined as:

$$C(x) = \overline{C}(x, r_1^0, r_2^0, ..., r_n^0).$$

For example, if the production function is of type

$$f(y_1, y_2, ..., y_n) = y_1^{\alpha_1}, y_2^{\alpha_2}, ..., y_n^{\alpha_n} \qquad \left(\sum_{i=1}^{n} \alpha_i \leqslant 1, \quad \alpha_i \geqslant 0\right),$$

then the cost functiom $C(x)$ can be easily found as:

$$C(x) = \left[x \cdot \prod_{i=1}^{n} \frac{r_i^0}{\alpha_i}\right]^{\frac{1}{n}} \cdot \sum_{i=1}^{n} \alpha_i.$$

2. Average Production Function

We consider some properties of the average production function $\varphi(x)$ defined as

$$\varphi(x) = \frac{f(x)}{x}, \quad x \in \mathbb{R}^+,$$

where $\mathbb{R}^+ = \{x \in \mathbb{R} | x > 0\}$.

Definition 2.1. [8] A differentiable function $h : \mathbb{R}^+ \to \mathbb{R}$ is pseudoconcave at $y \in \mathbb{R}^+$ if $h(x) - h(y) > 0$ implies $h'(y)(x - y) > 0$, $\forall x \in \mathbb{R}^+$. A function $h(\cdot)$ is pseudoconcave on $\mathbb{R}^+$ if it is pseudoconcave at each point $y \in \mathbb{R}^+$.

Lemma 2.1. *The average production function $\varphi(x)$ is pseudoconcave on $\mathbb{R}^+$.*

Proof. It is clear that $f(x) > 0$ for all $x \in \mathbb{R}^+$. Take any point $y \in \mathbb{R}^+$. Introduce the function $\psi(x) : \mathbb{R}^+ \to \mathbb{R}$ as follows:

$$\psi(x) = f(x)y - xf(y).$$

Clearly, $f(x)$ is concave and differentiable and $\psi(y) = 0$. On the other hand, it is obvious that the inequality $\varphi(x) > \varphi(y)$ is equivalent to $\psi(x) > \psi(y)$.

Since $\psi(\cdot)$ is concave and the differentiable, then we have:

$$0 < \psi(x) - \psi(y) \leqslant \psi'(y)(x - y).$$

Taking into account

$$\varphi'(y)(x - y) = \left\{ \frac{f'(y)y - f(y)}{y^2} \right\} (x - y) = \frac{\psi(y)(x - y)}{y^2} > 0,$$

we conclude that $\varphi(x) > \varphi(y)$ implies $\varphi'(y)(x - y) > 0$ which proves the assertion. ☐

3. Average Production Maximization Problem

Consider the average production maximization problem:

$$\max_{x \in \mathbb{R}^+} \left\{ \varphi(x) = \frac{f(x)}{x} \right\} \tag{3}$$

Lemma 3.1. *Any local maximizer x^* of $\varphi(x)$ on $\mathbb{R}^+$ is also a global maximizer.*

Proof. On the contrary, assume that x^* is not a global maximizer. Then there exists a point $u \in \mathbb{R}^+$ such that

$$\varphi(x^*) < \varphi(u). \tag{4}$$

Since $\mathbb{R}^+$ is a convex set, $x^*+\alpha(u-x^*) = \alpha u+(1-\alpha)x^* \in \mathbb{R}^+, \ \forall\alpha: \ 0 < \alpha < 1$.

By Taylor's expansion, we have

$$\varphi(x^* + \alpha(u - x^*)) = \varphi(x^*) + \alpha\varphi'(x^*)(u - x^*) + 0(\alpha\|u - x^*\|),$$

where

$$\lim_{\alpha\to 0+} \frac{0(\alpha\|u - x^*\|)}{\alpha} = 0.$$

Since x^* is a local maximizer of $\varphi(\cdot)$ on $\mathbb{R}^+$, there exists $0 < \alpha^* < 1$, so that

$$\varphi(x^* + \alpha(u - x^*)) - \varphi(x^*) < 0, \quad \forall\alpha, \ 0 < \alpha < \alpha^*.$$

which implies

$$\varphi'(x^*)(u - x^*) < 0.$$

Since $\varphi(\cdot)$ is pseudoconcave, $\varphi'(x^*)(u - x^*) < 0$ implies that $\varphi(u) < \varphi(x^*)$ contradicting $\varphi(x^*) < \varphi(u)$. This completes the proof. □

Lemma 3.2. *The average production maximization problem (3.1) has a unique solution.*

Proof. On the contrary, assume that there exist $x_1, x_2 \in \mathbb{R}^+$ such that $x_1 < x_2$ and

$$\max_{x\in\mathbb{R}^+} \varphi(x) = \varphi(x_1) = \varphi(x_2).$$

Take any point $x \in (x_1, x_2)$. Since $\varphi(x)$ is pseudoconcave, $\varphi(x_1) > \varphi(x)$ implies that $\varphi'(x)(x_1 - x) > 0$ or equivalently, $\varphi'(x) < 0$.

On the other hand, $\varphi(x_2) > \varphi(x)$ implies also $\varphi'(x)(x_2 - x) > 0$ or equivalently, $\varphi'(x) > 0$ which contradicts $\varphi'(x) < 0$. The proof is completed. □

4. Numerical Implementation

A unique global maximizer of the problem (3.1) can be computed as:

$$\varphi'(x) = \frac{f'(x)x - f(x)}{x^2} = 0.$$

Using a mongolian wool manufacturing company's data, we obtained the following production function.

$$f(x) = -34137x^3 + 21370x^2 + 10025x,$$

where f is output (yarns in thousand kg), x is costs in million tugrics. Solving by the bisection method, we found an optimal solution $x^* = 0.313003$.

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