POWER OF A GRAPH AND BINDING NUMBER

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Abstract: V.G. Kane (see [2]) asked to study the class of graphs satisfying $\text{bind}(G^k) \geq (\text{bind}(G))^k$. Here we determine binding number of power of various classes of graphs along with the solution for the parametric equation $\text{bind}(G^k) = (\text{bind}(G))^k$. We also show that $\text{bind}(G^2) = (\text{bind}(G))^2$ if and only if $G = H^+$, where $H^+$ is the graph obtained from $H$ by adjoining a pendant edge to each vertex of $H$, by proving that the diophantine equation $mq^2 - np^2 = 0$ has no nontrivial integral solution. In Theorem 3.4, we prove that there exists no graph $G$ such that $\text{bind}(G^k) = (\text{bind}(G))^k$ for $k \geq 3$ by seeking integral solution for $mq^k - np^k = 0$.

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1. Introduction

We consider only finite simple graphs $G$ with vertex set $V(G)$ and edge set $E(G)$. For a graph $G = (V,E)$ and a set $X \subseteq V$, we denote by $\Gamma(X)$ the set
of vertices joined to each vertex of \( X \). A set of independent edges which cover all vertices of a graph is called 1-factor of a graph. By (1,2)-factor of a graph \( G \) we mean, a set of independent edges or vertex disjoint cycles which cover all vertices of \( G \). Clearly the cycles in the definition are of odd length. A graph \( G \) is hallian, if \( |\Gamma(X)| \geq |X| \) for any set \( X \subseteq V \) or equivalently if \( G \) has a (1,2)-factor (see [1]). Thus, \( G \) is a hallian graph if its vertices can be covered by a set of vertex disjoint even paths or odd cycles. A graph \( G \) is \( k \)-hallian, if for any set \( A \) of vertices of order at most \( k \), the subgraph of \( G \) induced by the set \( V - A \) is hallian. The largest \( k \) such that \( G \) is \( k \)-hallian is called the hallian index of \( G \) and is denoted by \( h(G) \). Clearly, \( h(G) \leq \delta(G) - 1 \), where \( \delta(G) \) denotes the minimum degree among the vertices of \( G \). The graph \( G^k \) of \( G \) is a graph whose vertex set is same as that of \( G \) and two vertices \( u, v \) in \( G^k \) are adjacent if \( d(u, v) \leq k \). For concepts not defined here see ([5]). The binding number of a graph \( G \) is defined by D.R. Woodall (see [3]) as

\[
bind(G) = \min_{X \in \Sigma} \frac{|\Gamma(X)|}{|X|}
\]

where \( \Sigma \) is the set of all admissible sets of \( G \). Further an admissible set \( X \) is said to be a realizing set if \( bind(G) = \frac{|\Gamma(X)|}{|X|} \). We enlist few results from [1], [2], [3] and [4] and are as follows:

**Theorem 1.1.** (see [2]) If \( G = K_{n,n,\ldots,n} \) then

\[
bind(G) = \min_{i=1,2,\ldots,k} \left\{ \frac{\sum_{j=1}^{i} n_j}{\sum_{j \neq i} n_i} \right\}
\]

**Theorem 1.2.** (see [1]) If a graph \( G \) on \( n \) vertices has \( h(G) = \delta(G) - 1 \) and \( k(G) \geq h(G) \) then

\[
bind(G) = \frac{n-1}{n-\delta(G)}.
\]

**Theorem 1.3.** (see [3]) \( bind(K_n) = n - 1, \ n \geq 1 \).

**Proposition 1.4.** (see [2]) If \( G \) has a 1-factor then

\[
bind(G) \geq 1.
\]

**Proposition 1.5.** (see [2]) For any graph \( G \), \( bind(G) \leq \frac{n}{\beta_0 - 1} \), where \( \beta_0 \) denotes the vertex independence number of \( G \).

**Theorem 1.6.** (see [1]) For any graph \( G \), \( h(G) \leq \delta(G) - 1 \).
Lemma 1.7. (see [4]) If \( \text{bind}(G) = 1 \) then there exists a realizing set \( X \) such that \( X \cap \Gamma(X) = \phi \).

2. Results

Theorem 2.1.

\[
\text{bind}(P_n^2) = \begin{cases} \frac{n-1}{n-2} & \text{if } n \geq 5, n = 3, n \neq 2 \\ 1 & \text{if } n = 4 \end{cases}
\]

Proof. Let \( G = P_n^2 \). We consider two cases.

Case 1: If \( n \) is even, then for any vertex \( u \) of \( G \), \( G - u \) contains a triangle and \((n - 4)/2\) disjoint edges leading to hallian graph.

Case 2: If \( n \) is odd, then \( G - u \) contains a triangle and \((n - 1)/2\) disjoint edges leading to hallian graph. From above two cases, it is clear that hallian index \( h(G) \geq 1 \). Since \( \delta(G) = 2 \) and by proposition 1.6 (see [1]) we have \( h(G) = 1 \). For \( k(G) = 2 \), label the vertices of \( G \) as \( u_1, u_2, u_3, \ldots, u_{i-1}, u_i, u_{i+1}, \ldots, u_n \). Clearly removal of end vertices of \( G \) does not disconnect \( G \). Also removal of a vertex of degree 3 or 4 does not disconnect \( G \) as there exist an edge \( u_{i-1}u_{i+1} \in E(P_n^2) \) implying that \( k(G) \neq 1 \) and \( k(G) = 2 \). Thus, by Theorem 1.2 (see [1]), \( \text{bind}(P_n^2) = \frac{n-1}{n-2} \). For \( n = 4 \), the end vertices of \( P_4^2 \) form an independent set. Thus, by proposition 1.5 (see [2]), we have \( \text{bind}(P_4^2) \leq 1 \). Also \( P_4^2 \) contains 1-factor and hence by proposition 1.4 (see [2]) \( \text{bind}(P_4^2) \geq 1 \). Finally, \( \text{bind}(P_4^2) = 1 \).

Theorem 2.2.

\[
\text{bind}(C_n^2) = \begin{cases} \frac{n-1}{n-4} & \text{if } n \geq 7 \\ 2 & \text{if } n = 6 \\ 4 & \text{if } n = 5 \end{cases}
\]

Proof. For \( n = 5, C_5^2 = K_5 \) and by Theorem 1.3 (see [3]) the result follows. For \( n = 6, C_6^2 = K_{2,2,2} \) and by Theorem 1.1 (see [2]), \( \text{bind}(C_6^2) = 2 \). For \( n \geq 7 \) and let \( G = C_n^2 \), we have \( \delta(G) = 4, h(G) \leq 3 \). Further it is not difficult to show that \( h(G) \geq 3 \) since removal of any three consecutive vertices from \( G \) results into a hallian graph \( P_{n-3}^2 \). And removal of any three arbitrary vertices from \( G \) results into a graph having \((1,2)\)-factor which is hallian, therefore \( h(G) \geq 3 \) and hence \( h(G) = 3 \). Also from \( G \) one can easily see that \( k(G) \leq 4 \) and
δ(G) = 4. Further removal of any three vertices from G does not disconnect it and hence k(G) ≥ 4 implies that k(G) = 4. Thus, by Theorem 1.2 (see [1]) bind(C^2_n) = \frac{n-1}{n-4}.

3. Parametric Equation

**Lemma 3.1.** There is no integral solution for the equation \( mq^k - np^k = 0 \) for \( 0 < n < m \) and \( 0 < q < p \) where \( m, n, p \) and \( q \) are positive integers.

**Proof.** Suppose \( 0 < n < m \) and \( 0 < q < p \), then \( n + t = m \) and \( q + l = p \) for some integers \( t \geq 1 \), \( l \geq 1 \), and the equation \( mq^k - np^k = 0 \) reduces to \( m(p - 1)^k = p^k(m - t) \). Now applying binomial theorem we notice that

\[
m \left\{ p^k - kC_1 l + kC_2 p^{k-2} l^2 - \ldots + (-1)^{k-1} kpl^{k-1} + (-1)^{k} l^k \right\} = p^k m - p^k t \quad (1)
\]

\[
\Rightarrow tp^k - mkC_1 p^{k-1} l + mkC_2 p^{k-2} l^2 - \ldots + m(-1)^{k-1} kpl^{k-1} + m(-1)^k l^k = 0 \quad (2)
\]

On dividing (2) by \( l^k \), we obtain

\[
t(p/l)^k - mkC_1 (p/l)^{k-1} + mkC_2 (p/l)^{k-2} - \ldots + m(-1)^{k-1} k (p/l) + m(-1)^k = 0. \quad (3)
\]

Also on dividing (2) by \( p^k \), we obtain

\[
m(-1)^k (l/p)^k + m(-1)^{k-1} k (l/p)^{k-1} + \ldots - mk (l/p) + t = 0 \quad (4)
\]

\[
\Rightarrow (l/p)^k \left\{ m(-1)^k + m(-1)^{k-1} k (p/l) + \ldots + mkC_2 (p/l)^{k-2} - mC_1 (p/l)^{k-1} + t (p/l)^k \right\} = 0. \quad (5)
\]

Comparing the coefficient of \((p/l)^k\) in equation (3) and (5), we observe that \( t = (l/p)^k \times t \) that gives \( l = p \). Since \( q + l = p \), we get \( q = 0 \), a contradiction to the fact that \( q > 0 \). Similarly, we can prove that \( n = 0 \). □
**Corollary 3.1.1.** The only integral solution to the equation $mq^k - np^k = 0$ is $m = n = 1, p = q = 1$.

**Theorem 3.2.** $\text{bind}(G^2) = (\text{bind}(G))^2$ if and only if $G = H^+$ for some connected graph $H$.

Before proving Theorem 3.2, we prove following Lemma.

**Lemma 3.3.** If $\text{bind}(G^2) = (\text{bind}(G))^2$ then $\text{bind}(G) = 1$.

**Proof.** Case 1: Let $\text{bind}(G) < 1$, Suppose $\text{bind}(G) = \frac{m}{n}$ with $m < n$. Then $G$ is a spanning subgraph of $G^2$ and hence $\text{bind}(G) \leq (\text{bind}(G))^2$. But $\text{bind}(G^2) = (\text{bind}(G))^2 = m^2/n^2 < m/n = \text{bind}(G)$, a contradiction.

Case 2: Let $\text{bind}(G) > 1$. Let $X$ and $Y$ be the realizing sets of $G^2$ and $G$ respectively. Observe that $\frac{|\Gamma_{G^2}(X)|}{|X|} = \frac{|\Gamma_G(Y)|}{|Y|}$ with $|\Gamma_{G^2}(X)| > |X|$ and $|\Gamma_G(Y)| > |Y|$ so that $\text{bind}(G^2) > 1$ and by Lemma 3.1, $mq^k - np^k = 0$ for $0 < n < m$ and $0 < q < p$ has no integral solution at $k=2$. Thus, $\text{bind}(G) = 1$ holds.

**4. Proof of Theorem 3.2**

**Proof.** We prove this theorem in various steps as claims.

**Step 1:** Every realizing set in $G$ is a realizing set in $G^2$. By Lemma 1.7 (see [4]) there exists a realizing set in $G^2$ such that $X \cap \Gamma_{G^2}(X) = \emptyset$ and $|\Gamma_{G^2}(X)| = |X|$. But $\Gamma_G(X) \subseteq \Gamma_{G^2}(X)$ always holds and $X \cap \Gamma_G(X) \subseteq X \cap \Gamma_{G^2}(X) = \emptyset$ which forces us to conclude that $X \cap \Gamma_{G^2}(X) \neq \emptyset$. Clearly, $|\Gamma_G(X)| \leq |\Gamma_G| = |X|$ holds. If $|\Gamma_G X < |X|$ then, $\text{bind}(G) \leq \frac{|\Gamma_G(X)|}{|X|} < 1$, a contradiction to the fact that $\text{bind}(G) = 1$. Therefore, $|\Gamma_G(X)| = |X|$ holds and hence $1 = \text{bind}(G) \leq \frac{|\Gamma_G(X)|}{|X|} = 1$ implies $\text{bind}(G) = \frac{|\Gamma_G(X)|}{|X|}$ that proves that $X$ is a realizing set in $G$. Further, $X$ is an independent set as $X \cap \Gamma_G(X) = \emptyset$.

**Step 2:** $d_G(u, v) \geq 3$ for every $u, v \in X$. Assume that $d_G(u, v) \leq 2$ for some $u, v \in X$. Moreover $d_G(u, v) \neq 1$ and by step 1, $X$ is an independent set and hence $d_G(u, v) = 2$. But in $G^2$ the vertices $u$ and $v$ are adjacent giving $u, v \in \Gamma_{G^2}(X)$ and thereby $X \cap \Gamma_{G^2}(X) = \emptyset$, a contradiction.
Step 3: $X \cup \Gamma_G(X) = V(G)$. If $u \in V(G)$, $v \notin X \cup \Gamma_G(X)$ and a shortest path $\rho: u = u_1, u_2, u_3, \ldots, u_{t-1}, u_t = v$ for some vertex $v \in X$, then $u_{t-1} \in \Gamma_G(X)$ and $u_{t-2}$ must lie in $V(G) - (X \cup \Gamma_G(X))$. If $u_{t-2} \notin V(G) - (X \cup \Gamma_G(X))$ then $u_{t-2} \in X$ or $u_{t-2} \in \Gamma_G(X)$. If $u_{t-2} \in X$ then $v$ and $u_{t-2}$ belong to $X$ and they are adjacent in $G^2$. Hence they are in $X \cap \Gamma_G(X)$, a contradiction.

Step 4: $deg_G u = 1$ for all $u \in X$. Assume that $deg_G u \geq 2$ for some vertex $u \in X$, then there exists two vertices $v$ and $w$ in $\Gamma_G(X)$ such that $uv$ and $vw$ is an edge in $G$. Also, there exists at least one more vertex say $w'$ in $X$ such that $v$ is adjacent to $w'$ or $w$ is adjacent to $w'$. Since there exists one-to-one correspondence between the vertices of $X$ and those of $\Gamma_G(X)$, we take without loss of generality a vertex $w$ adjacent to $w'$. Consequently, $d(u, w) = 2$, a contradiction to claim 2. From all the claims 1 to 4, we conclude that $G = H^+$ where $H = \langle \Gamma_G(X) \rangle$ an induced subgraph induced by $\Gamma_G(X)$. Conversely, suppose $G = H^+$ for some connected graph $H$. Label the vertices of $H$ as $v_1, v_2, \ldots, v_{p/2}$ and end vertices of $G$ as $u_1, u_2, \ldots, u_{p/2}$ so that $G$ is a graph of order $p$ such that $u_i v_i$ is an edge on $G$ for $i = 1, 2, \ldots, p/2$. Clearly, $G$ has 1-factor so that $bind(G) \geq 1$. By taking $X = \{u_1, u_2, \ldots, u_{p/2}\}$, we have $\Gamma_G(X) = \{v_1, v_2, \ldots, v_{p/2}\}$ and hence $bind(G) \leq \frac{|\Gamma_G(X)|}{|X|} = 1$ which leads to $bind(G) = 1$. Also $1 = bind(G) \leq bind(G^2)$. From graph $G^2$ we have $\Gamma_G(X) = \{v_1, v_2, \ldots, v_{p/2}\}$ and $bind(G^2) \leq \frac{|\Gamma_G^2(X)|}{|X|} = 1$. Thus $bind(G^2) = (bind(G))^2$ holds.

**Theorem 4.1.** There does not exists a graph $G$ such that $bind(G^k) = (bind(G))^k$ for $k \geq 3$.

The proof follows from following Lemma

**Lemma 4.2.** If $bind(G) = 1$ then, $bind(G^k) > 1$ for $k \geq 3$.

**Proof.** Assume that $bind(G^k) \leq 1$, then following cases arise.

Case 1: If $bind(G^k) < 1$, then, $1 = bind(G) \leq bind(G^k) < 1$, a contradiction.

Case 2: Let $bind(G^k) = 1$ then, there exists a realizing set $X$ in $G^k$ such that $X \cap \Gamma_{G^k}(X) = \phi$ with $|X| = |\Gamma_{G^k}(X)|$. But $\Gamma_G(X) \subseteq \Gamma_{G^2}(X) \subseteq \ldots \subseteq \Gamma_{G^k}(X)$. And $|\Gamma_G(X)| = |X|$, $|\Gamma_{G^2}(X)| = |X|$, $|\Gamma_{G^3}(X)| = |X|$, $\ldots$, $|\Gamma_{G^{k-1}}(X)| = |X|$ holds. Otherwise $bind(G) \leq \frac{|\Gamma_G(X)|}{|X|} < 1$, a contradiction. Further $X \cap \Gamma_{G^i}(X) = \phi$ for $i = 1, 2, 3, \ldots, k$. For the claim $X \cup \Gamma_G(X) = V(G)$, assume there exist $u \in V(G)$
and \( u \notin X \cup \Gamma_G(X) \) and a shortest path \( \rho : u = u_1, u_2, u_3, \ldots, u_{t-2}, u_{t-1}, u_t = v \) for some vertex \( v \in X \), then \( u_{t-1} \in \Gamma_G(X) \) and \( u_{t-2} \) must lie in \( V(G) - (X \cup \Gamma_G(X)) \). If \( u_{t-2} \notin V(G) - (X \cup \Gamma_G(X)) \) then \( u_{t-2} \in X \) or \( u_{t-2} \in \Gamma_G(X) \). If \( u_{t-2} \in X \) then \( v \) and \( u_{t-2} \) belong to \( X \) and they are adjacent in \( G^2 \). Hence they are in \( X \cap \Gamma_G^2(X) \), a contradiction. On the other hand if \( u_{t-2} \in \Gamma_G(X) \) then there exist a vertex \( w \in X \) ( \( w \) may not be distinct from \( v \) ) such that \( u_{t-2} \) is an edge in \( G \), then the path \( \rho' : u = u_1, u_2, u_3, \ldots, u_{l-2}, u_{l-1}, u_l = v \) is shorter than \( \rho \), a contradiction. Thus \( X \cup \Gamma_G(X) = V(G) \) holds. Lastly we prove that \( \Gamma_G^k(X) = V(G) \) for \( k \geq 3 \) which shows that \( X \) is not a realizing set, a contradiction to the assumption that it is a realizing set. We claim that for every vertex \( u \in X \) there exist a vertex \( v \in X \) such that \( d_G(u, v) = 3 \). Clearly \( d_G(u, v) > 2 \) for every \( u, v \in X \), since \( X \cap \Gamma_G^i(X) = \emptyset \) for \( i = 1, 2 \). Let \( u \in X \) and \( v \) is any other vertex from \( G^k \neq K_p \) \( (k \geq 3) \) in \( X \) such that \( d_G(u, v) = l \geq 4 \). Further \( u = u_1, u_2, u_3, \ldots, u_{l-2}, u_{l-1}, u_l = v \) be path in \( G \). Hence by above finding the vertices \( u_2, u_3, \ldots, u_{l-2}, u_{l-1} \) are either in \( X \) or \( \Gamma_G(X) \), since \( X \cup \Gamma_G(X) = V(G) \). But \( u = u_1 \in X \) so \( u_2 \in \Gamma_G(X) \), \( u_3 \in \Gamma_G(X) \). Moreover \( X \cap \Gamma_G^i(X) = \emptyset \) for \( i = 1, 2 \) therefore there exist a vertex \( w \in X \) such that \( u_3w \) is an edge in \( G \). In fact vertex \( w \) plays the role of \( v \) in \( G \) and thus every pair of vertices in \( X \) are adjacent in \( G^k, k \geq 3 \). 

References


