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POWER OF A GRAPH AND BINDING NUMBER

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Abstract: V.G. Kane (see [2]) asked to study the class of graphs satisfying $bind(G^k) \geq (bind(G))^k$. Here we determine binding number of power of various classes of graphs along with the solution for the parametric equation $bind(G^k) = (bind(G))^k$. We also show that $bind(G^2) = (bind(G))^2$ if and only if $G = H^+$, where H^+ is the graph obtained from H by adjoining a pendant edge to each vertex of H, by proving that the diophantine equation $mq^2 - np^2 = 0$ has no nontrivial integral solution. In Theorem 3.4, we prove that there exists no graph G such that $bind(G^k) = (bind(G))^k$ for $k \geq 3$ by seeking integral solution for $mq^k - np^k = 0$.

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1. Introduction

We consider only finite simple graphs G with vertex set $V(G)$ and edge set $E(G)$. For a graph $G = (V, E)$ and a set $X \subseteq V$, we denote by $\Gamma(X)$ the set

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of vertices joined to each vertex of X. A set of independent edges which cover all vertices of a graph is called 1-factor of a graph. By (1,2)-factor of a graph G we mean, a set of independent edges or vertex disjoint cycles which cover all vertices of G. Clearly the cycles in the definition are of odd length. A graph G is hallian, if $|\Gamma(X)| \geq |X|$ for any set $X \subseteq V$ or equivalently if G has a (1,2)-factor (see [1]). Thus, G is a hallian graph if its vertices can be covered by a set of vertex disjoint even paths or odd cycles. A graph G is k-hallian, if for any set A of vertices of order at most k, the subgraph of G induced by the set $V - A$ is hallian. The largest k such that G is k-hallian is called the hallian index of G and is denoted by $h(G)$. Clearly, $h(G) \leq \delta(G) - 1$, where $\delta(G)$ denotes the minimum degree among the vertices of G. The graph G^k of G is a graph whose vertex set is same as that of G and two vertices u, v in G^k are adjacent if $d\,(u, v) \leq k$. For concepts not defined here see ([5]). The binding number of a graph G is defined by D.R. Woodall (see [3]) as $bind(G) = \min_{X \in \sum} \frac{|\Gamma(X)|}{|X|}$ where Σ is the set of all admissible sets of G. Further an admissible set X is said to be a realizing set if $bind(G) = \frac{|\Gamma(X)|}{|X|}$. We enlist few results from [1], [2], [3] and [4] and are as follows;

Theorem 1.1. **(see [2])** *If* $G = K_{n,n,\ldots,n}$ *then*

$$bind(G) = \min_{i=1,2,\ldots,k} \left\{ \Sigma_{\substack{j=1 \\ j\neq i}} \frac{n_j}{n_i} \right\}$$

Theorem 1.2. **(see [1])** *If a graph* G *on* n *vertices has* $h(G) = \delta(G) - 1$ *and* $k(G) \geq h(G)$ *then*

$$bind(G) = \frac{n-1}{n - \delta(G)}.$$

Theorem 1.3. **(see [3])** $bind(K_n) = n - 1$, $n \geq 1$.

Proposition 1.4. **(see [2])** *If* G *has a 1-factor then*

$$bind(G) \geq 1.$$

Proposition 1.5. **(see [2])** *For any graph* G, $bind(G) \leq \frac{n}{\beta_0 - 1}$, *where* β_0 *denotes the vertex independence number of* G.

Theorem 1.6. **(see [1])** *For any graph* G, $h(G) \leq \delta(G) - 1$.

Lemma 1.7. **(see [4])** *If* $bind(G) = 1$ *then there exists a realizing set* X *such that* $X \cap \Gamma(X) = \phi$.

2. Results

Theorem 2.1.

$$bind(P_n^2) = \begin{cases} \frac{n-1}{n-2} & \text{if } n \geq 5, n = 3, n \neq 2 \\ 1 & \text{if } n = 4 \end{cases}$$

Proof. Let $G = P_n^2$. We consider two cases.

Case 1: If n is even, then for any vertex u of G, $G - u$ contains a triangle and $(n-4)/2$ disjoint edges leading to hallian graph.

Case 2: If n is odd, then $G - u$ contains a triangle and $(n-1)/2$ disjoint edges leading to hallian graph. From above two cases, it is clear that hallian index $h(G) \geq 1$. Since $\delta(G) = 2$ and by proposition 1.6 (see [1]) we have $h(G) = 1$. For $k(G) = 2$, label the vertices of G as $u_1, u_2, u_3, \ldots, u_{i-1}, u_i, u_{i+1}, \ldots, u_n$. Clearly removal of end vertices of G does not disconnect G. Also removal of a vertex of degree 3 or 4 does not disconnect G as there exist an edge $u_{i-1}u_{i+1} \in E\left(P_n^2\right)$ implying that $k(G) \neq 1$ and $k(G) = 2$. Thus, by Theorem 1.2 (see [1]), $bind(P_n^2) = \frac{n-1}{n-2}$. For $n = 4$, the end vertices of P_4^2 form an independent set. Thus, by proposition 1.5 (see [2]), we have $bind\left(P_4^2\right) \leq 1$. Also P_4^2 contains 1-factor and hence by proposition 1.4 (see [2]) $bind\left(P_4^2\right) \geq 1$. Finally, $bind\left(P_4^2\right) = 1$. □

Theorem 2.2.

$$bind(C_n^2) = \begin{cases} \frac{n-1}{n-4} & \text{if } n \geq 7 \\ 2 & \text{if } n = 6 \\ 4 & \text{if } n = 5 \end{cases}$$

Proof. For $n = 5$, $C_5^2 = K_5$ and by Theorem 1.3 (see [3]) the result follows. For $n = 6$, $C_6^2 = K_{2,2,2}$ and by Theorem 1.1 (see [2]), $bind\left(C_6^2\right) = 2$. For $n \geq 7$ and let $G = C_n^2$, we have $\delta(G) = 4$, $h(G) \leq 3$. Further it is not difficult to show that $h(G) \geq 3$ since removal of any three consecutive vertices from G results into a hallian graph P_{n-3}^2. And removal of any three arbitrary vertices from G results into a graph having $(1,2)$-factor which is hallian, therefore $h(G) \geq 3$ and hence $h(G) = 3$. Also from G one can easily see that $k(G) \leq 4$ and

$\delta(G) = 4$. Further removal of any three vertices from G does not disconnect it and hence $k(G) \geq 4$ implies that $k(G) = 4$. Thus, by Theorem 1.2 (see [1]) $bind(C_n^2) = \frac{n-1}{n-4}$. □

3. Parametric Equation

Lemma 3.1. *There is no integral solution for the equation* $mq^k - np^k = 0$ *for* $0 < n < m$ *and* $0 < q < p$ *where* m, n, p *and* q *are positive integers.*

Proof. Suppose $0 < n < m$ and $0 < q < p$, then $n + t = m$ and $q + l = p$ for some integers $t \geq 1$, $l \geq 1$, and the equation $mq^k - np^k = 0$ reduces to $m(p-1)^k = p^k(m-t)$. Now applying binomial theorem we notice that

$$m\left\{p^k - {}^kC_1 l + {}^kC_2 p^{k-2} l^2 - \dots (-1)^{k-1} kpl^{k-1} + (-1)^1 l^k\right\} = p^k m - p^k t \quad (1)$$

$$\Rightarrow tp^k - m^kC_1 p^{k-1} l + m^kC_2 p^{k-2} l^2 - \dots + m(-1)^{k-1} kpl^{k-1} + m(-1)^k l^k = 0 \quad (2)$$

On dividing (2) by l^k, we obtain

$$t(p/l)^k - m^kC_1(p/l)^{k-1} + m^kC_2(p/l)^{k-2} - \dots + m(-1)^{k-1} k(p/l) + m(-1)^k = 0. \quad (3)$$

Also on dividing (2) by p^k, we obtain

$$m(-1)^k (l/p)^k + m(-1)^{k-1} k(l/p)^{k-1} + \dots - mk(l/p) + t = 0 \quad (4)$$

$$\Rightarrow (l/p)^k \left\{m(-1)^k + m(-1)^{k-1} k(p/l) + \dots + m^kC_2(p/l)^{k-2} - m^kC_1(p/l)^{k-1} + t(p/l)^k\right\} = 0. \quad (5)$$

Comparing the coefficient of $(p/l)^k$ in equation (3) and (5), we observe that $t = (l/p)^k \times t$ that gives $l = p$. Since $q + l = p$, we get $q = 0$, a contradiction to the fact that $q > 0$. Similarly, we can prove that $n = 0$. □

Corollary 3.1.1. *The only integral solution to the equation* $mq^k - np^k = 0$ *is* $m = n = 1$, $p = q = 1$.

Theorem 3.2. $bind\left(G^2\right) = (bind\,(G))^2$ *if and only if* $G = H^+$ *for some connected graph* H.

Before proving Theorem 3.2,we prove following Lemma.

Lemma 3.3. *If* $bind\left(G^2\right) = (bind\,(G))^2$ *then* $bind(G) = 1$.

Proof. Case 1: Let $bind(G) < 1$, Suppose $bind(G) = \frac{m}{n}$ with $m < n$. Then G is a spanning subgraph of G^2 and hence $bind\,(G) \leq (bind\,(G))^2$. But $bind\left(G^2\right) = (bind\,(G))^2 = m^2/n^2 < m/n = bind(G)$, a contradiction.

Case 2: Let $bind(G) > 1$. Let X and Y be the realizing sets of G^2 and G respectively. Observe that $\frac{|\Gamma_{G^2}(X)|}{|X|} = \frac{|\Gamma_G(Y)|^2}{|Y|^2}$ with $|\Gamma_{G^2}(X)| > |X|$ and $|\Gamma_G(Y)| > |Y|$ so that $bind(G^2) > 1$ and by Lemma 3.1, $mq^k - np^k = 0$ for $0 < n < m$ and $0 < q < p$ has no integral solution at k=2. Thus, $bind(G) = 1$ holds. □

4. Proof of Theorem 3.2

Proof. We prove this theorem in various steps as claims.

Step 1: Every realizing set in G is a realizing set in G^2. By Lemma 1.7 (see [4]) there exists a realizing set in G^2 such that $X \cap \Gamma_G^2\,(X) = \phi$ and $|\Gamma_G^2(X)| = |X|$. But $\Gamma_G\,(X) \subseteq \Gamma_{G^2}(X)$ always holds and $X \cap \Gamma_G\,(X) \subseteq X \cap \Gamma_{G^2}\,(X) = \phi$ which forces us to conclude that $X \cap \Gamma_{G^2}\,(X) \neq \phi$. Clearly, $|\Gamma_G(X)| \leq |\Gamma_G^2| = |X|$ holds. If $|\Gamma_G X < |X|$ then, $bind(G) \leq \frac{|\Gamma_G(X)|}{|X|} < 1$, a contradiction to the fact that $bind(G) = 1$. Therefore, $|\Gamma_G(X)| = |X|$ holds and hence $1 = bind(G) \leq \frac{|\Gamma_G(X)|}{|X|} = 1$ implies $bind(G) = \frac{|\Gamma_G(X)|}{|X|}$ that proves that X is a realizing set in G. Further, X is an independent set as $X \cap \Gamma_G(X) = \phi$.

Step 2: $d_G(u, v) \geq 3$ for every $u, v \in X$. Assume that $d_G(u, v) \leq 2$ for some $u, v \in X$. Moreover $d_G(u, v) \neq 1$ and by step 1, X is an independent set and hence $d_G(u, v) = 2$. But in G^2 the vertices u and v are adjacent giving $u, v \in \Gamma_{G^2}(X)$ and thereby $X \cap \Gamma_{G^2}(X) = \phi$, a contradiction.

Step 3: $X \cup \Gamma_G(X) = V(G)$. If $u \in V(G)$, $v \notin X \cup \Gamma_G(X)$ and a shortest path $\rho : u = u_1, u_2, u_3, \ldots, u_{t-2}, u_{t-1}, u_t = v$ for some vertex $v \in X$, then $u_{t-1} \in \Gamma_G(X)$ and u_{t-2} must lie in $V(G) - (X \cup \Gamma_G(X))$. If $u_{t-2} \notin V(G) - (X \cup \Gamma_G(X))$ then $u_{t-2} \in X$ or $u_{t-2} \in \Gamma_G(X)$. If $u_{t-2} \in X$ then v and u_{t-2} belong to X and they are adjacent in G^2. Hence they are in $X \cap \Gamma_{G^2}(X)$, a contradiction.

Step 4: $deg_G u = 1$ for all $u \in X$. Assume that $deg_G u \geq 2$ for some vertex $u \in X$, then there exists two vertices v and w in $\Gamma_G(X)$ such that uv and vw is an edge in G. Also, there exists at least one more vertex say $w^{'}$ in X such that v is adjacent to $w^{'}$ or w is adjacent to $w^{'}$. Since there exists one-to-one correspondence between the vertices of X and those of $\Gamma_G(X)$, we take without loss of generality a vertex w adjacent to w'. Consequently, $d(u, w) = 2$, a contradiction to claim 2. From all the claims 1 to 4, we conclude that $G = H^+$ where $H = \langle \Gamma_G(X) \rangle$ an induced subgraph induced by $\Gamma_G(X)$. Conversely, suppose $G = H^+$ for some connected graph H. Label the vertices of H as $v_1, v_2, \ldots, v_{p/2}$ and end vertices of G as $u_1, u_2, \ldots, u_{p/2}$ so that G is a graph of order p such that $u_i v_i$ is an edge on G for $i = 1, 2, \ldots, p/2$. Clearly, G has 1-factor so that $bind(G) \geq 1$. By taking $X = \{u_1, u_2, \ldots, u_{p/2}\}$, we have $\Gamma_G(X) = \{v_1, v_2, \ldots, v_{p/2}\}$ and hence $bind(G) \leq \frac{|\Gamma_G(X)|}{|X|} = 1$ which leads to $bind(G) = 1$. Also $1 = bind(G) \leq bind(G^2)$. From graph G^2 we have $\Gamma_{G^2}(X) = \{v_1, v_2, \ldots, v_{p/2}\}$ and $bind(G^2) \leq \frac{|\Gamma_{G^2}(X)|}{|X|} = 1$. Thus $bind\left(G^2\right) = (bind\,(G))^2$ holds. □

Theorem 4.1. *There does not exists a graph G such that $bind(G^k) = (bind(G))^k$ for $k \geq 3$.*

The proof follows from following Lemma

Lemma 4.2. *If $bind(G) = 1$ then, $bind(G^k) > 1$ for $k \geq 3$.*

Proof. Assume that $bind(G^k) \leq 1$, then following cases arise.

Case 1: If $bind(G^k) < 1$, then, $1 = bind(G) \leq bind(G^k) < 1$, a contradiction.

Case 2: Let $bind(G^k) = 1$ then, there exists a realizing set X in G^k such that $X \cap \Gamma_{G^k}(X) = \phi$ with $|X| = |\Gamma_{G^k}(X)|$. But $\Gamma_G(X) \subseteq \Gamma_{G^2}(X) \subseteq \ldots \subseteq \Gamma_{G^k}(X)$. And $|\Gamma_G(X)| = |X|$, $|\Gamma_{G^2}(X)| = |X|, \ldots, |\Gamma_{G^{k-1}}(X)| = |X|$ holds Otherwise $bind(G) \leq \frac{|\Gamma_G(X)|}{|X|} < 1$, a contradiction. Further $X \cap \Gamma_{G^i}(X) = \phi$ for $i = 1, 2, 3 \ldots, k$. For the claim $X \cup \Gamma_G(X) = V(G)$, assume there exist $u \in V(G)$

and $u \notin X \cup \Gamma_G(X)$ and a shortest path $\rho : u = u_1, u_2, u_3, \ldots, u_{t-2}, u_{t-1}, u_t = v$ for some vertex $v \in X$, then $u_{t-1} \in \Gamma_G(X)$ and u_{t-2} must lie in $V(G) - (X \cup \Gamma_G(X))$. If $u_{t-2} \notin V(G) - (X \cup \Gamma_G(X))$ then $u_{t-2} \in X$ or $u_{t-2} \in \Gamma_G(X)$. If $u_{t-2} \in X$ then v and u_{t-2} belong to X and they are adjacent in G^2. Hence they are in $X \cap \Gamma_{G^2}(X)$, a contradiction. On the other hand if $u_{t-2} \in \Gamma_G(X)$ then there exist a vertex $w \in X$ (w may not be distinct from v) such that u_{t-2} is an edge in G, then the path $\rho^{'} : u = u_1, u_2, u_3, \ldots, u_{t-2}, w$ is shorter than ρ, a contradiction. Thus $X \cup \Gamma_G(X) = V(G)$ holds. Lastly we prove that $\Gamma_{G^k}(X) = V(G)$ for $k \geq 3$ which shows that X is not a realizing set, a contradiction to the assumption that it is a realizing set. We claim that for every vertex $u \in X$ there exist a vertex $v \in X$ such that $d_G(u, v) = 3$.Clearly $d_G(u, v) > 2$ for every $u, v \in X$, since $X \cap \Gamma_{G^i}(X) = \phi$ for $i = 1, 2$. Let $u \in X$ and v is any other vertex from $G^k \neq K_p (k \geq 3)$ in X such that $d_G(u, v) = l \geq 4$. Further $u = u_1, u_2, u_3, \ldots, u_{l-2}, u_{l-1}, u_l = v$ be path in G. Hence by above finding the vertices $u_2, u_3, \ldots, u_{l-2}, u_{l-1}$ are either in X or $\Gamma_G(X)$, since $X \cup \Gamma_G(X) = V(G)$.But $u = u_1 \in X$ so $u_2 \in \Gamma_G(X), u_3 \in \Gamma_G(X)$. Moreover $X \cap \Gamma_{G^i}(X) = \phi$ for $i = 1, 2$ therefore there exist a vertex $w \in X$ such that $u_3 w$ is an edge in G. In fact vertex w plays the role of v in G and thus every pair of vertices in X are adjacent in $G^k, k \geq 3$. □

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