

**GENERIC LIGHTLIKE SUBMANIFOLDS OF
AN INDEFINITE KAEHLER MANIFOLD**

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Abstract: In this paper, we study generic lightlike submanifolds of an indefinite Kaehler manifold \bar{M} . We provide several new results on such a generic lightlike submanifold M equipped with an induced structure tensor F on M induced by the almost complex structure tensor J of \bar{M} .

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1. Introduction

In the classical theory of non-degenerate submanifolds, there exists an important class of submanifolds of an almost complex manifold: A submanifold (M, g) of an almost complex manifold (\bar{M}, J, \bar{g}) is called a *generic* (or, *anti-holomorphic*) *submanifold* [16, 17] if its normal bundle TM^\perp is mapped into its tangent bundle TM by action of the structure tensor J of \bar{M} , i.e., $J(TM^\perp) \subset TM$. Any real hypersurface of an almost complex manifold is a

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generic submanifold.

The theory of lightlike submanifolds is an important topic of research in differential geometry due to its application in mathematical physics, especially in the general relativity since lightlike submanifolds can be models of different types of horizons (event horizons, Cauchy's horizons, Kruskal's horizons *etc*). The study of such notion was initiated by Duggal-Bejancu [2] and later studied by many authors (see up-to date results in two books [4, 6]). For any lightlike submanifold M equipped with a screen distribution $S(TM)$, since TM is degenerate while $S(TM)$ is non-degenerate, we newly define the generic lightlike submanifold as follow: A lightlike submanifold M of an indefinite almost complex manifold \bar{M} equipped with an indefinite almost complex structure tensor J is called a *generic lightlike submanifold* [5, 13, 14] if there exists a screen distribution $S(TM)$ of M such that

$$J(S(TM)^\perp) \subset S(TM), \quad (1)$$

where $S(TM)^\perp$ is the orthogonal complement of $S(TM)$ in $T\bar{M}$. The geometry of generic lightlike submanifold is an extension of the geometry of lightlike hypersurface or half lightlike submanifold of codimension 2 of any indefinite almost complex manifold. Much of its theory will be immediately generalized in a formal way to general lightlike (*i.e.*, r -lightlike and coisotropic) submanifolds.

In this paper, we study generic lightlike submanifolds of an indefinite Kaehler manifold \bar{M} . In Section 2, we recall the structure equations of lightlike submanifolds which will be used in the sequel. In Section 3, using the above definition, we provide several new results on generic r -lightlike submanifolds of \bar{M} subject such that (1) the structure tensor F of M induced by the almost complex structure tensor J of \bar{M} is parallel with respect to ∇ , or (2) either M or $S(TM)$ is totally umbilical.

2. Lightlike Submanifolds

Let (M, g) be an m -dimensional lightlike submanifold of an $(m+n)$ -dimensional semi-Riemannian manifold (\bar{M}, \bar{g}) . Then the radical distribution $Rad(TM) = TM \cap TM^\perp$ is a subbundle of the tangent bundle TM and the normal bundle TM^\perp , of rank r ($1 \leq r \leq \min\{m, n\}$). In general, there exist two complementary non-degenerate distributions $S(TM)$ and $S(TM^\perp)$ of $Rad(TM)$ in TM and TM^\perp respectively, called the *screen* and *co-screen distributions* on M ,

such that

$$TM = Rad(TM) \oplus_{orth} S(TM), \quad TM^\perp = Rad(TM) \oplus_{orth} S(TM^\perp), \quad (2)$$

where \oplus_{orth} denotes the orthogonal direct sum. We denote such a lightlike submanifold by $(M, g, S(TM), S(TM^\perp))$. Denote by $F(M)$ the algebra of smooth functions on M , by $\Gamma(E)$ the $F(M)$ module of smooth sections of a vector bundle E over M and by $(*.*)_i$ the i -th equation of the equations $(*.*)$. We use the same notations for any others. We use the following range of indices:

$$i, j, k, \dots \in \{1, \dots, r\}, \quad \alpha, \beta, \gamma, \dots \in \{r + 1, \dots, n\}.$$

Let $tr(TM)$ and $ltr(TM)$ be complementary (but not orthogonal) vector bundles to TM in $T\bar{M}|_M$ and TM^\perp in $S(TM)^\perp$ respectively and let $\{N_1, \dots, N_r\}$ be a lightlike basis of $ltr(TM)$ consisting of smooth sections of $S(TM)^\perp$ such that

$$\bar{g}(N_i, \xi_j) = \delta_{ij}, \quad \bar{g}(N_i, N_j) = 0,$$

where $\{\xi_1, \dots, \xi_r\}$ is a lightlike basis of $\Gamma(Rad(TM))$. Then we have

$$\begin{aligned} T\bar{M} &= TM \oplus tr(TM) = \{Rad(TM) \oplus tr(TM)\} \oplus_{orth} S(TM) \\ &= \{Rad(TM) \oplus ltr(TM)\} \oplus_{orth} S(TM) \oplus_{orth} S(TM^\perp). \end{aligned} \quad (3)$$

We say that a lightlike submanifold $(M, g, S(TM), S(TM^\perp))$ of \bar{M} is

- (1) *r*-lightlike if $1 \leq r < \min\{m, n\}$;
- (2) *co-isotropic* if $1 \leq r = n < m$;
- (3) *isotropic* if $1 \leq r = m < n$;
- (4) *totally lightlike* if $1 \leq r = m = n$.

The last three classes (2)~(4) of the above definition are particular cases of the class (1) as follows $S(TM^\perp) = \{0\}$, $S(TM) = \{0\}$ and $S(TM) = S(TM^\perp) = \{0\}$ respectively. The geometry of *r*-lightlike submanifolds is more general form than that of the other submanifolds. For this reason, in this paper we consider only *r*-lightlike submanifolds with following local quasi-orthonormal field of frames:

$$\{\xi_1, \dots, \xi_r, N_1, \dots, N_r, F_{r+1}, \dots, F_m, E_{r+1}, \dots, E_n\},$$

where the sets $\{F_{r+1}, \dots, F_m\}$ and $\{E_{r+1}, \dots, E_n\}$ are orthonormal basis of $S(TM)$ and $S(TM^\perp)$ respectively. In the following, let X, Y, Z and W be the vector fields on M , unless otherwise specified.

Let $\bar{\nabla}$ be the Levi-Civita connection of \bar{M} and P the projection morphism of $\Gamma(TM)$ on $\Gamma(S(TM))$ with respect to (2). For an r -lightlike submanifold, the local Gauss-Weingartan formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + \sum_{i=1}^r h_i^\ell(X, Y)N_i + \sum_{\alpha=r+1}^n h_\alpha^s(X, Y)E_\alpha, \tag{4}$$

$$\bar{\nabla}_X N_i = -A_{N_i}X + \sum_{j=1}^r \tau_{ij}(X)N_j + \sum_{\alpha=r+1}^n \rho_{i\alpha}(X)E_\alpha, \tag{5}$$

$$\bar{\nabla}_X E_\alpha = -A_{E_\alpha}X + \sum_{i=1}^r \phi_{\alpha i}(X)N_i + \sum_{\beta=r+1}^n \sigma_{\alpha\beta}(X)E_\beta, \tag{6}$$

$$\nabla_X PY = \nabla_X^* PY + \sum_{i=1}^r h_i^*(X, PY)\xi_i, \tag{7}$$

$$\nabla_X \xi_i = -A_{\xi_i}^* X - \sum_{j=1}^r \tau_{ji}(X)\xi_j, \quad \forall X, Y \in \Gamma(TM), \tag{8}$$

where ∇ and ∇^* are induced linear connections on TM and $S(TM)$ respectively, the bilinear forms h_i^ℓ and h_α^s on M are called the *local lightlike* and *screen second fundamental forms* on TM respectively, h_i^* are called the *local radical second fundamental forms* on $S(TM)$. A_{N_i} , $A_{\xi_i}^*$ and A_{E_α} are linear operators on $\Gamma(TM)$ and τ_{ij} , $\rho_{i\alpha}$, $\phi_{\alpha i}$ and $\sigma_{\alpha\beta}$ are 1-forms on TM . Since $\bar{\nabla}$ is torsion-free, ∇ is also torsion-free, and h_i^ℓ and h_α^s are symmetric. From the fact $h_i^\ell(X, Y) = \bar{g}(\bar{\nabla}_X Y, \xi_i)$, we show that h_i^ℓ are independent of the choice of $S(TM)$. We say that

$$h(X, Y) = \sum_{i=1}^r h_i^\ell(X, Y)N_i + \sum_{\alpha=r+1}^n h_\alpha^s(X, Y)E_\alpha$$

is the *second fundamental tensor* of M .

The induced connection ∇ on TM is not metric and satisfies

$$(\nabla_X g)(Y, Z) = \sum_{i=1}^r \{h_i^\ell(X, Y)\eta_i(Z) + h_i^\ell(X, Z)\eta_i(Y)\}, \tag{9}$$

where η_i are the 1-forms such that

$$\eta_i(X) = \bar{g}(X, N_i), \quad i \in \{1, \dots, r\}. \tag{10}$$

However, the connection ∇^* on $S(TM)$ is metric. The above three local second

fundamental forms are related to their shape operators by

$$h_i^\ell(X, Y) = g(A_{\xi_i}^* X, Y) - \sum_{k=1}^r h_k^\ell(X, \xi_i) \eta_k(Y), \tag{11}$$

$$\bar{g}(A_{\xi_i}^* X, N_j) = 0;$$

$$\epsilon_\alpha h_\alpha^s(X, Y) = g(A_{E_\alpha} X, Y) - \sum_{i=1}^r \phi_{\alpha i}(X) \eta_i(Y), \tag{12}$$

$$\bar{g}(A_{E_\alpha} X, N_i) = \epsilon_\alpha \rho_{i\alpha}(X);$$

$$h_i^*(X, PY) = g(A_{N_i} X, PY), \quad \eta_j(A_{N_i} X) + \eta_i(A_{N_j} X) = 0, \tag{13}$$

and $\epsilon_\beta \sigma_{\alpha\beta} = -\epsilon_\alpha \sigma_{\beta\alpha}$, where $\epsilon_\alpha = \bar{g}(E_\alpha, E_\alpha)$ is the sign of E_α such that $\epsilon_\alpha = \pm 1$. For an r -lightlike submanifold, replace Y by ξ_i in (12), we have

$$h_\alpha^s(X, \xi_i) = -\epsilon_\alpha \phi_{\alpha i}(X). \tag{14}$$

From (4), (8) and (14), we have

$$\bar{\nabla}_X \xi_i = -A_{\xi_i}^* X + \sum_{j=1}^r \tau_{ji}(X) \xi_j + \sum_{j=1}^r h_j^\ell(X, \xi_i) N_j - \sum_{\alpha=r+1}^n \epsilon_\alpha \phi_{\alpha i}(X) E_\alpha. \tag{15}$$

Definition 1. A lightlike submanifold M of a semi-Riemannian manifold (\bar{M}, \bar{g}) is said to be *irrotational* if $\bar{\nabla}_X \xi_i \in \Gamma(TM)$ for all i .

Note 1. For an r -lightlike M , the above definition is equivalently to

$$h_j^\ell(X, \xi_i) = 0, \quad h_\alpha^s(X, \xi_i) = \phi_{\alpha i}(X) = 0. \tag{16}$$

Denote by \bar{R} and R the curvature tensors of $\bar{\nabla}$ and ∇ respectively. Using the local Gauss-Weingarten formulas for M , we obtain

$$\begin{aligned} \bar{R}(X, Y)Z &= R(X, Y)Z \\ &+ \sum_{i=1}^r \{h_i^\ell(X, Z)A_{N_i} Y - h_i^\ell(Y, Z)A_{N_i} X\} \\ &+ \sum_{\alpha=r+1}^n \{h_\alpha^s(X, Z)A_{E_\alpha} Y - h_\alpha^s(Y, Z)A_{E_\alpha} X\} \\ &+ \sum_{i=1}^r \{(\nabla_X h_i^\ell)(Y, Z) - (\nabla_Y h_i^\ell)(X, Z)\} \end{aligned} \tag{17}$$

$$\begin{aligned}
 & + \sum_{j=1}^r [\tau_{ji}(X)h_j^\ell(Y, Z) - \tau_{ji}(Y)h_j^\ell(X, Z)] \\
 & + \sum_{\alpha=r+1}^n [\phi_{\alpha i}(X)h_\alpha^s(Y, Z) - \phi_{\alpha i}(Y)h_\alpha^s(X, Z)]N_i \\
 & + \sum_{\alpha=r+1}^n \{(\nabla_X h_\alpha^s)(Y, Z) - (\nabla_Y h_\alpha^s)(X, Z) \\
 & + \sum_{i=1}^r [\rho_{i\alpha}(X)h_i^\ell(Y, Z) - \rho_{i\alpha}(Y)h_i^\ell(X, Z)] \\
 & + \sum_{\beta=r+1}^n [\sigma_{\beta\alpha}(X)h_\beta^s(Y, Z) - \sigma_{\beta\alpha}(Y)h_\beta^s(X, Z)]\}E_\alpha.
 \end{aligned}$$

Proposition 2. *Let M be an irrotational r -lightlike submanifold of a semi-Riemannian manifold (\bar{M}, \bar{g}) . Then we have the following equation :*

$$\bar{g}(\bar{R}(X, Y)Z, \xi_i) = 0, \quad \forall i. \tag{18}$$

Proof. Using the local Gauss-Weingarten formulas (7) and (8) for $S(TM)$, we have the following Codazzi equation for $S(TM)$:

$$\begin{aligned}
 R(X, Y)\xi_i & = -\nabla_X^*(A_{\xi_i}^*Y) + \nabla_Y^*(A_{\xi_i}^*X) + A_{\xi_i}^*[X, Y] \tag{19} \\
 & - \sum_{j=1}^r \tau_{ji}(X)A_{\xi_j}^*Y + \sum_{j=1}^r \tau_{ji}(Y)A_{\xi_j}^*X \\
 & + \sum_{j=1}^r \{h_j^*(Y, A_{\xi_i}^*X) - h_j^*(X, A_{\xi_i}^*Y) - 2d\tau_{ji}(X, Y) \\
 & + \sum_{k=1}^r [\tau_{jk}(X)\tau_{ki}(Y) - \tau_{jk}(Y)\tau_{ki}(X)]\}\xi_j.
 \end{aligned}$$

Taking the scalar product with Z to (19) and using (7), (9) and (11) with the facts $g(R(X, Y)Z, \xi_i) = 0$ and $S(TM)$ is non-degenerate, we have

$$\begin{aligned}
 & \nabla_X^*(A_{\xi_i}^*Y) - \nabla_Y^*(A_{\xi_i}^*X) - A_{\xi_i}^*[X, Y] \tag{20} \\
 & + \sum_{j=1}^r \{\tau_{ji}(X)A_{\xi_j}^*Y - \tau_{ji}(Y)A_{\xi_j}^*X\} = 0,
 \end{aligned}$$

$$\begin{aligned}
 &(\nabla_X h_i^\ell)(Y, Z) - (\nabla_Y h_i^\ell)(X, Z) \\
 &+ \sum_{j=1}^r \{ \tau_{ji}(X) h_j^\ell(Y, Z) - \tau_{ji}(Y) h_j^\ell(X, Z) \} = 0.
 \end{aligned}
 \tag{21}$$

Thus (19) is reduced to

$$\begin{aligned}
 R(X, Y)\xi_i &= \sum_{j=1}^r \{ h_j^*(Y, A_{\xi_i}^* X) - h_j^*(X, A_{\xi_i}^* Y) - 2d\tau_{ji}(X, Y) \\
 &+ \sum_{k=1}^r [\tau_{jk}(X)\tau_{ki}(Y) - \tau_{jk}(Y)\tau_{ki}(X)] \} \xi_j.
 \end{aligned}
 \tag{22}$$

Taking the scalar product with ξ_i to (17) and using (16) and (21), we have our proposition. □

3. Generic Lightlike Submanifolds

Let $\bar{M} = (\bar{M}, J, \bar{g})$ be a $2m$ -dimensional indefinite Kaehler manifold ([7] ~ [12]), where \bar{g} is a semi-Riemannian metric of index $q = 2v$, $0 < v < m$, and J is an almost complex structure on \bar{M} satisfying, for all $X, Y \in \Gamma(T\bar{M})$,

$$J^2 = -I, \quad \bar{g}(JX, JY) = \bar{g}(X, Y), \quad (\bar{\nabla}_X J)Y = 0.
 \tag{23}$$

An indefinite complex space form, denoted by $\bar{M}(c)$, is a connected indefinite Kaehler manifold of constant holomorphic sectional curvature c such that

$$\begin{aligned}
 \bar{R}(X, Y)Z &= \frac{c}{4} \{ \bar{g}(Y, Z)X - \bar{g}(X, Z)Y + \bar{g}(JY, Z)JX \\
 &- \bar{g}(JX, Z)JY + 2\bar{g}(X, JY)JZ \}, \quad \forall X, Y, Z \in \Gamma(T\bar{M}).
 \end{aligned}
 \tag{24}$$

Let M be a generic r -lightlike submanifold of an indefinite Kaehler manifold \bar{M} . Then the screen distribution $S(TM)$ of M is expressed as:

$$\begin{aligned}
 S(TM) &= J(S(TM)^\perp) \oplus_{orth} H_o, \\
 &= \{ J(Rad(TM)) \oplus J(ltr(TM)) \} \oplus_{orth} J(S(TM)^\perp) \oplus_{orth} H_o,
 \end{aligned}$$

where H_o is a non-degenerate almost complex distribution on M with respect to J , i.e., $J(H_o) = H_o$. Denote $H' = J(ltr(TM)) \oplus_{orth} J(S(TM)^\perp)$. Then H'

is a r -lightlike distribution on $S(TM)$ such that $J(H') \subset tr(TM)$. Using (25), the general decompositions of (2) and (3) reduce to

$$TM = H \oplus H', \quad T\bar{M} = H \oplus H' \oplus tr(TM),$$

where H is a $2r$ -lightlike almost complex distribution on M such that

$$H = Rad(TM) \oplus_{orth} J(Rad(TM)) \oplus_{orth} H_o.$$

Consider the null vector fields U_i and V_i such that $g(U_i, V_j) = \delta_{ij}$ and the non-null vector field W_α on $S(TM)$ defined by

$$U_i = -JN_i, \quad V_i = -J\xi_i, \quad W_\alpha = -JE_\alpha, \quad \forall i, \alpha. \tag{25}$$

Denote by S the projection morphism of TM on H . Using $TM = H \oplus H'$, for any vector field X on M , JX is expressed as:

$$JX = FX + \sum_{i=1}^r u_i(X)N_i + \sum_{\alpha=r+1}^n w_\alpha(X)E_\alpha, \tag{26}$$

where u_i, v_i and w_α are 1-forms locally defined on $\Gamma(TM)$ by

$$u_i(X) = g(X, V_i), \quad v_i(X) = g(X, U_i), \quad w_\alpha(X) = \epsilon_\alpha g(X, W_\alpha) \tag{27}$$

and F is a tensor field of type $(1, 1)$ globally defined on M by $F = J \circ S$. Apply J to (6), (7), (15) and (5) and use (23), (25) and (26), we have

$$\nabla_X U_i = F(A_{N_i}X) + \sum_{j=1}^r \tau_{ij}(X)U_j + \sum_{\alpha=r+1}^r \rho_{i\alpha}(X)W_\alpha, \tag{28}$$

$$h_j^\ell(X, U_i) = h_i^*(X, V_j), \quad h_i^*(X, W_\alpha) = \epsilon_\alpha h_\alpha^s(X, U_i); \tag{29}$$

$$\nabla_X W_\alpha = F(A_{E_\alpha}X) + \sum_{i=1}^r \phi_{\alpha i}(X)U_i + \sum_{\beta=r+1}^n \sigma_{\alpha\beta}(X)W_\beta, \tag{30}$$

$$h_i^\ell(X, W_\alpha) = \epsilon_\alpha h_\alpha^s(X, V_i), \quad \epsilon_\beta h_\beta^s(X, W_\alpha) = \epsilon_\alpha h_\alpha^s(X, W_\beta); \tag{31}$$

$$\nabla_X V_i = F(A_{\xi_i}^*X) - \sum_{j=1}^r \tau_{ji}(X)V_j - \sum_{\alpha=r+1}^r \epsilon_\alpha \phi_{\alpha i}(X)W_\alpha + \sum_{j=1}^r h_j^\ell(X, \xi_i)U_j, \tag{32}$$

$$h_j^\ell(X, V_i) = h_i^\ell(X, V_j), \tag{33}$$

$$(\nabla_X F)(Y) = \sum_{i=1}^r u_i(Y)A_{N_i}X + \sum_{\alpha=r+1}^n w_\alpha(Y)A_{E_\alpha}X \tag{34}$$

$$\begin{aligned}
 & - \sum_{i=1}^r h_i^\ell(X, Y)U_i - \sum_{\alpha=r+1}^n h_\alpha^s(X, Y)W_\alpha, \\
 (\nabla_X u_i)(Y) &= - \sum_{j=1}^r u_j(Y)\tau_{ji}(X) - \sum_{\alpha=r+1}^n w_\alpha(Y)\phi_{\alpha i}(X) - h_i^\ell(X, FY), \quad (35) \\
 (\nabla_X w_\alpha)(Y) &= - \sum_{i=1}^r u_i(Y)\rho_{i\alpha}(X) - \sum_{\beta=r+1}^n w_\beta(Y)\sigma_{\beta\alpha}(X) - h_\alpha^s(X, FY) \quad (36)
 \end{aligned}$$

Definition 3. An r -lightlike submanifold M of \bar{M} is said to be *totally umbilical* [2] if there is a smooth vector field $\mathcal{H} \in \Gamma(\text{tr}(TM))$ such that

$$h(X, Y) = \mathcal{H} g(X, Y).$$

In case $\mathcal{H} = 0$, we say that M is *totally geodesic*.

It is easy to see that M is totally umbilical if and only if, locally, there exist smooth functions \mathcal{A}_i and \mathcal{B}_α such that

$$h_i^\ell(X, Y) = \mathcal{A}_i g(X, Y), \quad h_\alpha^s(X, Y) = \mathcal{B}_\alpha g(X, Y), \quad \forall X, Y \in \Gamma(TM). \quad (37)$$

Theorem 4. Let M be a totally umbilical generic r -lightlike submanifold of an indefinite Kaehler manifold \bar{M} . Then M is totally geodesic.

Proof. From (31)₁ and (37), we obtain

$$\mathcal{A}_i g(X, W_\alpha) = \epsilon_\alpha \mathcal{B}_\alpha g(X, V_i), \quad \forall X \in \Gamma(TM).$$

Replace X by W_α and X by U_i to this equations by turns, we have $\mathcal{A}_i = 0$ for all i and $\mathcal{B}_\alpha = 0$ for all α . Thus M is totally geodesic. □

Theorem 5. Let M be a generic r -lightlike submanifold of an indefinite Kaehler manifold \bar{M} . Then H is an integrable distribution on M if and only if we have

$$h(X, FY) = h(FX, Y), \quad \forall X, Y \in \Gamma(H).$$

Moreover, if M is totally umbilical, then H is a parallel distribution on M .

Proof. Taking $Y \in \Gamma(H)$, we have $FY = JY \in \Gamma(H)$ and $u_i(Y) = w_\alpha(Y) = 0$ for all i and α . Applying J to (4) with $Y \in \Gamma(H)$ and using (34), we have

$$h_i^\ell(X, FY) = g(\nabla_X Y, V_i), \quad h_\alpha^s(X, FY) = \epsilon_\alpha g(\nabla_X Y, W_\alpha), \quad (38)$$

$$(\nabla_X F)(Y) = - \sum_{i=1}^r h_i^\ell(X, Y)U_i - \sum_{\alpha=r+1}^n h_\alpha^s(X, Y)W_\alpha. \tag{39}$$

By straightforward calculations from two equations of (38), we have

$$h(X, FY) - h(FX, Y) = \sum_{i=1}^r g([X, Y], V_i)N_i + \sum_{\alpha=r+1}^n \epsilon_\alpha g([X, Y], W_\alpha)E_\alpha.$$

If H is an integrable distribution on M , then $[X, Y] \in \Gamma(H)$ for all $X, Y \in \Gamma(H)$. This implies $g([X, Y], V_i) = g([X, Y], W_\alpha) = 0$. Thus we get $h(X, FY) = h(FX, Y)$. Conversely if $h(X, FY) = h(FX, Y)$ for all $X, Y \in \Gamma(H)$, then we have $g([X, Y], V_i) = g([X, Y], W_\alpha) = 0$. Thus $[X, Y] \in \Gamma(H)$ for all $X, Y \in \Gamma(H)$ and H is an integrable distribution on M .

If M is totally umbilical, then, from Theorem 3.1 and (38), we have

$$g(\nabla_X Y, V_i) = g(\nabla_X Y, W_\alpha) = 0.$$

This implies $\nabla_X Y \in \Gamma(H)$ for all $X, Y \in \Gamma(H)$, and hence H is a parallel distribution on M . □

Theorem 6. *Let M be a generic r -lightlike submanifold of an indefinite Kaehler manifold \bar{M} . Then F is parallel on H with respect to ∇ if and only if H is a parallel distribution on M .*

Proof. Assume that F is parallel on H with respect to ∇ . For any $X, Y \in \Gamma(H)$, we have $(\nabla_X F)Y = 0$. Taking the scalar product with V_j and E_β to (39) with $(\nabla_X F)Y = 0$, we have $h_i^\ell(X, Y) = 0$ for all i and $h_\alpha^s(X, Y) = 0$ for all α respectively. From (38), we have $g(\nabla_X Y, V_i) = 0$ for all i and $g(\nabla_X Y, W_\alpha) = 0$ for all α . This imply $\nabla_X Y \in \Gamma(H)$ for all $X, Y \in \Gamma(H)$. Thus H is a parallel distribution on M .

Conversely if H is a parallel distribution on M , then, from (38), we have

$$h_i^\ell(X, FY) = 0, \quad h_\alpha^s(X, FY) = 0, \quad \forall X, Y \in \Gamma(H), \quad i, \alpha.$$

For any $Y \in \Gamma(H)$, we show that $F^2Y = -Y$ by (26). Replace Y by FY to the equations, we have $h_i^\ell(X, Y) = 0$ for all i and $h_\alpha^s(X, Y) = 0$ for all α . From these results and (39), the proof is complete. □

Theorem 7. *Let M be a generic r -lightlike submanifold of an indefinite Kaehler manifold \bar{M} . If F is parallel with respect to ∇ with respect to ∇ , then $H, J(\text{ltr}(TM))$ and $J(S(TM^\perp))$ are parallel distributions on M and M is locally a product manifold $M^r \times M^{n-r} \times M^{m-n}$, where M^r, M^{n-r} and M^{m-n} are leaves of $J(\text{ltr}(TM)), J(S(TM^\perp))$ and H respectively.*

Proof. Assume that F is parallel with respect to ∇ . Then F is parallel on H with respect to ∇ . Theorem 3.3 implies H is a parallel distribution on M . Apply the operator F to (34) with $(\nabla_X F)Y = 0$ and use the facts $FU_i = FW_\alpha = 0$ for all i and α , we have

$$\sum_{i=1}^r u_i(Y)F(A_{N_i}X) + \sum_{\alpha=r+1}^n w_\alpha(Y)F(A_{E_\alpha}X) = 0.$$

Replacing Y by U_k and W_β to this by turns and using (27), we have

$$F(A_{N_i}X) = 0, \quad F(A_{E_\alpha}X) = 0.$$

Taking the scalar product with W_β and N_k to (34) by turns, we have

$$h_\alpha^s(X, Y) = \sum_{i=1}^r u_i(Y)w_\alpha(A_{N_i}X) + \sum_{\beta=r+1}^n w_\beta(Y)w_\alpha(A_{E_\beta}X), \quad (40)$$

$$\sum_{i=1}^r u_i(Y)g(A_{N_i}X, N_k) + \sum_{\alpha=r+1}^n w_\alpha(Y)g(A_{E_\alpha}X, N_k) = 0. \quad (41)$$

Replacing Y by ξ_j to (40), we get $\phi_{\alpha i}(X) = 0$ due to (14). Also replacing Y by W_β to (41), we have $\rho_{i\alpha}(X) = 0$ due to (12). From these results, (28) and (30), we get

$$\nabla_X U_i = \sum_{j=1}^r \tau_{ij}(X)U_j, \quad \nabla_X W_\alpha = \sum_{\beta=r+1}^n \sigma_{\alpha\beta}(X)W_\beta.$$

Thus $J(\text{ltr}(TM))$, $J(S(TM^\perp))$ and H are parallel distributions on M such that

$$TM = J(\text{ltr}(TM)) \oplus_{\text{orth}} J(S(TM^\perp)) \oplus_{\text{orth}} H.$$

By the decomposition theorem of de Rham [1], we show that $M = M^r \times M^{n-r} \times M^{m-n}$, where M^r , M^{n-r} and M^{m-n} are some leaves of $J(\text{ltr}(TM))$, $J(S(TM^\perp))$ and H respectively. \square

Definition 8. A screen distribution $S(TM)$ is called *totally umbilical* [2] if there exist smooth functions \mathcal{C}_i on any coordinate neighborhood \mathcal{U} in M such that

$$h_i^*(X, PY) = \mathcal{C}_i g(X, Y). \quad (42)$$

In case $\mathcal{C}_i = 0$ on \mathcal{U} for all i , we say that $S(TM)$ is *totally geodesic*.

Due to (13) and (42), we know that $S(TM)$ is totally umbilical in M if and only if each shape operator A_{N_i} of $S(TM)$ satisfies

$$g(A_{N_i}X, PY) = C_i g(X, PY), \tag{43}$$

for some smooth function C_i on $\mathcal{U} \subseteq M$.

Theorem 9. *Let M be a generic r -lightlike submanifold of an indefinite Kaehler manifold \bar{M} . We have the following assertions.*

(1) *If each V_i is parallel with respect to ∇ , then $\tau_{ij} = \phi_{\alpha i} = h^\ell(X, \xi_i) = 0$. In this case M is irrotational. Moreover, we have*

$$A_{\xi_i}^*X = \sum_{j=1}^r u_j(A_{\xi_i}^*X)U_j + \sum_{\alpha=r+1}^n w_\alpha(A_{\xi_i}^*X)W_\alpha, \quad \forall X \in \Gamma(TM).$$

(ii) *If each U_i is parallel with respect to ∇ , then $\tau_{ij} = \rho_{i\alpha} = 0$ and*

$$A_{N_i}X = \sum_{j=1}^r u_j(A_{N_i}X)U_j + \sum_{\alpha=r+1}^n w_\alpha(A_{N_i}X)W_\alpha, \quad \forall X \in \Gamma(TM).$$

(iii) *If each W_α is parallel with respect to ∇ , then $\phi_{\alpha i} = 0$ and*

$$A_{E_\alpha}X = \sum_{i=1}^r u_i(A_{E_\alpha}X)U_i + \sum_{\beta=r+1}^n w_\beta(A_{E_\alpha}X)W_\beta, \quad \forall X \in \Gamma(TM).$$

Moreover, if all of V_i, U_i and W_α are parallel on TM with respect to ∇ , then $S(TM)$ is totally geodesic in M and $\tau_{ij} = \phi_{\alpha i} = \rho_{i\alpha} = 0$ on $\Gamma(TM)$. In this case, each null transversal vector fields N_i of M is a constant on M .

Proof. If V_i is parallel with respect to ∇ , then, taking the scalar product with W_β and V_k to (32) by turns, we have $\phi_{\alpha i}(X) = 0$ and $h_j^\ell(X, \xi_i) = 0$ respectively. Thus M is irrotational. Also, taking the scalar product with U_k to (32), we have $\tau_{ji}(X) = 0$. Thus we have $F(A_{\xi_i}^*X) = 0$. From this result and (26), we obtain

$$J(A_{\xi_i}^*X) = \sum_{j=1}^r u_j(A_{\xi_i}^*X)N_j + \sum_{\alpha=r+1}^n w_\alpha(A_{\xi_i}^*X)E_\alpha.$$

Apply J to this equation and use (23) and (25), we obtain (i). In a similar way, by using (28) and (30) and by virtue of (25) and (26), we have (ii) and (iii).

Assume that all of V_i, U_i and W_α are parallel on TM with respect to ∇ . Substituting the equation of (i) into (29)₁, we have

$$u_j(A_{N_i}X) = v_i(A_{\xi_j}^*X) = g(A_{\xi_j}^*X, U_i) = 0.$$

Also, substituting the equation of (iii) into (29)₂, we have

$$w_\alpha(A_{N_i}X) = \epsilon_\alpha v_i(A_{E_\alpha}X) = g(A_{E_\alpha}X, U_i) = 0.$$

From the last two equations and the equation of (ii), we see that $A_{N_i}X = 0$ for all $X \in \Gamma(TM)$. From this and (43), we have $\mathcal{C}_i = 0$. Thus $S(TM)$ is totally geodesic in M and all 1-forms $\tau_{ij}, \phi_{\alpha i}$ and $\rho_{i\alpha}$, defined by (5) and (6), vanish identically. Using this results and (5), we show that N is a constant on M . \square

Theorem 10. *Let M be a totally umbilical generic r -lightlike submanifold of an indefinite Kaehler manifold \bar{M} such that $S(TM)$ is totally umbilical in M . Then $S(TM)$ is totally geodesic in M .*

Proof. Assume that M is totally umbilical. By Theorem 3.1, we have $h_i^\ell = h_\alpha^s = 0$ for all i and α . If $S(TM)$ is totally umbilical in M , we have $\mathcal{C}_i g(X, V_j) = 0$ and $\mathcal{C}_i g(X, W_\alpha) = 0$ due to (29). Replacing X by U_j to the first equation or replacing X by W_β to the second equation, we get $\mathcal{C}_i = 0$. Thus $S(TM)$ is totally geodesic in M . \square

Theorem 11. *Let M be a totally umbilical generic r -lightlike submanifold of an indefinite Kaehler manifold \bar{M} such that $S(TM)$ is totally umbilical. Then M is locally a product manifold $M^r \times M^s \times M^t$, where M^r, M^s and M^t are some leafs of $Rad(TM)$, $H_o^\perp = Span\{V_i, U_i, W_\alpha\}$ and H_o respectively and $s = n + r, t = m - n - 2r$.*

Proof. As M is totally umbilical. By Theorem 3.2, H is a parallel distribution M . Thus, for all $X, Y \in \Gamma(H_o)$, we have $\nabla_X Y \in \Gamma(H)$. From (7) and (39), we have

$$\begin{aligned} h_i^*(X, FY) &= g(\nabla_X FY, N_i) = g((\nabla_X F)Y + F(\nabla_X Y), N_i) \\ &= g(F(\nabla_X Y), N_i) = -g(\nabla_X Y, JN_i) = g(\nabla_X Y, U_i), \end{aligned} \tag{44}$$

due to $FY \in \Gamma(H_o)$. If $S(TM)$ is totally umbilical in M , then we have $h_i^* = 0$ due to Theorem 2.6. By (7) and (44), we get

$$g(\nabla_X Y, N_i) = 0, \quad g(\nabla_X Y, U_i) = 0, \quad \forall X \in \Gamma(TM), \quad \forall Y \in \Gamma(H_o).$$

This results and (38) imply $\nabla_X Y \in \Gamma(H_o)$ for all $X, Y \in \Gamma(H_o)$. Thus H_o is a parallel distribution on M . By Theorem 3.1 and 3.6, we have $h_i^\ell = h_\alpha^s = A_{N_i} = \phi_{\alpha i} = 0$ and $A_{E_\alpha} X = \sum_{i=1}^r \rho_{i\alpha}(X)\xi_i$. Thus (8) and (28), (30) and (32) deduce respectively to

$$\begin{aligned} \nabla_X \xi &= - \sum_{j=1}^r \tau_{ji}(X)\xi_j, \\ \nabla_X U_i &= \sum_{j=1}^r \tau_{ij}(X)U_j + \sum_{\alpha=r+1}^n \rho_{i\alpha}(X)W_\alpha, \\ \nabla_X W_\alpha &= - \sum_{i=1}^r \rho_{i\alpha}(X)V_j + \sum_{\beta=r+1}^n \sigma_{\alpha\beta}(X)W_\beta, \\ \nabla_X V_i &= - \sum_{j=1}^r \tau_{ji}(X)V_j + \sum_{j=1}^r h_j^\ell(X, \xi_i)U_j. \end{aligned}$$

Thus $Rad(TM)$, H_o^\perp and H_o are parallel distributions on M such that

$$TM = Rad(TM) \oplus_{orth} H_o^\perp \oplus_{orth} H_o,$$

where $H_o^\perp = Span\{V_i, U_i, W_\alpha\}$. Thus we have $M = M^r \times M^s \times M^t$, where M^r, M^s and M^t are some leaves of $Rad(TM)$, H_o^\perp and H_o respectively and $s = n + r, t = m - n - 2r$. □

Theorem 12. *Let M be an irrotational generic r -lightlike submanifold of an indefinite complex space form $\bar{M}(c)$. Then we have $c = 0$.*

Proof. Taking the scalar product with ξ_i to (24) and using (18) and (27), we have

$$\frac{c}{4}\{u_i(X)\bar{g}(JY, Z) - u_i(Y)\bar{g}(JX, Z) + 2u_i(Z)\bar{g}(X, JY)\} = 0.$$

Taking $X = Z = U_i$ and $Y = \xi_i$ and using (25) and (27), we have $c = 0$. □

Corollary 13. *There exist no irrotational generic r -lightlike submanifolds M of an indefinite complex space form $\bar{M}(c)$ with $c \neq 0$.*

Theorem 14. *Let M be a generic r -lightlike submanifold of an indefinite complex space form $\bar{M}(c)$. If $S(TM)$ is totally umbilical in M , then we have $c = 0$ and $C_i = 0$, on any coordinate neighborhood $\mathcal{U} \subset M$. Moreover,*

- (i) $c = 0$ implies the ambient space $\bar{M}(c)$ is a semi-Euclidean space,

(ii) $C_i = 0$, on any $\mathcal{U} \subset M$, implies $S(TM)$ is totally geodesic in M .

Proof. As $S(TM)$ is totally umbilical, using (29), (33) and (42), we have

$$h_j^\ell(X, U_i) = C_i g(X, V_j), \quad h_\alpha^s(X, U_i) = \epsilon_\alpha C_i g(X, W_\alpha), \quad \forall X \in \Gamma(TM). \quad (45)$$

Taking the scalar product with N_i to (17) and use (12), we have

$$\begin{aligned} \bar{g}(\bar{R}(X, Y)Z, N_i) &= \bar{g}(R(X, Y)Z, N_i) \\ &+ \sum_{j=1}^r \{h_j^\ell(X, Z)\eta_i(A_{N_j}Y) - h_j^\ell(Y, Z)\eta_i(A_{N_j}X)\}, \\ &+ \sum_{\alpha=r+1}^n \epsilon_\alpha \{h_\alpha^s(X, Z)\rho_{i\alpha}(Y) - h_\alpha^s(Y, Z)\rho_{i\alpha}(X)\}. \end{aligned} \quad (46)$$

Using the Gauss-Weingarten equations (7) and (8) for $S(TM)$, we obtain the Codazzi equation for $S(TM)$:

$$\begin{aligned} g(R(X, Y)PZ, N_i) &= (\nabla_X h_i^*)(Y, PZ) - (\nabla_Y h_i^*)(X, PZ) \\ &+ \sum_{j=1}^r \{h_j^*(X, PZ)\tau_{ij}(Y) - h_j^*(Y, PZ)\tau_{ij}(X)\}. \end{aligned} \quad (47)$$

Using (9), (23), (24), (27), (42), (46) and (47), we get

$$\begin{aligned} &C_i \sum_{j=1}^r \{h_j^\ell(X, PZ)\eta_j(Y) - h_j^\ell(Y, PZ)\eta_j(X)\} \\ &+ \sum_{\alpha=r+1}^n \epsilon_\alpha \{h_\alpha^s(X, PZ)\rho_{i\alpha}(Y) - h_\alpha^s(Y, PZ)\rho_{i\alpha}(X)\} \\ &+ \sum_{j=1}^r \{h_j^\ell(X, PZ)\eta_i(A_{N_j}Y) - h_j^\ell(Y, PZ)\eta_i(A_{N_j}X)\} \\ &= \{Y[C_i] - \sum_{j=1}^r C_j \tau_{ij}(Y) - \frac{c}{4} \eta_i(Y)\} g(X, PZ) \\ &- \{X[C_i] - \sum_{j=1}^r C_j \tau_{ij}(X) - \frac{c}{4} \eta_i(X)\} g(Y, PZ) \\ &- \frac{c}{4} \{\bar{g}(JX, PZ)v_i(Y) - \bar{g}(JY, PZ)v_i(X) - 2\bar{g}(X, JY)v_i(PZ)\}. \end{aligned} \quad (48)$$

Replacing X by ξ_k in this equation, we have

$$\begin{aligned}
 & C_i h_j^\ell(Y, PZ) + \sum_{\alpha=r+1}^n \epsilon_\alpha h_\alpha^s(Y, PZ) \rho_{i\alpha}(\xi_j) \\
 & + \sum_{k=1}^r h_k^\ell(Y, PZ) \eta_i(A_{N_k} \xi_j) + C_i \sum_{k=1}^r h_k^\ell(\xi_j, PZ) \eta_k(Y) \\
 & - \sum_{\alpha=r+1}^n \epsilon_\alpha h_\alpha^s(\xi_j, PZ) \rho_{i\alpha}(Y) - \sum_{k=1}^r h_k^\ell(\xi_j, PZ) \eta_i(A_{N_k} Y) \\
 & = \{ \xi_j [C_i] - \sum_{k=1}^r C_k \tau_{ik}(\xi_j) - \frac{c}{4} \delta_{ij} \} g(Y, PZ) \\
 & - \frac{c}{4} \{ u_j(PZ) v_i(Y) + 2u_j(Y) v_i(PZ) \},
 \end{aligned} \tag{49}$$

for all $Y, Z \in \Gamma(TM)$. Replacing PZ by U_k to (49) and using (26) and (45), we have

$$\begin{aligned}
 & C_i C_k g(Y, V_j) + \sum_{\alpha=r+1}^n \epsilon_\alpha C_k g(Y, W_\alpha) \rho_{i\alpha}(\xi_j) \\
 & + \sum_{h=1}^r C_k g(Y, V_h) \eta_i(A_{N_h} \xi_j) \\
 & = \{ \xi_j [C_i] - \sum_{k=1}^r C_k \tau_{ik}(\xi_j) - \frac{c}{4} \delta_{ij} \} g(Y, U_k) - \frac{c}{4} \delta_{jk} v_i(Y).
 \end{aligned} \tag{50}$$

Replacing Y with V_k, U_j and W_α to (50) by turns, we have

$$\xi_j [C_i] - \sum_{k=1}^r C_k \tau_{ik}(\xi_j) - \frac{c}{2} \delta_{ij} = 0, \quad C_k \{ C_i + \eta_i(A_{N_j} \xi_j) \} = 0, \quad C_k \rho_{i\alpha}(\xi_j) = 0,$$

respectively. This shows that $C_i = 0$ and $c = 0$. Thus we have our theorem. \square

Corollary 15. *There exist no generic r -lightlike submanifolds M of an indefinite complex space form $\bar{M}(c)$ with $c \neq 0$ such that $S(TM)$ is totally umbilical in M*

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