International Journal of Pure and Applied Mathematics
Volume 102 No. 1 2015, 1-7
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v102i1.1
ijpam.eu

MODIFIED RIEMANN-HILBERT BOUNDARY VALUE PROBLEM FOR NONLINEAR COMPLEX PARTIAL DIFFERENTIAL EQUATION

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Abstract: In this paper we discuss on the existence and uniqueness solution of the modified Riemann-Hilbert boundary value problem in the form:

$$\frac{\partial w}{\partial \bar{z}} = F(z, w, \frac{\partial w}{\partial z}), \quad z \in D, \tag{1}$$

$$Re(a + ib)w = g + \varphi \quad \text{on} \quad \partial D \tag{2}$$

in the $C_{1,\alpha}(\bar{D})$, wherer a, b, φ and g are given Holder continuously differentiable real-valued function of a real parameter t on ∂D. We shall assume that $a^2+b^2 = 1$ everywhere on ∂D, and φ is identically zero if $\chi \geq 0$ (χ is the index of the Riemann-Hilbert problem) and for $\chi < 0$

$$\varphi(z) = \sum_{k=\chi+1}^{-\chi-1} h_k \omega^k(z), \qquad z \in \bar{D},$$

the coefficients h_k are restricted to

Received: January 9, 2014

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$$h_{-k} = \bar{h_k} \qquad |k| \leq -\chi - 1,$$

and ω is the conformal map from D into unit disc $\mathcal{D}$.

AMS Subject Classification: 35D05, 35J55, 35j67
Key Words: boundary value problem, Banach space, fixed point theorem, complex differential equation, holomorphic function

1. Introduction

Suppose that D is a bounded domain belongs to the class $C_{1,\alpha}$ in the complex plane and $F = F(z, w, \frac{\partial w}{\partial z}) \in C_\alpha(\bar{D}), 0 < \alpha < 1$, by define the weakly and strongly singular operators T_D and $\prod_D$ in the form:

$$T_D f(z) = -\frac{1}{\pi}\int\int_D \frac{1}{\xi - z} f(\xi) d\zeta d\eta,$$

$$\prod_D f(z) = -\frac{1}{\pi}\int\int_D \frac{1}{(\xi - z)^2} f(\xi) d\zeta d\eta.$$

Here $\xi = \zeta + i\eta$, $z = x + iy$ and if $f \in L_p(D)$ then $T_D f$ is bounded and Holder continuouse, see [6].

So that

$$\frac{\partial T_D f(z)}{\partial \bar{z}} = f(z),$$

$$\frac{\partial T_D f(z)}{\partial z} = \prod_D f(z).$$

Furthermore, we assume that $w \in C_\alpha(\bar{D}), 0 < \alpha < 1$, is an arbitrary solution of:

$$\frac{\partial w}{\partial \bar{z}} = F(z, w, \frac{\partial w}{\partial z}).$$

We define a function Φ as follows:

$$\Phi(z) = w(z) - T_D F(z, w, \frac{\partial w}{\partial z}), \tag{3}$$

Differentiating Φ partially with respect to $\bar{z}$ and z respectively, we obtain the following

$$\begin{cases} \dfrac{\partial \Phi}{\partial \bar{z}} = \dfrac{\partial w}{\partial \bar{z}} - F(z, w, \dfrac{\partial w}{\partial z}), \\ \dfrac{\partial \Phi}{\partial z} = \dfrac{\partial w}{\partial z} - \prod_D F(z, w, \dfrac{\partial w}{\partial z}). \end{cases} \tag{4}$$

Furthermore, since $F = (z, w, \frac{\partial w}{\partial z}) \in C_\alpha(\bar{D}), 0 < \alpha < 1$, the following estimates hold:

$$\|\Phi\|_{\alpha,D} \leq \|w\|_{\alpha,D} + \|T_D F\|_{\alpha,D},$$

$$\|\frac{\partial \Phi}{\partial z}\|_{\alpha,D} \leq \|\frac{\partial w}{\partial z}\|_{\alpha,D} + \|\prod_D F\|_{\alpha,D}.$$

It follows from the first equation (4) and Wely's lemma [6] that Φ is holomorphic function in D, it belong to the class $C_\alpha(\bar{D}), 0 < \alpha < 1$. Moreover, we deduce that, if w is a solution of (1), then w necessarily is of the form

$$w(z) = \Phi(z) + T_D F(z, w, \frac{\partial w}{\partial z}), \tag{5}$$

where Φ is holomorphic in D. We now suppose that (w, h) is a solution of the following system:

$$\begin{cases} w(z) = \Phi(z) + T_D F(z, w, h), \\ h(z) = \Phi'(z) + \prod_D F(z, w, h). \end{cases} \tag{6}$$

Then w is a solution to the given differential equation(1). On substituting $h = \frac{\partial w}{\partial z}$ in (6) we obtain the following result.

Theorem 1.1. *A function $w \in C_{1,\alpha}(\bar{D})$ in the form in (5), is a solution to the partial differential equation* (1) *if and only if, for a holomorph function $\Phi \in C_{1,\alpha}(\bar{D})$, (w, h) is the solution of the system (6).*

2. Existence of a General Solution in $C_{1,\alpha}(D)$

In order to determine the existence of a solution $w \in C_{1,\alpha}(\bar{D})$, we shall work in the following space $\mathcal{J}_\alpha(\bar{D})$, $0 < \alpha < 1$:

We denote by $\mathcal{J}_\alpha(\bar{D})$ the set of all pairs (w, h) for which both w and h belong to the space $C_\alpha(\bar{D})$. The norm shall be defined as follows:

$$\|(w, h)\|_{\alpha,D} := max(\|w\|_{\alpha,D}, \|h\|_{\alpha,D}),$$

thus the $\mathcal{J}_\alpha(\bar{D})$ becomes a banach space. We impose the following assumptions on the complex differential equation (1):

I. The domain D is bounded and belongs to the class $C_{1,\alpha}$, $0 < \alpha < 1$.

II. The right hand side $F(z, w, h)$ is a continuouse function of $z \in D$, w, h.

III. The function $F(z, w, h) \in C_\alpha(\bar{D})$ if $w, h \in C_\alpha(\bar{D})$.

IV. $F(z,w,h)$ satisfies a Lipschitz condition in the metric of Holder norm:

$$\|F(z,w,h) - F(z,\tilde{w},\tilde{h})\|_{\alpha,D} \leq L\|(w,h) - (\tilde{w},\tilde{h})\|_{\alpha,D}.$$

The assupmtion III is satisfied if, the following Lipschitz condition is satisfied:

$$|F(z,w,h) - F(\tilde{z},\tilde{w},\tilde{h})| \leq L[|z-\tilde{z}|^{\alpha} + max(|w-\tilde{w}|, |h-\tilde{h}|)]$$

$$z, \tilde{z} \in \bar{D} \quad \text{and} \quad (w,h),(\tilde{w},\tilde{h}) \in \mathcal{J}_{\alpha}(\bar{D}).$$

In this case we have

$$|F(z,w,h) - F(\tilde{z},\tilde{w},\tilde{h})| \leq L[1 + max(H(\alpha,w), H(\alpha,h))]|z-\tilde{z}|^{\alpha},$$

where

$$H(\alpha,h) = sup\frac{|h(z) - h(\tilde{z})|}{|z-\tilde{z}|^{\alpha}} \; z,\tilde{z} \in D,$$

with the aid of the right hand side of (5) we now define an operator Q as follows:

For $(w,h) \in \mathcal{J}_{\alpha}(\bar{D})$. Let $Q(w,h) = (W,H)$:

$$W(z) = \Phi(z) + T_D F(.,w,h)(z)$$

$$H(z) = \Phi'(z) + \prod_D F(.,w,h)(z),$$

where Φ is a holomorphic function in D and $\Phi \in C_{1,\alpha}(\bar{D})$.

By the properties of the integral operators T_D, $\prod_D$ in $C_{\alpha}(\bar{D})$, we conclude that the Q maps $\mathcal{J}_{\alpha}(\bar{D})$ into itself. The following estimates holds:

$$\|W\|_{\alpha,D} \leq \|\Phi\| + \|T_D F\|_{\alpha,D}$$

$$\|H\|_{\alpha,D} \leq \|\Phi'\| + \|\prod_D F\|_{\alpha,D}.$$

In order to be able to use the Banach fixed point theorem, we now compare the distance between two elements $(w,h),(\tilde{w},\tilde{h}) \in \mathcal{J}_{\alpha}(\bar{D})$ and that between their corresponding image $(W,H),(\tilde{W},\tilde{H})$ under the mapping Q.

Thus

$$\begin{cases} W = \Phi + T_D F(.,w,h) \\ \tilde{W} = \Phi + T_D F(.,\tilde{w},\tilde{h}) \end{cases}$$

$$\begin{cases} H = \Phi^{\square} + \prod_D F(., w, h) \\ \tilde{H} = \Phi^{\square} + \prod_D F(., \tilde{w}, \tilde{h}) \end{cases}$$

It is an immediate consequence that

$$\|W - \tilde{W}\|_\alpha \leq \|T_D\|_\alpha \|F(z, w, h) - F(z, \tilde{w}, \tilde{h})\|_\alpha \leq K_1(\alpha, D)L\|(w, h) - (\tilde{w}, \tilde{h})\|_\alpha$$

$$\|H - \tilde{H}\|_\alpha \leq \|\prod_D \|_\alpha \|F(z, w, h) - F(z, \tilde{w}, \tilde{h})\|_\alpha \leq K_2(\alpha, D)L\|(w, h) - (\tilde{w}, \tilde{h})\|_\alpha$$

then

$$\|(W, H) - (\tilde{W}, \tilde{H})\|_\alpha \leq LMax(K_1, K_2)\|(w, h) - (\tilde{w}, \tilde{h})\|_\alpha.$$

If

$$0 < LMax(K_1(\alpha, D), K_2(\alpha, D)) < 1$$

then the Q is contractive in $\mathcal{J}_\alpha(\bar{D})$. Cosequently by fixed point theorem Q has exactly one fixed element (w, h), with the condition $Q(w, h) = (w, h)$ so that

$$w = \Phi + T_D F(., w, h), \quad h = \Phi^{\square} + \prod_D F(., w, h).$$

By theorem (1.1) the coresponding w is then a general solution of the partial complex differential equation(1). We now consider the modified Riemann-Hilbert boundary value problem. We determine the solution w satisfying the following conditions:

$$\frac{\partial w}{\partial \bar{z}} = F(z, w, \frac{\partial w}{\partial z}), \quad z \in D \tag{7}$$

$$Re(a + ib)w = g + \varphi \quad \text{on} \quad \partial D, \tag{8}$$

wherer a, b, φ and g are given Holder continuously differentiable real-valued function of a real parameter t on ∂D, we shall assume that $a^2 + b^2 = 1$ everywhere on ∂D, and φ is identically zero if $\chi \geq 0$ (χ is the index of the Riemann-Hilbert problem) and for $\chi < 0$

$$\varphi(z) = \sum_{k=\chi+1}^{-\chi-1} h_k \omega^k(z), \quad z \in \bar{D},$$

the coefficients h_k are restricted to

$$h_{-k} = \bar{h_k}, \quad |k| \leq -\chi - 1,$$

and ω is the conformal map from D into unit disc $\mathcal{D}$.

It was shown earlier that a general solution w of the partial differential equation (1) is the form

$$w = \Phi + T_D F(z, w, \frac{\partial w}{\partial z}), \tag{9}$$

where Φ is any function in $C_{1,\alpha}(\bar{D})$, $0 < \alpha < 1$, and holomorphic in D. We replace Φ in (9) by a sum of two holomorphic functions ϕ_g and $\phi_{(w,h)}$. It follows then from (7) and (8)

$$Re(a + ib)(\phi_g + \phi_{(w,h)} + T_D F) = g + \varphi$$

or

$$Re(a + ib)\phi_g + Re(a + ib)\phi_{(w,h)} = g + \varphi - Re(a + ib)T_D F.$$

Thus the modified Riemann-Hilbert problem for w reduces to a similar problem for the holomorphic functions ϕ_g and $\phi_{(w,h)}$. These are

I. $Re(a + ib)\phi_g = g$

II. $Re(a + ib)\phi_{(w,h)} = \varphi - Re(a + ib)T_D F(., w, h)$.

Since both g and $\varphi - Re(a+ib)T_D F(., w, h)$ are in $C_{1,\alpha}(\partial D)$, both problems have a unique solution in $C_{1,\alpha}(\bar{D})$, and the following estimates are valid (see [6]):

$$\|\phi_g\|_{\alpha,D} \leq C_1(\alpha, \chi, D)\|g\|_{\alpha,\partial D},$$

$$\|\phi_g'\|_{\alpha,D} \leq C_2(\alpha, \chi, D)\|g\|_{1,\alpha,\partial D},$$

$$\|\phi_{(w,h)}\|_{\alpha,D} \leq C_3(\alpha, \chi, D)\|h\|_{\alpha,\partial D} + C_4(\alpha, \chi, D)\|F(z, w, h)\|_{\alpha,D},$$

$$\|\phi_{(w,h)}'\|_{\alpha,D} \leq C_4(\alpha, \chi, D)\|h'\|_{\alpha,\partial D} + C_6(\alpha, \chi, D)\|F(z, w, h)\|_{\alpha,D}.$$

So that the corresponding w solves the modified Riemann-Hilbert problems (1) and (2).

3. Conclusion

In this paper we have discussed the modified Riemann-Hilbert boundary value problem for complex partial differential equation (1) and (2) in $C_{1,\alpha}$. We can discuss on the existence and uniqueness of the boundary value problem in the Sobolev space.

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