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RYCHLIK'S THEOREM AND INVARIANT MEASURES FOR RANDOM MAPS OF PIECEWISE EXPANDING C^1 MAPS SATISFYING SUMMABLE OSCILLATION CONDITION

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Abstract: We prove a Rychlik's type theorem for random maps where each of the component maps is piecewise C^1, piecewise expanding and satisfies summabale oscillation condition.

We prove the existence of absolutely continuous invariant measures for random maps. Our results are generalizations of results of Góra, Li and Boyarsky in [7] of single piecewise expanding maps to results of random maps. We present an example for the stability of absolutely continuous invariant measures.

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1. Introduction

The Lasota-Yorke inequality [9] plays an important role for the existence of absolutely continuous invariant measures for single maps using bounded variation techniques [1, 2, 6, 9, 10]. For bounded variation techniques, the Lasota –Yorke method requires that we use an iterate τ^n of the map τ for which the $\inf|(\tau^n)'| > 2$. Then the partition $\mathcal{P}^{(n)}$ of τ^n is used in an argument where the magnitude of the minimum length of $\mathcal{P}^{(n)}$ appears in the denominator of a

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term. However, in some situations, for example, piecewise expanding maps with periodic turning points (with slope ≤ 2), the standard Lasota–Yorke inequality cannot be applied because $1 \leq \inf|(\tau^n)'| \leq 2$ at turning periodic points and the iterate τ^n creates partition elements which go to 0 length (see for example, [7, 8]). In order to overcome this difficulty, P. Eslami and P. Gora [5] introduced a stronger Lasota–Yorke inequality for piecewise expanding $C^{1,1}$ maps and prove the existence of absolutely continuous invariant measures (acim). In [8], a stronger Lasota–Yorke inequality for random maps was introduced for the existence of acim where the component maps are piecewise expanding $C^{1,1}$. Recently, P. Góra, Z. Li and A. Boyarsky [7] have applied Rychlik's Theorem for piecewise expanding C^1 maps satisfying harmonic average of slopes condition and summabale oscillation conditions and prove the existence of acim. In this note, we generalize results [7] of single piecewise expanding maps to results of random maps.

In Section 2 we present the notation and summarize results we shall need in the sequel. In Section 3 we prove Rychlik's Theorem for random maps and apply our result for the existence of invariant measures for random maps where the component maps are piecewise C^1 and they satisfy harmonic average of slopes condition and summabale oscillation condition. We present an example in Section 4 for the stability of acims for random maps.

2. Notation and Preliminary Results

Let $(I = [0,1], \mathcal{B}, \lambda)$ be a measure space, where λ is the Lebesgue measure on I and $\mathcal{B}$ is a σ-algebra on I. We will be concerned with random maps where the component maps are piecewise expanding and satisfies harmonic average condition and summable oscillation condition (see [7]):

Definition 2.1. [7] A transformation $\tau : I \to I$ is in the class $\mathcal{T}_H(I)$, the class of piecewise expanding transformations satisfying harmonic average of slopes condition, if there exists a partition $\mathcal{P} = \{I_i := [a_{i-1}, a_i], i = 1, 2, \ldots, q\}$ of I such that $\tau : I \to I$ satisfies the following conditions.

1. $\tau_i = \tau_{|_{I_i}}$ is monotonic on each interval I_i;

2. $\tau_i := \tau|_{I_i}$ is C^1 and $\lim_{x\to a_{i-1}^+} \tau'(x), \lim_{x\to a_i^-} \tau'(x)$ exist; Let

$$M = \max_{x\in I} |\tau'(x)|;$$

3. $|\tau_i'(x)| \geq s_i > 1$ for any i and for all $x \in (a_{i-1}, a_i)$.

4. (Harmonic average of slopes condition):

$$s_H = \max_{i=1,2,\ldots,q-1} \{\frac{1}{s_i} + \frac{1}{s_{i+1}}\} < 1. \tag{2.1}$$

Let

$$\delta := \min_{2\leq i\leq q-1} \lambda(I_i), \; s := \min_{1\leq i\leq q} s_i. \tag{2.2}$$

For $n \geq 1$, we define the partition $\mathcal{Q}^{(n)}$ as follows:

$$\mathcal{Q}^{(n)} = \bigvee_{k=0}^{n-1} \tau^{-k}(\mathcal{P}) = \{I_{k_0} \cap \tau^{-1}(I_{k_1}) \cap \cdots \cap \tau^{-n+1}(I_{k_n}) : I_{k_j} \in \mathcal{P}\}.$$

For any measurable set $A \in [0,1]$, let $\mathcal{P}(A) = \{J \in \mathcal{P} : \lambda(J \cap A) > 0\}$. For $J \in \mathcal{Q}^{(n)}$, we define $\mathrm{osc}_{\mathrm{J}} \frac{1}{|\tau'|} = \max_{\mathrm{J}} \frac{1}{|\tau'|} - \min_{\mathrm{J}} \frac{1}{|\tau'|}$ and $d_n = \max_{J\in\mathcal{Q}^{(n)}} \mathrm{osc}_{\mathrm{J}} \frac{1}{|\tau'|}$.

Definition 2.2. [7] A transformation $\tau : I \to I$ satisfies the summable oscillation condition, or $\tau \in \mathcal{T}_{\sum}(I)$ if τ satisfies first 3 conditions of definition 2.1 and

$$\sum_{n\geq 1} d_n \leq D < +\infty.$$

Random maps: Let $\{\tau_1, \tau_2, \cdots, \tau_K\}$ be a collection of nonsingular maps on $[0,1]$ into $[0,1]$ and $\{p_1, p_2, \cdots, p_K\}$ be a collection of probabilities. A random map is a dynamical system in which one of the number of transformations $\{\tau_1, \tau_2, \cdots, \tau_K\}$ is randomly selected according probabilities $\{p_1, p_2, \cdots, p_K\}$ and applied at each iteration of the process. Random maps have application in the study of fractals [3], in computing metric entropy [12], in modeling interference effects in quantum mechanics [4] and in forecasting the financial markets [11, 13]. For any $x \in X$, $T(x) = \tau_k(x)$ with probability p_k and, for any non-negative integer N, $T^N(x) = \tau_{k_N} \circ \tau_{k_{N-1}} \circ \ldots \circ \tau_{k_1}(x)$ with probability $\prod_{i=1}^N p_{k_i}$. A measure μ is T invariant (see [10]) if and only if

$$\mu(A) = \sum_{k=1}^{K} p_k \mu(\tau_k^{-1}(A)).$$

for any $A \in \mathcal{B}$.

Let $T = \{\tau_1(x), \tau_2(x), \ldots, \tau_K(x); p_1, p_2, \ldots, p_K\}$ be a random map where the maps $\tau_k : I \to I$, $k = 1, 2, \ldots, K$ are in the class $\mathcal{T}_{\sum}(I)$ and satisfy harmonic average of slopes condition (2.1) on a common partition $\mathcal{P} = \{I_i := [a_{i-1}, a_i], i = 1, 2, \ldots, q\}$ of I.. For any k and for any i, let

$$\tau_{k,i} = \tau_{k_{|_{I_i}}}, |\tau'_{k,i}(x)| \geq s_{k,i} > 1 \text{ for all } \mathrm{x} \in (\mathrm{a_{i-1}}, \mathrm{a_i}).$$

Let

$$\begin{aligned} s_k : &= \min_{1\leq i\leq q} s_{k,i}, k = 1, 2, \ldots, K, \\ M_k : &= \max_{x\in I} |\tau'_k(x)|, k = 1, 2, \ldots, K \ \text{ and } \mathrm{M}^* = \max_{1\leq \mathrm{k}\leq \mathrm{K}} \mathrm{M_k}. \end{aligned}$$

Also, let

$$s_{H,k} = \max_{i=1,2,\ldots,q-1} \{\frac{1}{s_{k,i}} + \frac{1}{s_{k,i+1}}\}.$$

In [10], it was proved that the Frobenius–Perron operator $P_T : L^1(I) \to L^1(I)$ of a random map T is given by

$$P_T f(x) = \sum_{k=1}^{K} p_k P_{\tau_k}(f)(x), \tag{2.3}$$

where P_{τ_k} is the Frobenius-Perron operator (see [2]) associated with the transformation τ_k and P_{τ_k} is defined by

$$P_{\tau_k} f(x) = \sum_{y\in\tau_k^{-1}(x)} h_k(y) f(y) = \sum_{i=1}^{q} \frac{f(\tau_{k,i}^{-1}(x))}{|\tau'_k(\tau_{k,i}^{-1}(x))|} \chi_{\tau_k(I_i)}(x),$$

where $h_k(y) := \frac{1}{|\tau'_k(y)|}, k = 1, 2, \ldots, K$. For each $n \geq 1$ we define $\Omega_n = \{\omega_n = (k_1, k_2, \ldots, k_n) : k_j \in \{1, 2, \ldots, K\}, j = 1, 2, \ldots, n\}$. For $\omega_{n-1} \in \Omega_{n-1}$, $\omega_{n-1} = (k_1, k_2, \ldots, k_{n-1})$, let $\mathcal{P}^{(n)}_{\omega_{n-1}}$ be the the common refinement of $\mathcal{P}, \tau_{k_1}^{-1}(\mathcal{P}), (\tau_{k_2} \circ \tau_{k_1})^{-1}(\mathcal{P}), \ldots, (\tau_{k_{n-1}} \circ \tau_{k_{n-2}} \circ \ldots \circ \tau_{k_2} \circ \tau_{k_1})^{-1}(\mathcal{P})$, $n \geq 1$, where $\sigma^{-1}\mathcal{P} = \{\sigma^{-1}(J) : J \in \mathcal{P}\}$. Then, $\mathcal{P}^{(n)}$ is the union of all $\mathcal{P}^{(n)}_{\omega_{n-1}}$, $\omega_{n-1} \in \Omega_{n-1}$. Let $I^{(n)}$ denote a generic element of $\mathcal{P}^{(n)}$. We have $I^{(n)} \in \mathcal{P}^{(n)}_{\omega_{n-1}}$, for some $\omega_{n-1} = (k_1, k_2, \ldots, k_{n-1})$ or $I^{(n)} = (\tau_{k_{n-1}} \circ \tau_{k_{n-2}} \circ \ldots \circ \tau_{k_2} \circ \tau_{k_1})^{-1} I_s$, for some $I_s \in \mathcal{P}$. We will write $I^{(n)} = I^{(n)}_{\omega_{n-1}}$. Let $T^{i-1}_{\omega_{i-1}} = \tau_{k_{i-1}} \circ \tau_{k_{i-2}} \circ \ldots \circ \tau_{k_2} \circ \tau_{k_1}$. Then, $T^{n-1}_{\omega_{n-1}}$ is well defined on $I^{(n)}_{\omega_{n-1}}$. Moreover $T^{i-1}_{\omega_{i-1}}(I^{(n)}_{\omega_{n-1}}) \in \mathcal{P}^{(n-i)}_{\omega_{n-i-1}}$, where

$\omega_{n-i-1} = (k_{i+1}, k_{i+2}, \ldots, k_{n-1})$.

Let $g_{\omega_n} = \frac{1}{|(T^n_{\omega_n})'|} = \frac{1}{|(\tau_{k_n} \circ \tau_{k_{n-1}} \circ \ldots \circ \tau_{k_2} \circ \tau_{k_1})'|}$. For $J \in \mathcal{P}^{(n)}$, we define $\mathrm{osc_J} \frac{1}{|\tau'_\mathrm{k}|} = \mathrm{max_J} \frac{1}{|\tau'_\mathrm{k}|} - \min \frac{1}{|\tau'_\mathrm{k}|}$, $\mathrm{k} = 1, 2, \ldots, \mathrm{K}$, $d_{n,k} = \max_{J \in \mathcal{P}^{(n)}} \mathrm{osc_J} \frac{1}{|\tau'_\mathrm{k}|}$, $\mathrm{k} = 1, 2, \ldots, \mathrm{K}$ and $d^*_n = \max_{1 \le k \le K} d_{n,k}$.

3. Rychlik's Theorem for Random Maps

We now state and prove Rychlik's Theorem for random maps.

Theorem 3.1. *Let $T = \{\tau_1(x), \tau_2(x), \ldots, \tau_K(x); p_1, p_2, \ldots, p_K\}$ be a random map where the maps $\tau_k : I \to I$, $k = 1, 2, \ldots, K$ are piecewise monotonic on a common partition $\mathcal{P} = \{I_i := [a_{i-1}, a_i], i = 1, 2, \ldots, q\}$ of I and $\{p_k\}_{k=1}^K$ is a set of probabilities. Moreover, assume that the random map T satisfies the following conditions:*

(A) There exists a constant $d > 0$ such that for any $n \ge 1$ and any $J \in \mathcal{P}^{(n)}$,

$$\sup_J g_{\omega_n} \le d \cdot \inf_J g_{\omega_n};$$

(B) there exists $\epsilon > 0$ and $r \in (0, 1)$ such that for any $n \ge 1$ and any $J \in \mathcal{P}^{(n)}$,

$$\lambda(T^n_{\omega_n}(J)) < \epsilon \implies \sum_{J' \in \mathcal{P}(T^n_{\omega_n}(J))} \sup_{J'} g_{\omega_1} \le r;$$

(C)

$$\sum_{J \in \mathcal{P}} \sup_J g_{\omega_1} < +\infty.$$

Then, the random map T admits an absolutely continuous invariant measure.

Proof. Let $\mathbf{1}$ denotes the constant function equal to 1 everywhere on $[0, 1]$. Consider the sequence $f_n = P^n_T \mathbf{1}, n = 1, 2, \ldots$. We will prove that the functions $\{f_n\}_{n=1}^\infty = \{P^n_T \mathbf{1}\}_{n=1}^\infty$ are uniformly bounded. We have

$$\begin{aligned} f_1(x) &= P_T \mathbf{1}(x) = \sum_{k=1}^K p_k P_{\tau_k} \mathbf{1}(x) = \sum_{k=1}^K p_k \sum_{i=1}^q \frac{1}{|\tau'_k(\tau^{-1}_{k,i}(x))|} \chi_{\tau_k(I_i)}(x). \\ f_2(x) &= P^2_T \mathbf{1}(x) = P_T\left(P_T \mathbf{1}(x)\right) = P_T\left(\sum_{k=1}^K p_k P_{\tau_k} \mathbf{1}(x)\right) \\ &= \sum_{k=1}^K p_k P_T\left(P_{\tau_k} \mathbf{1}(x)\right) \end{aligned}$$

$$= \sum_{k=1}^{K} p_k \sum_{l=1}^{K} p_l P_{\tau_l}\left(P_{\tau_k}\mathbf{1}(x)\right)$$
$$= \sum_{k=1}^{K}\sum_{l=1}^{K}\sum_{j=1}^{q}\sum_{i=1}^{q} \frac{p_k p_l}{|\tau_l'(\tau_{l,j}^{-1}(\tau_{k,i}^{-1}(x)))||\tau_k'(\tau_{k,i}^{-1}(x))|}\chi_{\tau_l(I_j)}(\tau_{k,i}^{-1}(x))\chi_{\tau_k(I_i)}(x).$$

Thus, we have

$$f_2(x) = \sum_{1\le k_1,k_2\le K}\ \sum_{1\le i_1,i_2\le q} \frac{p_{k_1}p_{k_2}}{|\tau_{k_2}'(\tau_{k_2,i_2}^{-1}(\tau_{k_1,i_1}^{-1}(x)))||\tau_{k_1}'(\tau_{k_1,i_1}^{-1}(x))|} \times \chi_{\tau_{k_2}(I_{i_2})}(\tau_{k_1,i_1}^{-1}(x))\chi_{\tau_{k_1}(I_{i_1})}(x).$$

In general, we have

$$\begin{aligned} f_n(x) = & \sum_{1\le k_1,k_2,\ldots,k_n\le K}\ \sum_{1\le i_1,i_2,\ldots,i_n\le q} \\ & \left[\frac{p_{k_1}p_{k_2}\cdots p_{k_n}}{|\tau_{k_n}'(\tau_{k_n,i_n}^{-1}(\tau_{k_{n-1},i_{n-1}}^{-1}(\ldots(\tau_{k_2,i_2}^{-1}(\tau_{k_1,i_1}^{-1}(x)))\ldots))))|}\right. \\ & \left.\frac{1}{|\tau_{k_{n-1}}'(\tau_{k_{n-1},i_{n-1}}^{-1}(\ldots(\tau_{k_2,i_2}^{-1}(\tau_{k_1,i_1}^{-1}(x)))\ldots))|\ldots|\tau_{k_1}'(\tau_{k_1,i_1}^{-1}(x))|}\right]\times \\ & \chi_{\tau_{k_n}(I_{i_n})}(\tau_{k_{n-1},i_{n-1}}^{-1}\circ\tau_{k_{n-2},i_{n-2}}^{-1}\ldots\tau_{k_2,i_2}^{-1} \\ & \circ\tau_{k_1,i_1}^{-1}(x))\ldots\chi_{\tau_{k_2}(I_{i_2})}(\tau_{k_1,i_1}^{-1}(x))\chi_{\tau_{k_1}(I_{i_1})}(x) \\ = & \sum_{1\le k_1,k_2,\ldots,k_n\le K}\ \sum_{1\le i_1,i_2,\ldots,i_n\le q} \\ & \left[\frac{p_{k_1}p_{k_2}\cdots p_{k_n}}{|(\tau_{k_n,i_n}\circ\tau_{k_{n-1},i_{n-1}}\circ\ldots\circ\tau_{k_2,i_2}\circ\tau_{k_1,i_1})'(\phi_{n,\mathbf{k}_n,\mathbf{j}_n}(x))|}\right]\times \\ & \chi_{\tau_{k_n}(I_{i_n})}(\tau_{k_{n-1},i_{n-1}}^{-1}\circ\tau_{k_{n-2},i_{n-2}}^{-1}\ldots\tau_{k_2,i_2}^{-1} \\ & \circ\tau_{k_1,i_1}^{-1}(x))\ldots\chi_{\tau_{k_2}(I_{i_2})}(\tau_{k_1,i_1}^{-1}(x))\chi_{\tau_{k_1}(I_{i_1})}(x), \end{aligned}$$

where

$$\phi_{n,\mathbf{k}_n,\mathbf{j}_n} = (\tau_{k_n,i_n}\circ\tau_{k_{n-1},i_{n-1}}\circ\ldots\circ\tau_{k_2,i_2}\circ\tau_{k_1,i_1})^{-1},$$

with $\mathbf{j}_n = (i_n, i_{n-1},\ldots,i_2,i_1) \in \{1,2\ldots,q\}^n, \mathbf{k}_n = (k_n,k_{n-1},\ldots,k_2,k_1) \in \{1,2,\ldots,K\}^n$. Recall that $g_{\omega_n} = \frac{1}{|(T_{\omega_n}^n)'|} = \frac{1}{|(\tau_{k_n}\circ\tau_{k_{n-1}}\circ\ldots\circ\tau_{k_2}\circ\tau_{k_1})'|}$. Let $\gamma_{\omega_n} = \sum_{J\in\mathcal{P}^{(n)}}\sup_J g_{\omega_n}$. It can be easily shown that

$$\| f_n \|_\infty = \| P_T^n\mathbf{1} \|_\infty$$

$$= \inf\{c : \lambda\{x \in I : P_T^n \mathbf{1}(x) > c\} = 0\}$$
$$\leq \gamma_{\omega_n}.$$

Thus, in order to prove that the functions $\{f_n\}_{n=1}^{\infty} = \{P_T^n \mathbf{1}\}_{n=1}^{\infty}$ are uniformly bounded, it is enough to prove that $\{\gamma_{\omega_n}\}_{n=1}^{\infty}$ is bounded. By condition (C), $\gamma_{\omega_1} = \sum_{J \in \mathcal{P}} \sup_J g_{\omega_1}$ is bounded. We divide the sets of $\mathcal{P}^{(n)}$ into two groups:

$$\mathcal{P}_1 = \{J' \in \mathcal{P}^{(n)} : \lambda(T_{\omega_n}^n J') < \epsilon\}$$

and

$$\mathcal{P}_2 = \{J' \in \mathcal{P}^{(n)} : \lambda(T_{\omega_n}^n J') \geq \epsilon\}.$$

By following the analogous method similar to the proof of Theorem 6.2.1 in [2], it can be easily show that

$$\gamma_{\omega_{n+1}} \leq r^n \gamma_{\omega_n} + C_1,$$

where $C_1 = \frac{\gamma_{\omega_1 d}}{\epsilon}$. Using this estimate, it can show that

$$\gamma_{\omega_{n+1}} \leq r^n \gamma_{\omega_1} + C_1 \frac{1}{1-r}.$$

Thus, the functions $\{f_n\}_{n=1}^{\infty} = \{P_T^n \mathbf{1}\}_{n=1}^{\infty}$ are uniformly bounded in L^{∞} and hence $\{P_T^n \mathbf{1}\}_{n=1}^{\infty}$ is weakly compact in L^1. Therefore, any weak limit points is a P_T invariant density, by virtue of Kakutani–Yoshida Theorem. □

The following Theorem is an application of the above Rychlik's Theorem (Theorem 3.1) to random maps satisfying harmonic average condition.

Theorem 3.2. *Let $T = \{\tau_1(x), \tau_2(x), \ldots, \tau_K(x); p_1, p_2, \ldots, p_K\}$ be a random map such that $\tau_k \in \mathcal{T}_\Sigma, k = 1, 2, \ldots K$ are defined on a common partition $\mathcal{P} = \{I_i := [a_{i-1}, a_i], i = 1, 2, \ldots, q\}$ of I and satisfy the average of slopes conditions $s_{k,H} < 1, k = 1, 2, \ldots, K$. Then the random map T satisfies the assumptions of the above Rychlik's Theorem (Theorem 3.1).*

Proof. Let $s = \min_{1 \leq k \leq K} s_k$. Then $\sup g_{\omega_1} \leq \frac{1}{s}$. Let $J \in \mathcal{P}^{(n)}$ and $x, y \in J$. We have

$$\frac{g_{\omega_n}(x)}{g_{\omega_n}(y)} = \frac{g_{\omega_1}(T_{\omega_{n-1}}^{n-1}(x) g_{\omega_1}(T_{\omega_{n-2}}^{n-2}(x)) \ldots g_{\omega_1}(T_{\omega_1}^{1}(x)) g_{\omega_1}(x)}{g_{\omega_1}(T_{\omega_{n-1}}^{n-1}(y)) g_{\omega_1}(T_{\omega_{n-2}}^{n-2}(y)) \ldots g_{\omega_1}(T_{\omega_1}^{n}(y)) g_{\omega_1}(y)}.$$

For any $i = 0, 1, 2, \ldots n-1$, $T_{\omega_i}^i(x)$ and $T_{\omega_i}^i(y)$ belong to the same element J_i of $\mathcal{P}^{(n-i)}$. Using the inequality

$$\frac{a}{b} = 1 + \frac{a-b}{b} \leq \exp\left(\left|\frac{a-b}{b}\right|\right),$$

we get

$$
\begin{aligned}
\frac{g_{\omega_1}(T^i_{\omega_i}(x))}{g_{\omega_1}(T^i_{\omega_i}(y))} &= \frac{\frac{1}{g_{\omega_1}(T^i_{\omega_i}(y))}}{\frac{1}{g_{\omega_1}(T^i_{\omega_i}(x))}} \\
&\leq \exp\left(g_{\omega_1}(T^i_{\omega_i}(x))\Big|\frac{1}{g_{\omega_1}(T^i_{\omega_i}(x))} - \frac{1}{g_{\omega_1}(T^i_{\omega_i}(y))}\Big|\right) \\
&\leq \exp\left(M^* \cdot d^*_{n-i}\right)
\end{aligned}
$$

and thus,

$$
\frac{g_{\omega_n}(x)}{g_{\omega_n}(y)} \leq \exp\left(M^* \cdot \sum_{i=0}^{n-1} d^*_{n-i}\right).
$$

Thus, condition (A) is satisfied. Now, we invoke the harmonic average of slopes conditions of component maps of the random map to prove condition (B). Let $\epsilon = \frac{1}{2}\delta$ and $r = \max_{1\leq k\leq K} s_{H,k} < 1$. For any $J' \in \mathcal{P}^{(n)}$, $T^n_{\omega_n}(J')$ is an interval and if $\lambda(T^n_{\omega_n}(J') < \epsilon$, then $T^n_{\omega_n}(J')$ can intersect at most two intervals of $\mathcal{P}$. Therefore, $\sum_{J'\in\mathcal{P}(T^n_{\omega_n}(J))} \sup_{J'} g_{\omega_1} \leq r$. Condition (C) is obvious. This complete the proof. □

4. Examples

We present an example to show that Theorem 3.1 and Theorem 3.2 in Section 3 are useful when the component maps with a fixed (or periodic) turning point is perturbed and standard Lasota – Yorke inequality (for example, in [10] or in [1]) and the new (stronger) Lasota – Yorke inequality in [8] fails in the limit.

Example 4.1. Consider the random map $T = \{\tau_1, \tau_2; \frac{2}{5}, \frac{3}{5}\}$, where $\tau_1 : [0,1] \to [0,1]$ is defined by

$$
\tau_1(x) = \begin{cases} 1 - 5x, & 0 \leq x < \frac{1}{6}, \\ 5x - \frac{2}{3}, & \frac{1}{6} \leq x < \frac{1}{3}, \\ -9x + 4, & \frac{1}{3} \leq x < \frac{4}{9}, \\ \frac{9}{5}x - \frac{4}{5}, & \frac{4}{9} \leq x \leq 1, \end{cases}
$$

and $\tau_2 : [0,1] \to [0,1]$ is from [7] which is defined by

$$\tau_2(x) = \begin{cases} 1-6x, & 0 \le x < \frac{1}{6}, \\ -\frac{2}{9}+\frac{5}{3}x - \frac{\sqrt{6}}{81}(3-9x)^{\frac{3}{2}}, & \frac{1}{6} \le x < \frac{1}{3}, \\ -3(\frac{1}{3}+\frac{1}{n})(x-\frac{4}{9}), & \frac{1}{3} \le x < \frac{4}{9}, \\ \frac{9}{5}x - \frac{4}{5}, & \frac{4}{9} \le x \le 1, \end{cases}$$

Here, $\frac{1}{6}$ is a fixed turning point for τ_1 and $\frac{1}{3}$ is a fixed turning point for τ_2. The

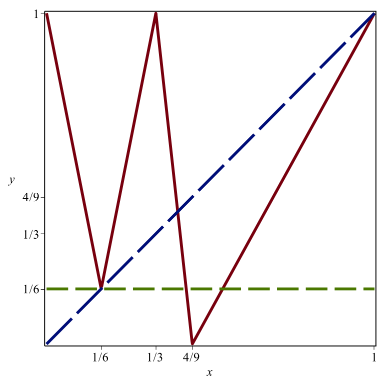

Figure 1: Map τ_1 .

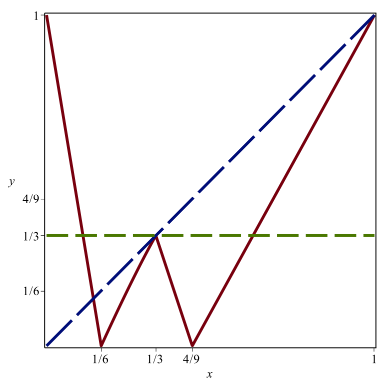

Figure 2: Map τ_2 .

map τ_2 is not C^2 and it is also not C^{1+1}. Moreover,

$$s_{H,1} = \max\{\frac{1}{5} + \frac{1}{5}, \frac{1}{5} + \frac{1}{9}, \frac{1}{9} + \frac{5}{9}\} = \frac{6}{9} < 1.$$

$$s_{H,2} = \max\{\frac{1}{6} + \frac{3}{5}, \frac{3}{5} + \frac{1}{3}, \frac{1}{3} + \frac{5}{9}\} = \frac{24}{27} < 1.$$

It can be easily shown that the random map T satisfies the hypothesis of Theorem 3.2. Thus, by Theorem 3.2, the random map T satisfies the assumptions of Rychlik's Theorem 3.1 and thus by Rychlik's Theorem 3.1, the random map T has an absolutely continuous invariant measure. Note that, Pelikan's [10] existence theorem can not be applied to T because τ_2 is not a Lasota-Yorke type map.

Now, consider the one parameter family $\{T_n\}$ of perturbations

$$T_n = \{\tau_{1,n}(x), \tau_{2,n}(x); p_1 = \frac{2}{5}, p_2 = \frac{3}{5}\},$$

where $\tau_{1,n}$ is a perturbation of τ_1 by moving the fixed turning point $\frac{1}{6}$ by $\frac{1}{n}$ downward and $\tau_{2,n}$ is a perturbation of τ_2 by moving the fixed turning point $\frac{1}{3}$ by $\frac{1}{n}$ upward:

$$\tau_{1,n}(x) = \begin{cases} -(5 + \frac{6}{n})x + 1, & 0 \leq x < \frac{1}{6}, \\ (5 + \frac{6}{n})(x - \frac{1}{3}) + 1, & \frac{1}{6} \leq x < \frac{1}{3}, \\ -9x + 4, & \frac{1}{3} \leq x < \frac{4}{9}, \\ \frac{9}{5}x - \frac{4}{5}, & \frac{4}{9} \leq x \leq 1, \end{cases}$$

$$\tau_{2,n}(x) = \begin{cases} 1 - 6x, & 0 \leq x < \frac{1}{6}, \\ -\frac{2}{9} + \frac{5}{3}x - \frac{\sqrt{6}}{81}(3 - 9x)^{\frac{3}{2}}, & \frac{1}{6} \leq x < \frac{1}{3}, \\ -3x + \frac{4}{3}, & \frac{1}{3} \leq x < \frac{4}{9}, \\ \frac{9}{5}x - \frac{4}{5}, & \frac{4}{9} \leq x \leq 1, \end{cases}$$

Theorem 3.1 and Theorem 3.2 in Section 3 are useful to prove the stability of acim under the above perturbations. The new (stronger) Lasota–Yorke inequality in [8] is not useful to prove the stability of acim under the above perturbations because the map τ_2 is not C^{1+1}. It is not difficult to show that the

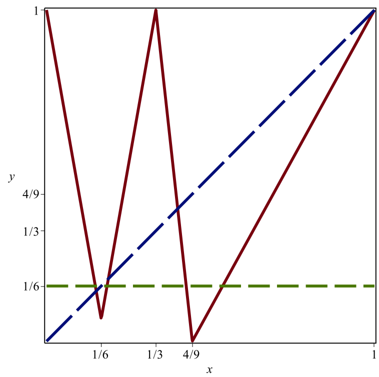

Figure 3: A perturbation $\tau_{1,n}$ of the map τ_1 with $n = 10$.

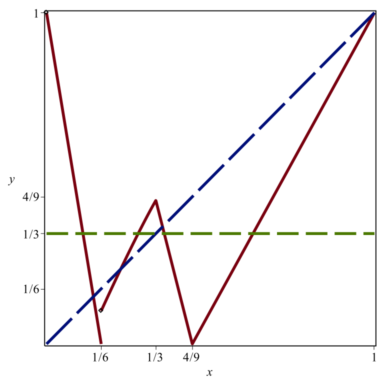

Figure 4: A perturbation $\tau_{2,n}$ of the map τ_2 with $n = 10$.

standard Lasota – Yorke inequality in [10] or in [1] blows up as $n \to \infty$. Iterates cannot be used as they produce arbitrary small intervals of the partition and second constant in the classical Lasota – Yorke inequality goes to ∞.

References

[1] Bahsoun, W. and Góra, P., *Position Dependent Random Maps in One and Higher Dimensions*, Studia Math., 166 (2005), 271-286.

[2] Boyarsky, A. and Góra, P., *Laws of Chaos*, Birkhaüser, 1997.

[3] Barnsley, M. , *Fractals everywhere*, Academic Press, London, 1998.

[4] Boyarsky, A. and Góra, P., *A dynamical model for interference effects and two slit experiment of quantum physics*, Phys, Lett. A 168 (1992), 103-112.

[5] Eslami, P. and Góra, P., *Stronger Lasota-Yorke inequality for one dimensional piecewise expanding maps*, Proc. Amer. Math. Soc. (2012) (to appear).

[6] Góra, P. and Boyarsky, A., *Absolutely continuous invariant measures for random maps with position dependent probabilities*, Math. Anal. and Appl. 278 (2003), 225-242.

[7] Góra, P., Li, Z. and Boyarsky, A., *Harmonic averages of slopes and the stability of absolutely continuous invariant measure*, J. Math. Anal. and Appl. (2003),doi: 10.1016/j.jmaa.2012.05.067.

[8] Islam, M. S., *A Lasota –Yorke type iniquality for position dependent random maps of one dimensional piecewise expanding* $\mathcal{C}^{1,1}$ *transformations,* accepted for publication in Dynamics of continuous, Discrete and Impulsive Systems- Series A.

[9] Lasota, A. and Yorke, J.A., *On the existence of invariant measures for piecewise monotonic transformations*, Trans. Amer. Math. Soc. 186 (1973),481-488.

[10] Pelikan, S., *Invariant densities for random maps of the interval*, Proc. Amer. Math. Soc. 281 (1984), 813-825.

[11] Schenk-Hoppe, K. R., *Random Dynamical Systems in Economics*, working paper series, Institute of empirical research in economics, University of Zurich (2000), 1424-0459,

[12] Slomczynski, W., Kwapien, J. and Zyczkowski, K., *Entropy computing via integration over fractal measures*, Chaos 10 (2000), 180-188.

[13] Swishchuk, A.V. and Islam, M. S. [2011], *The Geometric Markov Renewal Processes with application to Finance*, Stochastic analysis and application, Vol 29, Number 4 (2011), 684–705.