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SUBSPACE-COUNTABLY HYPERCYCLICITY CRITERION FOR TUPLES OF OPERATORS

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Abstract: In this paper, we give a criterion under which a tuple being countably subspace-hypercyclic.

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1. Introduction

By an n-tuple of operators we mean a finite sequence of length n of commuting continuous linear operators on a Banach space X.

Definition 1.1. Let $\mathcal{T} = (T_1, T_2, ..., T_n)$ be an n-tuple of operators acting on an infinite dimensional Banach space X and let M be a nonzero subspace of X. We will let

$$\mathcal{F} = \{{T_1}^{k_1}{T_2}^{k_2}...{T_n}^{k_n} : k_i \geq 0, i = 1, ..., n\}$$

be the semigroup generated by $\mathcal{T}$. For $x \in X$, the orbit of x under the tuple $\mathcal{T}$ is the set $Orb(\mathcal{T}, x) = \{Sx : S \in \mathcal{F}\}$. A vector x is called a subspace-hypercyclic (or M-hypercyclic) for $\mathcal{T}$ if $Orb(\mathcal{T}, x) \cap M$ is dense in M and in this case the tuple $\mathcal{T}$ is called M-hypercyclic.

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Definition 1.2. A sequence $\{x_k\}_k$ in a Banach space X is called separated if there exists $\epsilon > 0$ such that $\|x_n - x_k\| \geq \epsilon$ for all $n \neq k$.

Definition 1.3. Let $\mathcal{T} = (T_1, T_2)$ be a pair of continuous linear operators acting on a separable infinite dimensional Banach space X and M be a nonzero closed subspace of X. The pair $\mathcal{T}$ is called finitely M-hypercyclic if there exists a finite set $\{x_1, ..., x_n\}$ such that $\cup_{k=1}^{n} Orb(\mathcal{T}, x_k) \cap M$ is dense in M. Also, $\mathcal{T}$ is called countably hypercyclic if there exists a bounded separated sequence $\{x_k\}_k$ such that $\cup_{k=1}^{\infty} Orb(\mathcal{T}, x_k) \cap M$ is dense in M.

Suprisingly, there are something that does not happen for single operators. For example, hypercyclic tuples can arise in finite dimensional, and there are operators that have somewhere dense orbits that are not everywhere dense ([5]). Also, we note that there are subspace-hypercyclic operators that are not hypercyclic ([7]). Here, we want to extend the countably hypercyclicity criterion to the countably subspace-hypercyclicity criterion for a pair of commuting operators. For some topics we refer to [1–14]. The papers [4, 7] have an important role to prove our main results.

2. Main Results

In this section, we will give the countably subspace-hypercyclicity criterion for tuples of operators.

Lemma 2.1. *If Y is an infinite dimensional manifold in a Banach space, then there exists a sequence $\{x_n\} \subset Y$ such that $\|x_n\| = 1$ for all $n \geq 1$, and $\|x_n - x_k\| > 1$ for all $n \neq k$.*

Proof. By an application of the Hahn-Banach Theorem the proof is clear. □

Theorem 2.2. *Let $\mathcal{T} = (T_1, T_2)$ be a pair of continuous linear operators acting on a separable infinite dimensional Banach space X and M be a nonzero closed subspace of X. If T_i is invertible and $\|T_i\| < 1$ for $i = 1, 2$, then $\mathcal{T}$ is countably M-hypercyclic.*

Proof. Set $U = \{x \in X : \|x\| > 1\}$ and $Z = \{z \in \mathbf{C} : |z| > 1\} \cap Q_c = \{z_m\}_m$ where Q_c is the set of all complex rational numbers. Also, let $Y = \{y_n : n \geq 0\}$ be a countable dense subset of U. If $Z = \{z_m : m \geq 0\}$, then clearly

$\{e_{(m,n)} : m, n \geq 0\}$ is dense in U, where $e_{(m,n)} = z_m y_n$ for all $m, n \geq 0$. Define $x^{p,q}_{(m,n)} = T_1^{-p} T_2^{-q} e_{(m,n)}$ for all $p, q \geq 0$. Thus

$$x^{p,q}_{(m,n)} \in \cup_{m,n} Orb(\mathcal{T}, e_{(m,n)}) \cap M = \cup_{m,n}\{T_1^i T_2^j e_{(m,n)} : i, j \geq 0\} \cap M$$

for all $p, q \geq 0$. Hence

$$U \subseteq \overline{\cup_{m,n} Orb(\mathcal{T}, e_{(m,n)})} \cap M.$$

Note that if x is a nonzero vector in M, then $\|T_1^{-i} T_2^{-j} x\| \to \infty$ as $i, j \to \infty$. Thus, for i, j large enough, $T_1^{-i} T_2^{-j} x \in U$, which implies that

$$x \in \overline{\cup_{m,n} Orb(\mathcal{T}, e_{(m,n)})} \cap M.$$

Thus, indeed $\mathcal{T}$ is countably hypercyclic and so the proof is complete. □

Theorem 2.3. (The Countably Subspace-Hypercyclicity Criterion for Tuples) *Suppose that $\mathcal{T} = (T_1, T_2)$ is a pair of continuous linear operators acting on a separable infinite dimensional Banach space X and let M be a nonzero closed subspace of X. Let there exists a pair of nonnegative sequences $(\{m_k\}, \{n_k\})$ of integers, and also there exist two subspaces Y and Z in M, where Y is infinite dimensional and Z is dense in M such that:*

1. $T_1^{m_k} T_2^{n_k} y \to 0$ for all $y \in Y$, and

2. For every $z \in Z$, there exists a sequence $\{x_k\}k$ in M such that $x_k \to 0$ and $T_1^{m_k} T_2^{n_k} x_k \to z$,

3. $T_1^{m_k} T_2^{n_k} M \subset M$ for all k.

Then T is countably M- hypercyclic.

Proof. Let $Z = \{z_n : n = 1, 2, ...\}$ and fix $0 < r < 1$. By the Lemma 2.1, there exists a sequence $\{y_n\} \subset Y$ such that $\|y_n\| = 1$ for all $n \geq 1$, and $\|y_n - y_k\| > 1$ for all $n \neq k$. Since $x_k \to 0$, $T_1^{m_k} T_2^{n_k} \to 0$ on Y, and $T_1^{m_k} T_2^{n_k} x_k - z_j \to 0$, thus for all j, there exists k_j such that $\|x_j\| \leq r/3j$, $\|T_1^{m_{k_j}} T_2^{n_{k_j}} y_j\| \leq r/3j$, and $\|T_1^{m_{k_j}} T_2^{n_{k_j}} x_j - z_j\| \leq r/3j$. Define $w_j = y_j + x_j$, then $\|w_j\| \leq 2$ for all j. If $i \neq j$, we have

$$\begin{aligned} \|w_i - w_j\| &= \|(y_i - y_j) + x_i - x_j\| \\ &\geq \|y_i - y_j\| - \|x_i - x_j\| \\ &> 1 - r. \end{aligned}$$

Thus $\{w_j\}_j$ is bounded and separated by $1 - r$. Now, we want to show that $\cup_{j=1}^{\infty} Orb(\mathcal{T}, w_j) \cap M$ is dense in M. For this, let $x \in M$ and $\delta > 0$. Since Z is dense in M, so there exists j large enough such that $1/j < \delta$ and $\|z_j - x\| < \delta/3$. Note that

$$\begin{aligned} T_1^{m_{k_j}} T_2^{n_{k_j}} w_j &= T_1^{m_{k_j}} T_2^{n_{k_j}} (y_j + x_j) \\ &= T_1^{m_{k_j}} T_2^{n_{k_j}} y_j + T_1^{m_{k_j}} T_2^{n_{k_j}} x_j. \end{aligned}$$

Therefore,

$$\begin{aligned} \|x - T_1^{m_{k_j}} T_2^{n_{k_j}} w_j\| &\leq \|x - z_j\| + \|T_1^{m_{k_j}} T_2^{n_{k_j}} y_j\| + \|T_1^{m_{k_j}} T_2^{n_{k_j}} x_j - z_j\| \\ &< \delta/3 + \delta/3j + \delta/3j < \delta. \end{aligned}$$

Thus,

$$B(x, \delta) \cap Orb(\mathcal{T}, w_j) \neq \emptyset.$$

Note that $w_j \in M$ and $Sw_j \in M$ for all $S \in \mathcal{F}$. Hence $Orb(\mathcal{T}, w_j) \subset M$ and this implies that indeed $\cup_{j=1}^{\infty} Orb(\mathcal{T}, w_j)$ is dense in M and so $\mathcal{T}$ is countably M-hypercyclic. Now the proof is complete. □

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