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NECESSARY CONDITIONS FOR A CLASS OF QUASIDIFFERENTIABLE OPTIMIZATION

Si-Da Lin[1 §], Wei Xia[2], Zun-Quan Xia[3]
[1,3]School of Mathematical Sciences
Dalian University of Technology
Dalian 116024, Liaoning Province, P. R. CHINA
[2]Department of Hydrography & Cartography
PLA Dalian Naval Academy
Dalian 116018, Liaoning Province, P. R. CHINA

Abstract: The necessary optimality conditions for the unconstrained kernelled quasidifferentiable optimization is given. The problem of minimizing a kernelled quasidifferentiable function on a set described by equality-type quasidifferentiable constraints is considered and the first-order necessary optimality condition for a minimum is derived under regularity condition.

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1. Introduction

Quasidifferential calculus, developed by Demyanov and Rubinov [1], is of considerable importance in nonsmooth analysis and optimization. Necessary optimality conditions, in the multiplier form, of unconstrained and constrained quasidifferentiable optimization have being studied since the early of 1980's. The necessary optimality conditions in geometric form for quasidifferentiable optimization were proposed by Polyakova [6] and Shapiro [7]. The versions of

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§Correspondence author

the Lagrange multiplier type in the inequality constrained case were developed by Eppler [3] and Luderer [5]. For the equality and inequality constrained case, optimality conditions with Lagrange multipliers were studied by Gao [4].

It is well known that the quasidifferential is not uniquely defined. Xia [8] introduced the notion of the kernelled quasidifferential, which can be taken as a representative of the equivalent class of quasidifferentials. The purpose of this paper is to explore necessary optimality conditions for unconstrained and constrained kernelled quasidifferentiable optimization. The rest of the paper is organized as follows. In Section 2, some preliminary definitions and results used in the paper are provided. In Section 3, the first-order necessary optimality conditions for unconstrained kernelled quasidifferentiable optimization is proposed, the problem of minimizing a kernelled quasidifferentiable function on a set described by equality-type quasidifferentiable constraints is considered and the first-order necessary optimality condition for a minimum is derived under regularity condition.

2. Preliminaries

A function f defined on an open set $\mathcal{O} \subset R^n$ is called quasidifferentiable at a point $x \in \mathcal{O}$, in the sense of Demyanov and Rubinov [1], if it is directionally differentiable at x and there exist two nonempty convex compact sets $\underline{\partial} f(x)$ and $\overline{\partial} f(x)$ such that the directional derivative can be represented as

$$f'(x;d) = \max_{u \in \underline{\partial} f(x)} \langle u, d\rangle + \min_{v \in \overline{\partial} f(x)} \langle v, d\rangle, \qquad \forall d \in R^n.$$

where $\langle \cdot, \cdot \rangle$ denotes the usual inner product in R^n. The pair of sets $Df(x) = [\underline{\partial} f(x), \overline{\partial} f(x)]$ is called a quasidifferential of f at x, $\underline{\partial} f(x)$ and $\overline{\partial} f(x)$ are called a subdifferential and a superdifferential, respectively. Denote by $\mathscr{D} f(x)$ the set of quasidifferentials of f at x, then $Df(x) \in \mathscr{D} f(x)$.

A set $K \subset R^n$ is called a cone if $\lambda x \in K$ when $x \in K$ and $\lambda > 0$. The set

$$K^* = \{w \in R^n \mid \langle w, v\rangle \geq 0, \forall v \in K\}$$

is called the cone conjugate to K. If K is a convex cone, the set

$$K^\circ = \{w \in R^n \mid \langle w, v\rangle \leq 0, \forall v \in K\}$$

is called the polar of K.

Let $\Omega \subset R^n$ and $x \in \operatorname{cl} \Omega$, where 'cl' refers to the closure. The set

$$\gamma(x, \Omega) = \{d \in R^n \mid \exists\, \alpha_d > 0 \text{ such that } x + \alpha d \in \Omega, \forall \alpha \in (0, \alpha_d]\}$$

is called the cone of admissible directions with respect to Ω at x.

Let Y_n be the set of all nonempty convex compact sets in R^n. Denote $A \pm B = \{a \pm b \,|\, a \in A, b \in B\}$ and $\lambda A = \{\lambda a \,|\, a \in A\}$, where $A, B \in Y_n$ and $\lambda \geq 0$.

Definition 2.1. *Suppose that the function f is defined on an open set $\mathcal{O} \subset R^n$ and quasidifferentiable at a point $x \in \mathcal{O}$, and let*

$$S = \bigcap_{Df(x) \in \mathscr{D}f(x)} (\underline{\partial} f(x) + \overline{\partial} f(x)), \quad \overline{S} = \bigcap_{Df(x) \in \mathscr{D}f(x)} (\overline{\partial} f(x) - \overline{\partial} f(x)).$$

The sets S and $\overline{S}$ are called kernel and super-kernel, respectively. $[S, \overline{S}]$ is called a quasi-kernel of $\mathscr{D}f(x)$.

Let $\Delta_n(x)$ denote the set of all functions defined on an open set $\mathcal{O} \subset R^n$ and quasidifferentiable at a point $x \in \mathcal{O}$.

Definition 2.2. *Let $f \in \Delta_n(x)$. The quasi-kernel of $\mathscr{D}f(x)$ is said to be an kernelled quasidifferential of f at x if and only if*

$$[S, \overline{S}] \in \mathscr{D}f(x),$$

and the quasi-kernel $[S, \overline{S}]$ is a quasidifferential, denoted by

$$D_k f(x) = [\underline{\partial}_k f(x), \overline{\partial}_k f(x)].$$

If f has an kernelled quasidifferential at $x \in R^n$, then f is said to be an kernelled quasidifferentiable function at x.

3. Necessary Optimality Conditions

In this section, first-order necessary optimality conditions for the unconstrained and constrained kernelled quasidifferentiable optimization are presented.

Theorem 3.1. *Let a function f be kernelled quasidifferentiable on R^n, then for $x^* \in R^n$ to be a local minimizer of the function f on R^n it is necessary that $0 \in \underline{\partial}_k f(x^*)$. If x^* is a strict local minimizer of f on R^n, then $0 \in \operatorname{int} \underline{\partial}_k f(x^*)$.*

Proof. Since f is kernelled quasidifferentiable on R^n, then $D_k f(x^*)$ exists. If x^* is a local minimizer of the function f on R^n, then by Theorem 3.1 in Chapter V of [2], one has that $-\overline{\partial}_k f(x^*) \subset \underline{\partial}_k f(x^*)$. By the Definition 2.1, it is easy to check that $0 \in \overline{\partial}_k f(x^*)$, then $0 \in \underline{\partial}_k f(x^*)$. If x^* is a strict local minimizer of f on R^n, then, according to Theorem 3.1 in Chapter V of [2], $-\overline{\partial}_k f(x^*) \subset \operatorname{int} \underline{\partial}_k f(x^*)$, hence $0 \in \operatorname{int} \underline{\partial}_k f(x^*)$. □

Theorem 3.2. *Let a function f be kernelled quasidifferentiable on R^n and K be a convex cone in R^n. If $\min_{d\in K} f'(x;d) = 0$ then $0 \in \underline{\partial}_k f(x) + K^\circ$.*

Proof. Since f is kernelled quasidifferentiable on R^n, then by Theorem 16.3 in [1], one has that $\min_{d\in K} f'(x;d) = 0$ if and only if $-\overline{\partial}_k f(x) \subset \underline{\partial}_k f(x) - K^*$. It follows immediately from the definitions of K^* and K° that $-K^* = K^\circ$, and by the Definition 2.1, it is easy to check that $0 \in \overline{\partial}_k f(x)$. Then if $-\overline{\partial}_k f(x) \subset \underline{\partial}_k f(x) - K^*$, one has that $0 \in \underline{\partial}_k f(x) + K^\circ$. □

Theorem 3.3. *Let f be a kernelled quasidifferentiable function on R^n. For f to attain its local minimum on a set $\Omega \subset R^n$ at the point $x^* \in \operatorname{int}\Omega$ it is necessary that $0 \in \underline{\partial}_k f(x^*)$.*

Proof. If f attains its local minimum on $\Omega \subset R^n$ at x^*, then by Theorem 16.1 in [1] $\min_{d\in\gamma(x^*,\Omega)} f'(x^*;d) = 0$. It follows from Theorem 3.2 that $0 \in \underline{\partial}_k f(x^*) + \gamma^\circ(x^*,\Omega)$. Since $x^* \in \operatorname{int}\Omega$, the cone $\gamma(x^*,\Omega) = R^n$, and therefore $\gamma^\circ(x^*,\Omega) = \{0\}$. Hence $0 \in \underline{\partial}_k f(x^*)$. □

A set $\Omega \subset R^n$ is called quasidifferentiable if it can be represented in the form

$$\Omega = \{x \in R^n \mid h(x) \le 0\},$$

where h is quasidifferentiable on R^n. Take $x \in R^n$ and consider the cones

$$\gamma_1(x) = \{g \in R^n \mid h'(x;g) < 0\}, \quad \Gamma_1(x) = \{g \in R^n \mid h'(x;g) \le 0\}.$$

Let $h(x) = 0$. We say that the regularity condition is satisfied at the point x if $\operatorname{cl}\gamma_1(x) = \Gamma_1(x)$.

Theorem 3.4. *Let f be a kernelled quasidifferentiable function on R^n and h be a quasidifferentiable function on R^n. Let the point $x^* \in \Omega$ and $h(x^*) = 0$. Assume also that the regularity condition holds at x^*. For f to attain its local minimum on Ω at x^* it is necessary that*

$$0 \in \bigcap_{v\in\overline{\partial} h(x^*)} (\underline{\partial}_k f(x^*) + \operatorname{cl}\operatorname{cone}(\underline{\partial} h(x^*) + v)),$$

where 'cone' *denotes the convex conical hull.*

Proof. Since f is kernelled quasidifferentiable on R^n and $x^* \in \Omega \subset R^n$, $D_k f(x^*)$ exists. Note that the regularity condition holds at x^* and $h(x^*) = 0$, then, according to Corollary 3.1 in Chapter V of [2],

$$-\overline{\partial}_k f(x^*) \subset \bigcap_{v \in \overline{\partial} h(x^*)} (\underline{\partial}_k f(x^*) + \mathrm{cl}\,\mathrm{cone}(\underline{\partial} h(x^*) + v)).$$

Since $0 \in \overline{\partial}_k f(x^*)$, then

$$0 \in \bigcap_{v \in \overline{\partial} h(x^*)} (\underline{\partial}_k f(x^*) + \mathrm{cl}\,\mathrm{cone}(\underline{\partial} h(x^*) + v)). \qquad \square$$

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