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ON FIXED POINT THEOREMS FOR (ξ,α,η)-EXPANSIVE MAPPINGS IN COMPLETE METRIC SPACES

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Abstract: In this paper, a new class of (ξ,α,η)-expansive mappings is introduced which contains the class of generalized (ξ,α)-expansive mappings has been posed by Karapinar et al. [Generalized (ξ,α)-expansive mappings and related fixed-point theorems, Journal of Inequalities and Applications 2014, 2014:22]. Moreover, a representation for a generalized (ξ,α)-expansive mapping is established and by applying it some conditions given in the literature by many authors in order to guarantee the existence of a fixed point are relaxed and by using some suitable conditions the existence of fixed points, in the setting of complete metric spaces as well as ordered complete metric spaces, for the new class of (ξ,α,η)-expansive mappings is investigated. Furthermore, the relationship between having a fixed point for a map and being one-to-one of the map is discussed.

The results of this note can be viewed as an extension of the corresponding results have been presented in [3, 5, 7, 10, 11].

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1. Introduction and Preliminaries

Fixed point theory is one of the most important tools for proving the existence and uniqueness of the solutions to various mathematical models. It has been continually studied since 1922 (see, for examples, [1, 2, 4, 9, 10] and the references contained therein). Samet et al. [10] introduced a new category of contractive type mappings known as α-ψ-contractive type mappings where $\psi \in \Psi_1$ and assured some fixed point theorems for such mappings. Shahi et al. [11] introduced a new category as a complement of the concept of α-ψ-contractive type mappings which is called (ξ, α)-expansive mappings . Karapinar et al. [5] introduced the definition of generalized (ξ, α)-expansive mappings and assured the existence of fixed point theorems for such mappings in complete metric spaces. Recently, Farajzadeh et al. [3] introduced a new family Ψ_2 of mappings and proved the existence of fixed point theorems for α-ψ-contractive type mappings where $\psi \in \Psi_2$.

In this paper, we introduce a new class of (ξ, α, η)-expansive mappings which contains all the classes of (ξ, α)-expansive mappings introduced until now, especially generalized (ξ, α)-expansive mappings has been posed by Karapinar et al. [5]. Moreover, a representation for a generalized (ξ,α)-expansive mapping is established and by applying it some conditions given in the literature by many authors in order to guarantee the existence of a fixed point are relaxed and by using some suitable conditions the existence of fixed points, in the setting of complete metric spaces as well as ordered complete metric spaces, for the new class of (ξ, α, η)-expansive mappings is investigated. Furthermore, the relationship between having a fixed point for a map and being one-to-one of the map is discussed. The results of this note can be viewed as an extension of the corresponding results have been presented in [3, 5, 7, 10, 11].

The rest of this section will deal with the definitions and preliminary results which will be needed in the next sections.

Notation 1.1. ([10]). Let Ψ_1 denote the family of all mappings $\psi : [0, +\infty) \to [0, +\infty)$ which satisfy the following properties:

(i) ψ is nondecreasing;

(ii) $\sum_{n=1}^{+\infty} \psi^n(t) < +\infty$ for each $t > 0$, where ψ^n is the nth iterate of ψ.

Lemma 1.2. *([10]). Suppose that* $\psi : [0, +\infty) \to [0, +\infty)$. *If* ψ *is nondecreasing, then for each* $t \in (0, +\infty), \lim_{n\to\infty} \psi^n(t) = 0$ *implies* $\psi(t) < t$.

Remark 1.3. It is easily seen that if $\psi : [0, +\infty) \to [0, +\infty)$ is nondecreasing and $\psi(t) < t$ for all $t \in (0, +\infty)$, then $\psi(0) = 0$.

Definition 1.4. ([10]). Let (X, d) be a metric space and $T : X \to X$ be a given self mapping. We say that T is an α-ψ-contractive mapping if there exist two mappings $\alpha : X \times X \to [0, +\infty)$ and $\psi \in \Psi_1$ such that

$$\alpha(x, y)d(Tx, Ty) \leq \psi(d(x, y))$$

for all $x, y \in X$.

Definition 1.5. ([10]). Let $T : X \to X$ and $\alpha : X \times X \to [0, +\infty)$. We say that T is α-admissible if for all $x, y \in X$,

$$\alpha(x, y) \geq 1 \text{ implies } \alpha(Tx, Ty) \geq 1.$$

Theorem 1.6. *([10]). Let (X, d) be a complete metric space and $T : X \to X$ be an α-ψ-contractive mapping satisfying the following conditions:*

(i) T is α-admissible;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;

(iii) T is continuous or if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to +\infty$, then $\alpha(x_n, x) \geq 1$ for all n.

Then T has a fixed point.

On the other hand, Shahi et al. [11] introduced a new category as a complement of the concept of α-ψ-contractive type mappings which is called (ξ, α)-expansive mappings. They proved the existence of several fixed-point theorems for such mappings in complete metric spaces.

The following family of the mappings was first introduced by Shahi et al. [11] in order to define the concept of (ξ, α)-expansive mappings.

Notation 1.7. ([11]). Let χ denote the family of all mappings $\xi : [0, +\infty) \to [0, +\infty)$ which satisfy the following properties:

(ξ_i) ξ is nondecreasing;

(ξ_{ii}) $\sum_{n=1}^{+\infty} \xi^n(t) < +\infty$ for each $t > 0$, where ξ^n is the nth iterate of ξ;

(ξ_{iii}) $\xi(s+t) = \xi(s) + \xi(t)$, for all $s, t \in [0, +\infty)$.

Definition 1.8. ([11]). Let (X, d) be a metric space and $T : X \to X$ be a given mapping. We say that T is a (ξ, α)-expansive mapping if there exist two mappings $\xi \in \chi$ and $\alpha : X \times X \to [0, +\infty)$ such that

$$\xi(d(Tx, Ty)) \geq \alpha(x, y)d(x, y) \tag{1}$$

for all $x, y \in X$.

Remark 1.9. ([5]). If $T : X \to X$ is an expansion mapping, then T is an (ξ, α)-expansive mapping, where $\alpha(x, y) = 1$ for all $x, y \in X$ and $\xi(a) = ka$ for all $a \geq 0$ and some $k \in [0, 1)$.

The following question has been raised by Remark 1.9 as follows:

Question: Is there a representation for a mapping that fulfills the conditions (ξ_i)-(ξ_{iii}) of Notation 1.7 ?

The answer is affirmative and in what follows we are going to reply it by relaxing the conditions (ξ_i) and (ξ_{ii}) for a mapping $\xi : [0, \infty) \to [0, \infty)$ that satisfies the condition (ξ_{iii}) of Notation 1.7 and is continuous at least one point in $[0, \infty)$.

Proposition 1.10. Let $\xi : [0, +\infty) \to [0, +\infty)$. If ξ is continuous at a point $a \in [0, \infty)$ and satisfies the condition (ξ_{iii}) of Notation 1.7, then $\xi(x) = x\xi(1)$ for all $x \in [0, \infty)$.

Proof. We divide the proof into four cases as follows.

Case I: Let $x \in \mathbb{N}$. Thus $\xi(x) = \xi\underbrace{(1 + 1 + \cdots + 1)}_{x \text{ times}} = x\xi(1)$.

Case II: Let $x = 0$. Therefore $\xi(0) = \xi(0 + 0) = \xi(0) + \xi(0)$ and so $\xi(0) = 0 = 0 \cdot \xi(1)$.

Case III: Let $x \in (0, \infty)$ be a rational number. Thus $x = \frac{m}{n}$ where m and n are positive integers. Since $\xi(1) = \xi(n \cdot \frac{1}{n}) = n\xi(\frac{1}{n})$, we obtain that $\xi(\frac{1}{n}) = \frac{1}{n}\xi(1)$. This implies that $\xi(x) = \xi(\frac{m}{n}) = \xi(m \cdot \frac{1}{n}) = m\xi(\frac{1}{n}) = \frac{m}{n}\xi(1) = x\xi(1)$.

Case IV: Let $x \in (0, \infty)$ be an irrational number. Since ξ is continuous at a. It follows that the exists a sequence of rational numbers $\{r_n\}$ such that $r_n > x$ for all $n \in \mathbb{N}$ converging to x. Using the continuity of ξ at a and $r_n - x + a \to a$ as $n \to \infty$, we obtain that $\xi(r_n - x + a) \to \xi(a)$ as $n \to \infty$. Applying (ξ_{iii}),

this yields $\xi(r_n - x) \to 0$ as $n \to \infty$. Since $\xi(r_n) = \xi(r_n - x) + \xi(x) \to \xi(x)$ and $\xi(r_n) = r_n\xi(1) \to x\xi(1)$ as $n \to \infty$, we have $\xi(x) = x\xi(1)$.
Hence $\xi(x) = x\xi(1)$ for all $x \in [0, \infty)$. □

Remark 1.11. It is obvious that if $\xi(x) = x\xi(1)$ for all $x \in [0, \infty)$, then ξ is nondecreasing and $\sum_{n=1}^{\infty} \xi^n(x) = x \sum_{n=1}^{\infty} \xi^n(1)$. Furthermore, the series $\sum_{n=1}^{\infty} \xi^n(x) < \infty$ for each $x \in (0, \infty)$ if and only if $0 \leq \xi(1) < 1$. Since every nondecreasing mapping is differentiable almost everywhere (see [8]), then it is continuous almost everywhere. Consequently, if we assume that $\xi : [0, \infty) \to [0, \infty)$ satisfies condition (ξ_{iii}) of Notation 1.7 and is continuous at some point in $[0, \infty)$, then we can relax the conditions (ξ_i) and (ξ_{ii}) from the Notation 1.7.

Shahi et al. [11] proved the fixed point results for (ξ, α)-expansive mappings in the setting of complete metric spaces.

Theorem 1.12. *([11]). Let (X, d) be a complete metric space and $T : X \to X$ be a bijective (ξ, α)-expansive mapping satisfying the following conditions:*

(i) T^{-1} is α-admissible;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, T^{-1}x_0) \geq 1$;

(iii) T is continuous.

Then T has a fixed point.

Karapinar et al. [5] introduced the definition of generalized (ξ, α)-expansive mappings as follows.

Definition 1.13. ([5]). Let (X, d) be a metric space and $T : X \to X$ be a given mapping. We say that T is a generalized (ξ, α)-expansive mapping if there exist two mappings $\xi \in \chi$ and $\alpha : X \times X \to [0, +\infty)$ such that for all $x, y \in X$,

$$\xi(d(Tx, Ty)) \geq \alpha(x, y)m(x, y), \tag{2}$$

where $m(x, y) = \min\{d(x, y), d(x, Tx), d(y, Ty)\}$.

Remark 1.14. (i) If $\alpha(x, y) = 0$ for all $x, y \in X$, then the inequality defined by (2) of Definition 1.13 automatically holds and thus T is a generalized (ξ, α)-expansive mapping. Hence, the Definition 1.13 is still true if we replace $\alpha : X \times X \to [0, +\infty)$ by $\alpha : X \times X \to (0, +\infty)$.

(ii) In the Definition 1.13, if $\alpha : X \times X \to (0, +\infty)$ and T does not have a fixed point, then T is one-to-one. Indeed, let $x, y \in X$ and $Tx = Ty$. Hence, $d(Tx, Ty) = 0$. It means that

$$0 = \xi(d(Tx, Ty)) \geq \alpha(x, y)m(x, y).$$

This implies that $m(x, y) = 0$, note that the range of α is a subset of $(0, +\infty)$ and T does not have any fixed point, and so T is one-to-one. This completes the proof of the assertion.

By Remark 1.14, we immediately obtain the following proposition.

Proposition 1.15. *Let (X, d) be a metric space. If $T : X \to X$ is a generalized (ξ, α)-expansive mapping and T is not one-to-one, then T has a fixed point.*

It is natural to ask a question that what assumptions will give the confirmation of the existence a fixed point for a generalized (ξ, α)-expansive mapping T where T is one-to-one. Karapinar et al. [5] assured the existence of fixed point theorems for generalized (ξ, α)-expansive mappings in complete metric spaces as follows:

Theorem 1.16. *([5]). Let (X, d) be a complete metric space and $T : X \to X$ be a bijective generalized (ξ, α)-expansive mapping satisfying the following conditions:*

(i) T^{-1} is α-admissible;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, T^{-1}x_0) \geq 1$;

(iii) T is continuous.

Then T has a fixed point.

Salimi et al. [9] modified the concept of α-admissibility.

Definition 1.17. ([9]). Let $T : X \to X$ and $\alpha, \eta : X \times X \to [0, +\infty)$. We say that T is α-admissible with respect to η if for all $x, y \in X$,

$$\alpha(x, y) \geq \eta(x, y) \text{ implies } \alpha(Tx, Ty) \geq \eta(Tx, Ty).$$

Recently, Farajzadeh et al. [3] introduced a new family Ψ_2 of mappings and proved the existence of fixed point results for α-ψ-contractive type mappings where $\psi \in \Psi_2$.

Notation 1.18. ([3]). Let Ψ_2 denote the family of all mappings $\psi : [0, +\infty) \to [0, +\infty)$ which satisfy the following properties:

(i) ψ is an upper semicontinuous mapping from the right;

(ii) $\psi(t) < t$ for all $t \in (0, +\infty)$;

(iii) $\psi(0) = 0$.

Remark 1.19. ([3]). By Lemma 1.2 for each $\psi \in \Psi_1$, we have $\psi(t) < t$ for all $t \in (0, +\infty)$ and by Remark 1.3 we obtain that $\psi(0) = 0$.

Farajzadeh et al. [3] gave some examples verifying Ψ_1 and Ψ_2 do not contain each other and proved the existence of the fixed point theorem for α-admissible mappings with respect to η where $\psi \in \Psi_2$.

Example 1.20. ([3]). The floor function $f(x) = \lfloor x \rfloor = \max\{n \in \mathbb{Z} : n \leq x\}$, where $\mathbb{Z}$ if the set of all integer numbers, is an upper semicontinuous mapping from the right and nondecreasing but is not continuous.

Example 1.21. ([3]). Let $\psi : [0, +\infty) \to [0, +\infty)$ be a mapping defined by

$$\psi(t) = \begin{cases} \frac{1}{2}, & t \in \mathbb{N}; \\ 0, & \text{otherwise.} \end{cases}$$

Thus ψ is upper semicontinuous from the right, $\psi(t) < t$ for all $t \in (0, +\infty)$ and $\psi(0) = 0$. Moreover ψ is not nondecreasing.

It is clear that Ψ_1 contains χ (see Notation 1.7). On the other hand, if $\xi \in \chi$, then $\xi(t) < t$ for all $t \in (0, +\infty)$ by Remark 1.2 and $\xi(0) = 0$ by Remark 1.19. Since ξ is nondecreasing, we can conclude that $\xi \in \Psi_2$. Hence χ is a subset of Ψ_2. Consequently, $\chi \subset \Psi_1, \Psi_2$ and the inclusion is strict.

2. Main Results

We now introduce a new class of (ξ, α, η)-expansive mappings which contains a class of generalized (ξ, α)-expansive mappings introduced by Karapinar et al. [5] and proved the fixed point theorems for (ξ, α, η)-expansive mappings satisfying α-admissibility with respect to η in complete metric spaces where $\xi \in \Psi_1$ and $\xi \in \Psi_2$.

Definition 2.1. Let (X, d) be a metric space and $T : X \to X$ be a given mapping. We say that T is a (ξ, α, η)-expansive mapping of type A if there exist three mappings $\xi \in \Psi_1$, $\alpha : X \times X \to [0, +\infty)$ and $\eta : X \times X \to (0, +\infty)$ such that for all $x, y \in X$,

$$\alpha(x, y) \geq \eta(x, y) \text{ implies } \eta(x, y)\xi(d(Tx, Ty)) \geq \alpha(x, y)m(x, y), \tag{3}$$

where $m(x, y) = \min\{d(x, y), d(x, Tx), d(y, Ty)\}$.

Remark 2.2. (i) In the Definition 2.1, if $\alpha(x, y) = 0$ for all $x, y \in X$, then since the range of η is a subset of $(0, +\infty)$ the inequality defined by (3) trivially holds. Therefore T is a (ξ, α, η)-expansive mapping of type A for any $\eta : X \times X \to (0, +\infty)$. Thus the Definition 2.1 is still true if we replace $\alpha : X \times X \to [0, +\infty)$ by $\alpha : X \times X \to (0, +\infty)$.

(ii) In the Definition 2.1, if $\alpha : X \times X \to (0, +\infty)$ and T does not have a fixed point, then T is not necessarily one-to-one. Hence there is no relationship between T does not have a fixed point and injectiveness of T as seen in Remark 1.14.

We next prove our main result to obtain the fixed point for (ξ, α, η)-expansive mappings of type A.

Theorem 2.3. *Let (X, d) be a complete metric space and $T : X \to X$ be a bijective (ξ, α, η)-expansive mapping of type A satisfying the following conditions:*

(i) T^{-1} is α-admissible with respect to η,

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, T^{-1}x_0) \geq \eta(x_0, T^{-1}x_0)$,

(iii) T is continuous.

Then T has a fixed point.

Proof. By (ii), there exists $x_0 \in X$ such that $\alpha(x_0, T^{-1}x_0) \geq \eta(x_0, T^{-1}x_0)$. Let us define the sequence $\{x_n\}$ in X by

$$x_n = Tx_{n+1}, \text{ for all } n \in \mathbb{N}.$$

Now, if $x_n = x_{n+1}$ for some $n \in \mathbb{N}$, then x_{n+1} is a fixed point of T. Without loss of generality, we can suppose $x_n \neq x_{n+1}$ for each $n \in \mathbb{N}$. Since T^{-1} is an α-admissible mapping with respect to η and $\alpha(x_0, T^{-1}x_0) \geq \eta(x_0, T^{-1}x_0)$, we deduce that $\alpha(x_1, x_2) = \alpha(T^{-1}x_0, T^{-1}x_1) \geq \eta(T^{-1}x_0, T^{-1}x_1) = \eta(x_1, x_2)$. Continuing this process, we get that

$$\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1}), \tag{4}$$

for all $n \in \mathbb{N}$.
By (3) we obtain that

$$\eta(x_n, x_{n+1})d(x_{n-1}, x_n) > \eta(x_n, x_{n+1})\xi(d(Tx_n, Tx_{n+1})) \geq \alpha(x_n, x_{n+1})m(x_n, x_{n+1})$$

owing to the fact that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all n, we have

$$\begin{aligned} d(x_{n-1}, x_n) > \xi(d(Tx_n, Tx_{n+1})) &\geq \frac{\alpha(x_n, x_{n+1})}{\eta(x_n, x_{n+1})} m(x_n, x_{n+1}) \geq m(x_n, x_{n+1}) \\ &= \min\{d(x_n, x_{n+1}), d(x_n, Tx_n), d(x_{n+1}, Tx_{n+1})\} \\ &= \min\{d(x_n, x_{n+1}), d(x_{n-1}, x_n), d(x_n, x_{n+1})\} \\ &= \min\{d(x_n, x_{n+1}), d(x_{n-1}, x_n)\}. \end{aligned}$$

Now, if $\min\{d(x_n, x_{n+1}), d(x_{n-1}, x_n)\} = d(x_{n-1}, x_n)$ for some $n \in \mathbb{N}$, then

$$d(x_{n-1}, x_n) > \xi(d(Tx_n, Tx_{n+1})) \geq d(x_{n-1}, x_n).$$

which is a contradiction. Hence for all $n \in \mathbb{N}$, we obtain that

$$\xi(d(x_{n-1}, x_n)) \geq d(x_n, x_{n+1}).$$

By induction, we have

$$\xi^n(d(x_0, x_1)) \geq d(x_n, x_{n+1}).$$

Since $\xi^n(d(x_0, x_1)) < \infty$, for each $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\sum_{n \geq N} \xi^n(d(x_0, x_1)) < \varepsilon.$$

For any $m, n \in \mathbb{N}$ and $m > n \geq N$, we obtain that

$$d(x_n, x_m) \leq \sum_{k=n}^{m-1} (d(x_k, x_{k+1})) \leq \sum_{n \geq N} \xi^n(d(x_0, x_1)) < \varepsilon.$$

It follows that $\{x_n\}$ is a Cauchy sequence in the complete metric space (X, d). So, there exists $u \in X$ such that $x_n \to u$ as $n \to +\infty$. From the continuity of T, it follows that $Tu = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n-1} = u$. So u is a fixed point of T. □

Definition 2.4. Let (X, d) be a metric space and $T : X \to X$ be a given mapping. We say that T is a (ξ, α, η)-expansive mapping of type B if there exist three mappings $\xi \in \Psi_2$, $\alpha : X \times X \to [0, +\infty)$ and $\eta : X \times X \to (0, +\infty)$ such that for all $x, y \in X$,

$$\alpha(x, y) \geq \eta(x, y) \text{ implies } \eta(x, y)\xi(d(Tx, Ty)) \geq \alpha(x, y)m(x, y), \tag{5}$$

where $m(x, y) = \min\{d(x, y), d(x, Tx), d(y, Ty)\}$.

Remark 2.5. (i) In the Definition 2.4, if $\alpha(x, y) = 0$ for all $x, y \in X$, then the inequality (5) automatically holds and therefore T is a (ξ, α, η)-expansive mapping of type B for any $\eta : X \times X \to (0, +\infty)$. Thus the Definition 2.4 is still true if we replace $\alpha : X \times X \to [0, +\infty)$ by $\alpha : X \times X \to (0, +\infty)$.

(ii) In the Definition 2.4, if $\alpha : X \times X \to (0, +\infty)$ and T does not have a fixed point, then T is not necessarily one-to-one. Hence there is no relationship between T does not have a fixed point and injectiveness of T as seen in Remark 1.14.

We will prove the existence of the fixed point theorem for a (ξ, α, η)-expansive mapping of type B.

Theorem 2.6. *Let (X, d) be a complete metric space and $T : X \to X$ be a bijective (ξ, α, η)-expansive mapping of type B satisfying the following conditions:*

(i) T^{-1} is α-admissible with respect to η,

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, T^{-1}x_0) \geq \eta(x_0, T^{-1}x_0)$,

(iii) T is continuous.

Then T has a fixed point.

Proof. By(ii), there exists $x_0 \in X$ such that $\alpha(x_0, T^{-1}x_0) \geq \eta(x_0, T^{-1}x_0)$. Let us define the sequence $\{x_n\}$ in X by

$$x_n = Tx_{n+1}, \text{ for all } n \in \mathbb{N}.$$

Now, if $x_n = x_{n+1}$ for some $n \in \mathbb{N}$, then x_{n+1} is a fixed point of T. Without loss of generality, we can suppose $x_n \neq x_{n+1}$ for each $n \in \mathbb{N}$. Since T^{-1} is an α-admissible mapping respect to η and $\alpha(x_0, T^{-1}x_0) \geq \eta(x_0, T^{-1}x_0)$, we deduce that $\alpha(x_1, x_2) = \alpha(T^{-1}x_0, T^{-1}x_1) \geq \eta(T^{-1}x_0, T^{-1}x_1) = \eta(x_1, x_2)$. Continuing this process, we get

$$\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1}), \text{ for all } n \in \mathbb{N}.$$

Now, by (5), we get that

$$\eta(x_n, x_{n+1})d(x_{n-1}, x_n) > \eta(x_n, x_{n+1})\xi(d(Tx_n, Tx_{n+1})) \geq \alpha(x_n, x_{n+1})m(x_n, x_{n+1})$$

owing to the fact that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all n, we have

$$\begin{aligned} \eta(x_n, x_{n+1})d(x_{n-1}, x_n) &> \eta(x_n, x_{n+1})\xi(d(Tx_n, Tx_{n+1})) \\ &\geq \eta(x_n, x_{n+1})m(x_n, x_{n+1}). \end{aligned}$$

Therefore

$$\begin{aligned} d(x_{n-1}, x_n) > \xi(d(Tx_n, Tx_{n+1})) &\geq m(x_n, x_{n+1}) \\ &= \min\{d(x_n, x_{n+1}), d(x_n, Tx_n), d(x_{n+1}, Tx_{n+1})\} \\ &= \min\{d(x_n, x_{n+1}), d(x_{n-1}, x_n), d(x_n, x_{n+1})\} \\ &= \min\{d(x_n, x_{n+1}), d(x_{n-1}, x_n)\}. \end{aligned}$$

Now, if $\min\{d(x_n, x_{n+1}), d(x_{n-1}, x_n)\} = d(x_{n-1}, x_n)$ for some $n \in \mathbb{N}$, then

$$d(x_{n-1}, x_n) > \xi(d(Tx_n, Tx_{n+1})) \geq d(x_{n-1}, x_n).$$

which is a contradiction. Hence for all $n \in \mathbb{N}$, we obtain that

$$d(x_{n-1}, x_n) > \xi(d(Tx_n, Tx_{n+1})) \geq d(x_n, x_{n+1}). \tag{6}$$

for all $n \in \mathbb{N}$. Therefore $\{d(x_n, x_{n+1})\}$ is nonincreasing sequence. It follows that there exists $c \geq 0$ such that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = c.$$

We will prove that $c = 0$. Suppose that $c > 0$. Since ξ is upper semicontinuous from the right and by using (6), we have

$$c = \limsup_{n\to\infty} d(x_n, x_{n+1}) \leq \limsup_{n\to\infty} \xi(d(x_{n-1}, x_n)) \leq \xi(c) < c,$$

which leads to a contradiction. Therefore

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

This implies that for each $k \in \mathbb{N}$, there exists $n_k \in \mathbb{N}$ such that

$$d(x_{n_k}, x_{n_k+1}) < \frac{1}{2^k}.$$

We obtain that $\sum_{k=1}^{\infty} d(x_{n_k}, x_{n_k+1}) < \infty$.
Therefore $\{x_{n_k}\}$ is a Cauchy sequence and so converges to some $x \in X$. By continuity of T, we have

$$\lim_{n\to\infty} x_{n_k-1} = \lim_{n\to\infty} Tx_{n_k} = Tx.$$

This implies that x is a fixed point of T. □

We now prove that Theorem 2.3 and Theorem 2.7 still hold by replacing the continuity of T by the following condition:

(P) If $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all n and $\{x_n\} \to x \in X$ as $n \to +\infty$, then

$$\alpha(T^{-1}x_n, T^{-1}x) \geq \eta(T^{-1}x_n, T^{-1}x) \text{ for all } n.$$

Theorem 2.7. *If in Theorem 2.3 we replace the continuity of T by the condition (P), then the result holds.*

Proof. As in the proof of Theorem 2.3, we can construct a sequence $\{x_n\}$ in X such that

$$x_n = Tx_{n+1},\ \alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1}) \text{ for all } n \in \mathbb{N}$$

and $x_n \to u \in X$ as $n \to +\infty$. Using condition (P), we obtain that

$$\alpha(T^{-1}x_n, T^{-1}u) \geq \eta(T^{-1}x_n, T^{-1}u) \text{ for all } n \in \mathbb{N}. \tag{7}$$

Applying (3) and (7), we have

$$\begin{aligned}\eta(T^{-1}x_n, T^{-1}u)m(T^{-1}x_n, T^{-1}u) &\leq \alpha(T^{-1}x_n, T^{-1}u)m(T^{-1}x_n, T^{-1}u)\\ &\leq \eta(T^{-1}x_n, T^{-1}u)\xi(d(x_n, u)),\end{aligned}$$

for all $n \in \mathbb{N}$. This implies that $m(T^{-1}x_n, T^{-1}u) \leq \xi(d(x_n, u))$ for all $n \in \mathbb{N}$. Taking $n \to +\infty$ in the above inequality and using the continuity of ξ at 0, we can conclude that $m(u, T^{-1}u) = 0$. Since

$$\begin{aligned}m(u, T^{-1}u) &= \min\{d(u, T^{-1}u), d(u, Tu), d(u, u))\}\\ &= \min\{d(u, T^{-1}u), d(u, Tu)\},\end{aligned}$$

we have $d(u, T^{-1}u) = 0$ or $d(u, Tu) = 0$. Hence u is a fixed point of T. □

Theorem 2.8. *If in Theorem 2.6 we replace the continuity of T by the condition P, then the result holds.*

Proof. As in the proof of Theorem 2.6, we can construct a sequence $\{x_n\}$ in X such that

$$x_n = Tx_{n+1},\ \alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1}) \text{ for all } n \in \mathbb{N}$$

and $x_n \to u \in X$ as $n \to +\infty$. Using condition (P), we obtain that

$$\alpha(T^{-1}x_n, T^{-1}u) \geq \eta(T^{-1}x_n, T^{-1}u) \text{ for all } n \in \mathbb{N}, \tag{8}$$

for all $n \in \mathbb{N}$. Applying (3) and (8), we have

$$\begin{aligned}\eta(T^{-1}x_n, T^{-1}u)m(T^{-1}x_n, T^{-1}u) &\leq \alpha(T^{-1}x_n, T^{-1}u)m(T^{-1}x_n, T^{-1}u)\\ &\leq \eta(T^{-1}x_n, T^{-1}u)\xi(d(x_n, u)),\end{aligned}$$

for all $n \in \mathbb{N}$. This implies that $m(T^{-1}x_n, T^{-1}u) \leq \xi(d(x_n, u))$ for all $n \in \mathbb{N}$. Using upper semicontinuity from the right of ξ, we obtain that

$$\limsup_{n\to\infty} m(T^{-1}x_n, T^{-1}u) \leq \limsup_{n\to\infty} \xi(d(x_n, u)) \leq \xi(0) = 0.$$

This implies that $m(u, T^{-1}u) = 0$. Since

$$\begin{aligned}m(u, T^{-1}u) &= \min\{d(u, T^{-1}u), d(u, Tu), d(u, u))\}\\ &= \min\{d(u, T^{-1}u), d(u, Tu)\},\end{aligned}$$

we have $d(u, T^{-1}u) = 0$ or $d(u, Tu) = 0$. Hence u is a fixed point of T. □

3. Consequences

3.1. Fixed Point Results on Partially Ordered Metric Spaces

In this section, we present the fixed point results for some expansive mappings in metric spaces endowed with partial orders.

Theorem 3.1. *Let $(X, \preccurlyeq)$ be a partially ordered set and suppose that there exists a metric d in X such that the metric space (X, d) is complete. Let $T : X \to X$ be a bijective mapping such that T^{-1} is a nondecreasing mapping with respect to $\preccurlyeq$. Assume that the following conditions hold:*

(i) for all $x, y \in X$ with $x \preccurlyeq y$, $\xi(d(Tx, Ty)) \geq m(x, y)$ where $m(x, y) = \min\{d(x, y), d(x, Tx), d(y, Ty)\}$ and $\xi \in \Psi_1$;

(ii) there exists $x_0 \in X$ such that $x_0 \preccurlyeq T^{-1}x_0$;

(iii) T is continuous.

Then T has a fixed point.

Proof. Suppose that $\alpha : X \times X \to [0, +\infty)$ and $\eta : X \times X \to (0, +\infty)$ are mappings defined by

$$\alpha(x, y) = \begin{cases} 1, & x \preccurlyeq y; \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \eta(x, y) = \begin{cases} 1, & x \preccurlyeq y; \\ 2, & \text{otherwise.} \end{cases}$$

Let $x, y \in X$ such that $\alpha(x, y) \geq \eta(x, y)$. This implies that $x \preccurlyeq y$. Since T^{-1} is nondecreasing with respect to $\preccurlyeq$, we obtain that $T^{-1}x \preccurlyeq T^{-1}y$. Therefore $\alpha(T^{-1}x, T^{-1}y) \geq \eta(T^{-1}x, T^{-1}y)$. It follows that T^{-1} is α-admissible with respect to η. For each $x, y \in X$ with $\alpha(x, y) \geq \eta(x, y)$, we obtain that $x \preccurlyeq y$ and this yields

$$\eta(x, y)\xi(d(Tx, Ty)) = \xi(d(Tx, Ty)) \geq m(x, y) = \alpha(x, y)m(x, y).$$

Therefore T is a (ξ, α, η)-expansive mapping of type A. By using (ii), we have $\alpha(x_0, T^{-1}x_0) \geq \eta(x_0, T^{-1}x_0)$. Hence all assumptions in Theorem 2.3 are now satisfied. Thus we obtain the desired result. □

Theorem 3.2. *Let $(X, \preccurlyeq)$ be a partially ordered set and suppose that there exists a metric d in X such that the metric space (X, d) is complete. Let $T : X \to X$ be a bijective mapping such that T^{-1} is a nondecreasing mapping with respect to $\preccurlyeq$. Assume that the following conditions hold:*

(i) *for all* $x, y \in X$ *with* $x \preccurlyeq y$, $\xi(d(Tx, Ty)) \geq m(x, y)$ *where* $m(x, y) = \min\{d(x, y), d(x, Tx), d(y, Ty)\}$ *and* $\xi \in \Psi_1$;

(ii) *there exists* $x_0 \in X$ *such that* $x_0 \preccurlyeq T^{-1}x_0$;

(iii) *if* $\{x_n\}$ *is a nondecreasing sequence in* X *such that* $x_n \to x$ *as* $n \to \infty$, *then* $x_n \preccurlyeq x$ *for all* $n \in \mathbb{N}$.

Then T *has a fixed point.*

Proof. Suppose that $\alpha : X \times X \to [0, +\infty)$ and $\eta : X \times X \to (0, +\infty)$ are mappings defined as in the proof of Theorem 3.1. Assume that $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$ and $x_n \to x \in X$ as $n \to \infty$. This implies that $x_n \preccurlyeq x_{n+1}$ for all $n \in \mathbb{N}$. Using (iii), this yield $x_n \preccurlyeq x$ for all $n \in \mathbb{N}$. Therefore $\alpha(x_n, x) \geq \eta(x_n, x)$ for all $n \in \mathbb{N}$. Hence all assumptions in Theorem 2.7 are now satisfied. Thus we obtain the desired result. □

If ξ in Theorem 3.1 and Theorem 3.2 belong to Ψ_2, then the results still hold.

Theorem 3.3. *Let* $(X, \preccurlyeq)$ *be a partially ordered set and suppose that there exists a metric* d *in* X *such that the metric space* (X, d) *is complete. Let* $T : X \to X$ *be a bijective mapping such that* T^{-1} *is a nondecreasing mapping with respect to* $\preccurlyeq$. *Assume that the following conditions hold:*

(i) *for all* $x, y \in X$ *with* $x \preccurlyeq y$, $\xi(d(Tx, Ty)) \geq m(x, y)$ *where* $m(x, y) = \min\{d(x, y), d(x, Tx), d(y, Ty)\}$ *and* $\xi \in \Psi_2$;

(ii) *there exists* $x_0 \in X$ *such that* $x_0 \preccurlyeq T^{-1}x_0$;

(iii) T *is continuous or if* $\{x_n\}$ *is a nondecreasing sequence in* X *such that* $x_n \to x$ *as* $n \to \infty$, *then* $x_n \preccurlyeq x$ *for all* $n \in \mathbb{N}$.

Then T *has a fixed point.*

3.2. Fixed Point Results for Cyclic Contractive Mappings on Complete Metric Spaces

The cyclic contractive mappings have been introduced by Kirk et al. [6]. Such mappings can be applying to generalize the Banach contraction mappings principle.

Definition 3.4. ([6]). Let (X, d) be a metric space and A, B are nonempty subsets of X. A mapping $T : A \cup B \to A \cup B$ is cyclic if $T(A) \subseteq B$ and $T(B) \subseteq A$. A mapping T is called a cyclic contractive mapping if there exists $k \in (0, 1)$ such that

$$d(Tx, Ty) \leq kd(x, y),$$

for all $x \in A$ and $y \in B$.

We now prove the following fixed point theorems using our main theorems.

Theorem 3.5. *Let (X, d) be a complete metric space. Suppose that A and B are nonempty closed subsets of X and $T : Y \to Y$ is a bijective mapping where $Y = A \cup B$. Assume that the following conditions hold:*

(i) for all $(x, y) \in A \times B$, $\xi(d(Tx, Ty)) \geq m(x, y)$ where $m(x, y) = \min\{d(x, y), d(x, Tx), d(y, Ty)\}$ and $\xi \in \Psi_1$;

(ii) $T^{-1}(A) \subseteq B$ and $T^{-1}(B) \subseteq A$.

Then T has a fixed point.

Proof. Since A and B are closed subsets of a complete metric space (X, d) we obtain that (Y, d) is complete. Suppose that $\alpha : Y \times Y \to [0, +\infty)$ and $\eta : Y \times Y \to (0, +\infty)$ are mappings defined by

$$\alpha(x, y) = \begin{cases} 1, & (x, y) \in (A \times B) \cup (B \times A); \\ 0, & \text{otherwise} \end{cases}$$

and

$$\eta(x, y) = \begin{cases} 1, & (x, y) \in (A \times B) \cup (B \times A); \\ 2, & \text{otherwise.} \end{cases}$$

Let $x, y \in Y$ such that $\alpha(x, y) \geq \eta(x, y)$. This implies that $(x, y) \in (A \times B) \cup (B \times A)$. If $(x, y) \in A \times B$, then $(T^{-1}(x), T^{-1}(y)) \in B \times A$. Therefore $\alpha(T^{-1}(x), T^{-1}(y)) \geq \eta(T^{-1}(x), T^{-1}(y))$. On the other hand, if $(x, y) \in B \times A$, then $(T^{-1}(x), T^{-1}(y)) \in A \times B$. Therefore $\alpha(T^{-1}(x), T^{-1}(y)) \geq \eta(T^{-1}(x), T^{-1}(y))$. It follows that T^{-1} is α-admissible with respect to η. For each $x, y \in Y$ with $\alpha(x, y) \geq \eta(x, y)$, we obtain that

$$\eta(x, y)\xi(d(Tx, Ty)) = \xi(d(Tx, Ty)) \geq m(x, y) = \alpha(x, y)m(x, y).$$

Therefore T is a (ξ, α, η)-expansive mapping of type A. Assume that $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$ and $x_n \to x \in$

X as $n \to \infty$. This implies that

$$(x_n, x_{n+1}) \in (A \times B) \cup (B \times A) \quad \text{for all } n \in \mathbb{N}.$$

Using the closedness of $(A \times B) \cup (B \times A)$, this yields $(x, x) \in (A \times B) \cup (B \times A)$. This implies that $x \in A \cap B$ and so $\alpha(x_n, x) \geq \eta(x_n, x)$ for all $n \in \mathbb{N}$. Furthermore, if $z \in A$, then by using (i) we can conclude that $(z, T^{-1}(z)) \in A \times B$. This means that $\alpha(z, T^{-1}(z)) \geq \eta(z, T^{-1}(z))$. Hence all assumptions in Theorem 2.7 are now satisfied. Thus we obtain the desired result. □

If ξ in Theorem 3.5 belongs to Ψ_2, then the result still holds.

Theorem 3.6. *Let (X, d) be a complete metric space. Suppose that A and B are nonempty closed subsets of X and $T : Y \to Y$ is a bijective mapping where $Y = A \cup B$. Assume that the following conditions hold:*

(i) for all $(x, y) \in A \times B$, $\xi(d(Tx, Ty)) \geq m(x, y)$ where $m(x, y) = \min\{d(x, y), d(x, Tx), d(y, Ty)\}$ and $\xi \in \Psi_2$;

(ii) $T^{-1}(A) \subseteq B$ and $T^{-1}(B) \subseteq A$.

Then T has a fixed point.

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